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LOGHARMONIC MAPPINGS WITH TYPICALLY REAL ANALYTIC COMPONENTS

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Abstract. This paper treats the class of normalized logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ in the unit disk satisfying $\varphi(z) = zh(z)g(z)$ is analytically typically real. Every such mapping f admits an integral representation in terms of its second dilatation function and a function of positive real part with real coefficients. The radius of starlikeness and an upper estimate for arclength are obtained. Additionally, it is shown that f maps the unit disk into a domain symmetric with respect to the real axis when its second dilatation has real coefficients.

1. Introduction

Let $\mathcal{H}(U)$ be the linear space of analytic functions defined in the unit disk $U = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. Let B denote the set of self-maps $a \in \mathcal{H}(U)$, and B_0 its subclass consisting of $a \in B(U)$ with $a(0) = 0$. A logharmonic mapping in U is a solution of the nonlinear elliptic partial differential equation

$$\overline{\left(\frac{f_{\overline{z}}(z)}{f(z)}\right)} = a(z)\frac{f_z(z)}{f(z)}, \tag{1.1}$$

where the second dilatation function a lies in B. Thus the Jacobian

$$J_f = |f_z|^2 (1 - |a|^2)$$

is positive, and all non-constant logharmonic mappings are sense-preserving and open in U.

If f is a non-constant logharmonic mapping which vanishes only at $z = 0$, then [4] shows that f admits the representation

$$f(z) = z^m|z|^{2\beta m}h(z)\overline{g(z)}, \tag{1.2}$$

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where m is a positive integer, Re $\beta > -1/2$, and $h, g \in \mathcal{H}(U)$ satisfy $g(0) = 1$ and $h(0) \neq 0$. The exponent β in (1.2) depends only on $a(0)$ and is given by

$$\beta = \overline{a(0)} \frac{1 + a(0)}{1 - |a(0)|^2}.$$

Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent logharmonic mapping vanishes at the origin if and only if $m = 1$, that is, f has the form

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)},$$

where $0 \notin (hg)(U)$. This class has been studied extensively over recent years in [1–8].

As further evidence of its importance, note that $F(\zeta) = \log f(e^\zeta)$ are univalent harmonic mappings of the half-plane $\{\zeta : \text{Re } \zeta < 0\}$. Studies on univalent harmonic mappings can be found in [9–13, 15–17], which are closely related to the theory of minimal surfaces (see [19, 20]).

An analytic function φ in U is typically real if $\varphi(z)$ is real whenever z is real and nonreal elsewhere. Similarly, a logharmonic mapping f in U is typically real if $f(z)$ is real whenever z is real and nonreal elsewhere. Investigations into typically real logharmonic mappings was initiated by Abdulhadi in [2].

Denote by HG the class of analytic functions $\varphi(z) = zh(z)g(z)$, where h and g in $\mathcal{H}(U)$ are normalized by $h(0) = 1 = g(0)$, and $0 \notin (hg)(U)$. This paper treats the class T_{Ra} of logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ satisfying $\varphi(z) = zh(z)g(z) \in HG$ is analytically typically real in U.

In Section 2, every mapping $f \in T_{Ra}$ is shown to admit an integral representation in terms of its second dilatation function and a function of positive real part with real coefficients. The radius of starlikeness is also obtained for the class T_{Ra}, as well as an upper estimate for its arclength.

For an analytic univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, it is known [2] that f is typically real if and only if the image $f(U)$ is a domain symmetric with respect to the real axis. However, this characterization no longer holds for logharmonic maps, that is, it is not true that a univalent logharmonic mapping $F(z) = zh(z)\overline{g(z)} \in T_{Ra}$ if and only if the image $F(U)$ is a symmetric domain with respect to the real axis.

As an illustration, Figure 1 shows the mapping $F(z) = z(1+iz/3)(1+i\overline{z}/3) \in T_{Ra}$ but $F(U)$ is not a symmetric domain with respect to the real axis.

On the other hand, Figure 2 shows the mapping

$$F(z) = z \exp\{\text{Re } (4z/(1-z))\} (1-\overline{z})/(1-z)$$

which does not belong to the class T_{Ra}, but yet maps U onto a symmetric domain with respect to the real axis $F(U)$.

In Section 3 we explore conditions on the dilatation a that would ensure a logharmonic mapping $f(z) = zh(z)\overline{g(z)} \in T_{Ra}$ necessarily satisfies $f(U)$ is symmetric with respect to the real axis. Sufficient conditions for univalent

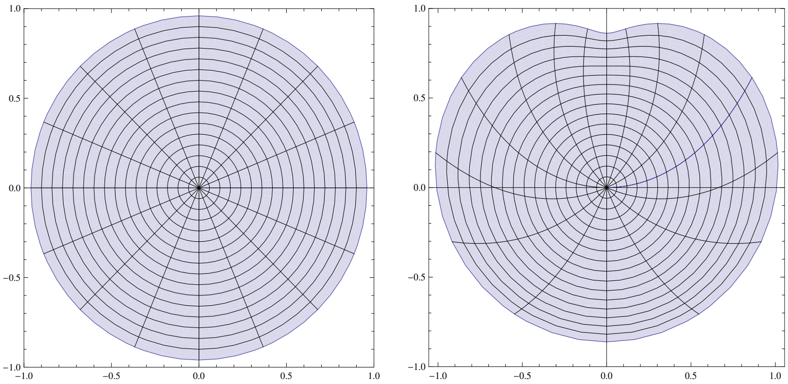

FIGURE 1. Graph of $F(z) = z(1 + \frac{iz}{3})(1 + \frac{i\overline{z}}{3})$.

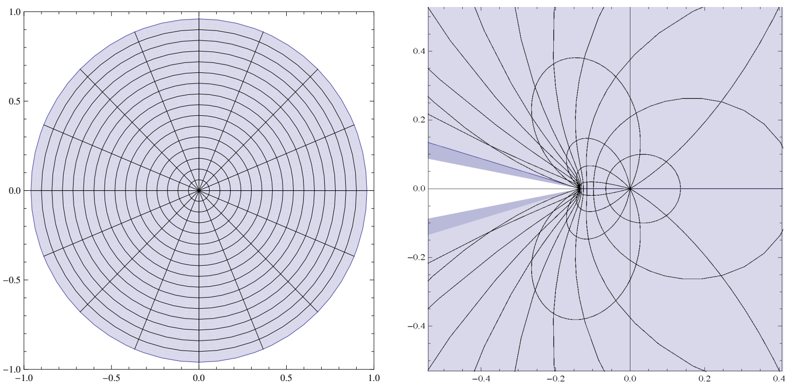

FIGURE 2. Graph of $F(z) = z\frac{1-\overline{z}}{1-z} \exp\left\{\mathrm{Re}\left(\frac{4z}{1-z}\right)\right\}$.

logharmonic mappings to belong to the class T_{Ra} in some subdisk of U are also determined.

2. An integral representation and radius of starlikeness

Let us denote by $\mathcal{P}_{\mathbb{R}}$ the class of normalized analytic functions with positive real part and with real coefficients in U. The following result gives a representation of $f \in T_{Ra}$ in terms of the dilatation a and $p \in \mathcal{P}_{\mathbb{R}}$.

Theorem 1. *Let $f = zh(z)\overline{g(z)}$ belongs to T_{Ra} with respect to $a \in B_0$. Then*

$$f(z) = \frac{zp(z)}{1-z^2} \exp\left(-2i\,\mathrm{Im} \int_0^z \frac{a(s)}{1+a(s)} \left(\frac{1+s^2}{s(1-s^2)} + \frac{p'(s)}{p(s)}\right) ds\right)$$

for some $p \in \mathcal{P}_{\mathbb{R}}$.

Proof. Let $\varphi(z) = zh(z)g(z)$, and

$$f(z) = \varphi(z)\frac{\overline{g(z)}}{g(z)}. \tag{2.1}$$

It follows from (1.1) that

$$\frac{g'(z)}{g(z)} = \frac{a(z)}{1+a(z)}\frac{\varphi'(z)}{\varphi(z)},$$

which readily yields

$$g(z) = \exp\int_0^z \frac{a(s)}{1+a(s)}\frac{\varphi'(s)}{\varphi(s)}ds. \tag{2.2}$$

Substituting (2.2) into (2.1) yields

$$f(z) = \varphi(z)\exp\left(-2i\,\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{\varphi'(s)}{\varphi(s)}ds\right).$$

It is known [21] that every typically real analytic function φ has the form $(1-z^2)\varphi(z) = zp(z)$ for some $p \in \mathcal{P}_{\mathbb{R}}$, which yields the desired result. □

In the next result, we obtain an estimate on the radius of starlikeness for the class T_{Ra}.

Theorem 2. *Let $f(z) = zh(z)\overline{g(z)} \in T_{Ra}$ with respect to $a \in B_0$. Then f maps the disk $|z| < 3 - 2\sqrt{2}$ onto a starlike domain.*

Proof. The function f maps the circle $|z| = r$ onto a starlike curve provided

$$\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) = \mathrm{Im}\left(\frac{\partial}{\partial\theta}\log f(re^{i\theta})\right) = \mathrm{Re}\,\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > 0.$$

With $\varphi(z) = zh(z)g(z)$, a short computation gives

$$\mathrm{Re}\,\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = \mathrm{Re}\,\left(\frac{1-a(z)}{1+a(z)}\frac{z\varphi'(z)}{\varphi(z)}\right)$$

for some $a \in B_0$.

Next let

$$q(z) = \frac{1-a(z)}{1+a(z)}\frac{z\varphi'(z)}{\varphi(z)},$$

and $\sigma(z) = \rho_0 z$. Kirwan [18] has shown that the radius of starlikeness for typically real analytic functions φ is $\rho_0 = \sqrt{2}-1$. Thus $\mathrm{Re}\,\{\zeta\varphi'(\zeta)/\varphi(\zeta)|_{(\sigma(z))}\} > 0$, and so $q(\sigma(z))$ is subordinated to $((1+z)/(1-z))^2$ in U.

Writing $p(z) = (1+z)/(1-z)$, it follows from [14, p. 84] that

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}.$$

Thus $|\arg(p(z))| < \pi/4$ provided $|z| < \rho_0$, where ρ_0 is a smallest positive root of the equation $r^2 - 2\sqrt{2}r + 1 = 0$. The function $f(z) = zh(z)\overline{g(z)}$ is thus starlike in the disk $|z| < \rho_0^2 = 3 - 2\sqrt{2}$. □

The next result gives an upper estimate for arclength of all mappings f in the class T_{Ra}.

Theorem 3. *Let $f(z) = zh(z)\overline{g(z)} \in T_{Ra}$ with respect to $a \in B_0$. Then for $|z| = r$, an upper bound for arclength $L(r)$ is given by*

$$L(r) \leq \frac{2\pi r(1+r)^2}{(1-r)^4}.$$

Proof. Let C_r denote the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$. Then

$$\begin{aligned} L(r) &= \int_{C_r} |df| \; = \int_0^{2\pi} |zf_z - \overline{z}f_{\overline{z}}|d\theta \\ &\leq M(r)\int_0^{2\pi} \left|\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right| d\theta, \end{aligned}$$

where $|f(z)| \leq M(r)$, $0 < r < 1$. Let $\varphi(z) = zh(z)g(z)$. Since φ is a typically real analytic function, and $|f| = |\varphi|$, then $|z\varphi'(z)/\varphi(z)| \leq (1+r)/(1-r)$ and $M(r) \leq r/(1-r)^2$.

Further

$$\begin{aligned} \left|\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right| &= \left|\frac{z\varphi'(z)}{\varphi(z)} - 2\mathrm{Re}\,\left(\frac{a(z)}{1+a(z)}\frac{z\varphi'(z)}{\varphi(z)}\right)\right| \\ &\leq \frac{1+r}{1-r} + 2\frac{r}{1-r}\frac{1+r}{1-r}, \end{aligned}$$

and thus,

$$L(r) \leq \frac{2\pi r(1+r)^2}{(1-r)^4}. \qquad \square$$

3. Logharmonic mappings in the class T_{Ra}

The following result is readily established, and thus the proof is omitted. It describes the geometry of a logharmonic function in the class T_{Ra} when its second dilatation has real coefficients.

Theorem 4. *Let $f(z) = zh(z)\overline{g(z)} \in T_{Ra}$ be a sense-preserving logharmonic mapping in U. If the second dilatation function a has real coefficients, that is, $a(\overline{z}) = \overline{a(z)}$, then $f(U)$ is symmetric with respect to the real axis.*

The final result derives sufficient conditions for $f \in T_{Ra}$ in some subdisk of U.

Theorem 5. *Let $f(z) = zh(z)\overline{g(z)}$ be a univalent sense-preserving logharmonic mapping in U normalized by $h(0) = 1 = g(0)$, where its second dilatation function a has real coefficients. Further, suppose $f(U) = \Omega$, $\Omega \neq \mathbb{C}$, is a strictly starlike Jordan domain. If $f(U)$ is a symmetric domain with respect to the real axis, and $|a(z)| \leq k < 1$ in U, then $\varphi(z) = zh(z)g(z)$ is typically real in the disk $|z| < \sqrt{2} - 1$.*

Proof. The domain Ω is a strictly starlike Jordan domain provided each radial ray from 0 intersects the boundary $\partial\Omega$ of $\Omega = f(U)$ at exactly one point of $\mathbb{C}$. Further, since $|a(z)| \leq k < 1$ in U, it follows from [7, Lemma 2.4] that there is only one univalent logharmonic mapping from U onto Ω which is a solution of (1.1) normalized by $f(0) = 0$ and $h(0) = 1 = g(0)$.

Since a has real coefficients, then $\overline{a(\overline{z})} = a(z)$. On the other hand, the mapping $F(z) = \overline{f(\overline{z})}$ is also univalent and logharmonic in U, where $F(U) = f(U)$.

If $F(z) = zH(z)\overline{G(z)} = z\overline{h(\overline{z})}g(\overline{z})$ with $H(z) = \overline{h(\overline{z})}$ and $G(z) = \overline{g(\overline{z})}$, then F satisfies the normalization $F(0) = 0, H(0) = 1 = G(0)$, and F is a solution of

$$\frac{\overline{F_{\overline{z}}(z)}}{\overline{F(z)}} = \overline{a(\overline{z})}\frac{F_z(z)}{F(z)} = a(z)\frac{F_z(z)}{F(z)}.$$

Thus, F is a logharmonic mapping with respect to the same a, and consequently, $f(z) \equiv F(z)$ in U. This implies f has real coefficients, and so $\psi(z) = zh(z)/g(z) = f(z)/|g(z)|^2$ has real coefficients.

Direct calculations yield

$$\frac{g'(z)}{g(z)} = \frac{a(z)}{1 - a(z)}\frac{\psi'(z)}{\psi(z)},$$

which upon integrating leads to

$$g(z) = \exp \int_0^z \frac{a(t)}{1 - a(t)}\frac{\psi'(t)}{\psi(t)}dt.$$

Then g, and so does h, have real coefficients, and thus $\varphi(z) = zh(z)g(z)$ has real coefficients. Furthermore, [5, Theorem 3.1] shows that φ is starlike univalent in the disk $|z| < \rho$, where $\rho = \sqrt{2} - 1$. Thus φ is typically real in the disk $|z| < \sqrt{2} - 1$. □

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