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THE SHARP BOUND OF THE THIRD HANKEL DETERMINANT FOR SOME CLASSES OF ANALYTIC FUNCTIONS

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Abstract. In the present paper, we have proved the sharp inequality $|H_{3,1}(f)| \le 4$ and $|H_{3,1}(f)| \le 1$ for analytic functions f with $a_n := f^{(n)}(0)/n!$, $n \in \mathbb{N}$, such that

$$\operatorname{Re} \frac{f(z)}{z} > \alpha, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

for $\alpha = 0$ and $\alpha = 1/2$, respectively, where

$$H_{3,1}(f) := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

is the third Hankel determinant.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be its subclass normalized by $f(0) = 0$, $f'(0) = 1$, i.e., functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1, \ z \in \mathbb{D}. \tag{1.1}$$

For $q, n \in \mathbb{N}$, the Hankel determinants $H_{q,n}(f)$ of functions $f \in \mathcal{A}$ of the form (1.1) are defined by

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

Computing the upper bound of $H_{q,n}$ over subfamilies of $\mathcal{A}$ is an interesting problem to study. Recently many authors have examined the Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$ of order 2 (see e.g., [4, 5, 9, 13, 19]). Note also

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that $H_{2,1}(f) = a_3 - a_2^2$ is the well-known coefficient functional which for $\mathcal{S}$ was estimated in 1916 by Bieberbach (see e.g., [8, Vol. I, p. 35]). To find the upper bound of the Hankel determinant

$$(1.2)\quad H_{3,1}(f) = \begin{vmatrix} a_1 & a_1 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

of order 3, is more difficult if we expect to get a sharp estimate. Results in this direction, however not sharp, were obtained by various authors, e.g., [1, 2, 4, 5, 21, 23, 25, 29, 30]). If a subclass $\mathcal{F}$ of $\mathcal{A}$ has a representation involving the Carathéodory class $\mathcal{P}$, i.e., the class of functions $p \in \mathcal{H}$ of the form

$$(1.3)\qquad p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D},$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{F}$ have a suitable representation expressed by coefficients of functions in $\mathcal{P}$. Therefore, to get the upper bound of $H_{3,1}$ over $\mathcal{F}$, the authors based their computing on the well-known formulas on coefficient c_2 (e.g., [20, p. 166]) and the formula c_3 due to Libera and Zlotkiewicz [14, 15]. The formula for c_4 which was recently found in [12] allows to reach sharpness of bound of $H_{3,1}$. It was done in [10] and [11] for convex functions and starlike functions of order 1/2.

Given $\alpha \in [0, 1)$, let $\mathcal{T}(\alpha)$ be the class of $f \in \mathcal{A}$ such that

$$(1.4)\qquad \operatorname{Re} \frac{f(z)}{z} > \alpha, \quad z \in \mathbb{D}.$$

Let $\mathcal{T} := \mathcal{T}(0)$. In this paper, we found the sharp bound of $H_{3,1}$ over the classes $\mathcal{T}$ and $\mathcal{T}(1/2)$, namely, we proved that $|H_{3,1}(f)| \le 4$ for $f \in \mathcal{T}$ and $|H_{3,1}(f)| \le 1$ for $f \in \mathcal{T}(1/2)$.

The families $\mathcal{T}$ and $\mathcal{T}(1/2)$ play important roles in the theory of univalent functions although their elements are functions which are not necessarily univalent. One of the significant results belongs to Marx [17] and Strohhäcker [27]. They proved that

$$(1.5)\qquad \mathcal{S}^c \subset \mathcal{S}^*(1/2) \subset \mathcal{T}(1/2)$$

(see also [18, Theorem 2.6a, p. 57]), where $\mathcal{S}^c$ is the class of convex functions introduced by Study [28], i.e., the family of all univalent functions in $\mathcal{A}$ which map $\mathbb{D}$ onto convex domains, and $\mathcal{S}^*(1/2)$ is the class of starlike functions of order 1/2. An idea of starlikeness of order α ($\alpha \in [0, 1)$) belongs to Robertson [24]. By the well known result due to Study ([28], see also [6, p. 42]) a function f is in $\mathcal{S}^c$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D},$$

and a function f is in $\mathcal{S}^*(1/2)$ ([24], see also [8, Vol. I, p. 138]) if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2}, \quad z \in \mathbb{D}.$$

What is interesting, a function

$$f(z) = \frac{z}{1-z}, \quad z \in \mathbb{D}, \tag{1.6}$$

is extremal for many computational problems in these three classes, i.e., in $\mathcal{S}^c$, $\mathcal{S}^*(1/2)$ and $\mathcal{T}(1/2)$. The class $\mathcal{T}$ plays a fundamental role in the theory of semigroups of analytic functions as a generator of one-parameter continues semigroups studied by Berkson, Porta, Shoikhet, Elin and others (e.g., [26], [7]). For other classical results concerning the classes $\mathcal{T}$ and $\mathcal{T}(1/2)$ see e.g., [16, 22].

At the end let us mention that in [10] and [11] it was shown that $|H_{3,1}(f)| \leq 4/135$ for $f \in \mathcal{S}^c$ and $|H_{3,1}(f)| \leq 1/9$ for $f \in \mathcal{S}^*(1/2)$, respectively, with sharpness of both results. In view of the inclusion (1.5) we can say that the corresponding bounds of $H_{3,1}$ carry some information about the richness of classes. Coefficient bounds does not necessarily include such a distinction, namely, for all three classes i.e., $\mathcal{S}^c$, $\mathcal{S}^*(1/2)$ and $\mathcal{T}(1/2)$ modules of all coefficients are bounded by 1 (see [8, Theorem 7, p. 117; Theorem 2, p. 140]) with the extremal function given by (1.6).

2. Main results

The basis for proof of the main result is the following lemma. It contains the well-known formula for c_2 (e.g., [20, p. 166]), the formula for c_3 due to Libera and Zlotkiewicz [14, 15] and the formula for c_4 found in [12].

Lemma 2.1. *If $p \in \mathcal{P}$ is of the form* (1.3) *with $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta, \tag{2.1}$$

$$4c_3 = c_1^3 + (4 - c_1^2)c_1\zeta(2 - \zeta) + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{2.2}$$

and

$$\begin{aligned} 8c_4 = {} & c_1^4 + (4 - c_1^2)\zeta \left[c_1^2(\zeta^2 - 3\zeta + 3) + 4\zeta\right] \\ & - 4(4 - c_1^2)(1 - |\zeta|^2)\left[c_1(\zeta - 1)\eta + \overline{\zeta}\eta^2 - \left(1 - |\eta|^2\right)\xi\right] \end{aligned} \tag{2.3}$$

for some $\zeta, \eta, \xi \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

We will now estimate $H_{3,1}(f)$ for $f \in \mathcal{T}$.

Theorem 2.2.

$$\max\{|H_{3,1}(f)| : f \in \mathcal{T}\} = 4 \tag{2.4}$$

with the extremal function

$$f(z) = \frac{z + z^3}{1 - z^2}, \quad z \in \mathbb{D}. \tag{2.5}$$

Proof. Let $f \in \mathcal{T}$ be of the form (1.1). Then by (1.4),

$$f(z) = zp(z), \quad z \in \mathbb{D}, \tag{2.6}$$

for some function $p \in \mathcal{P}$ of the form (1.3). Since the class $\mathcal{P}$ is invariant under the rotations, by Carathéodory Theorem we may assume that $c := c_1 \in [0, 2]$ ([3], see also [8, Vol. I, p. 80, Theorem 3]). Substituting the series (1.1) and (1.3) into (2.6) and equating the coefficients we get $a_2 = c$, $a_3 = c_2$, $a_4 = c_3$ and $a_5 = c_4$. Hence, and by (1.2) we have

$$H_{3,1}(f) = 2cc_2c_3 - c_2^3 - c_3^2 + c_4(c_2 - c^2). \tag{2.7}$$

To simplify the computation, let $t := 4 - c^2$. By using the equalities (2.1)-(2.3) we have

$$\begin{aligned} c_2 &= \frac{1}{2}(c^2 + t\zeta), \quad c_3 = \frac{1}{4}\left(c^3 + 2ct\zeta - ct\zeta^2 + 2t(1 - |\zeta|^2)\eta\right), \\ c_4 &= \frac{1}{8}\left[c^4 + 3c^2t\zeta + (4 - 3c^2)t\zeta^2 + c^2t\zeta^3 + 4t(1 - |\zeta|^2)\left(c\eta - c\zeta\eta - \overline{\zeta}\eta^2\right)\right. \\ &\quad \left. + 4t(1 - |\zeta|^2)(1 - |\eta|^2)\xi\right]. \end{aligned} \tag{2.8}$$

Hence by straightforward algebraic computation we have

$$\begin{aligned} 2cc_2c_3 &= \frac{1}{16}\left[4c^6 + 12c^4t\zeta - 4c^4t\zeta^2 + 8c^2t^2\zeta^2 - 4c^2t^2\zeta^3\right. \\ &\quad \left. + 8c^3t(1 - |\zeta|^2)\eta + 8ct^2(1 - |\zeta|^2)\zeta\eta\right], \\ c_2^3 &= \frac{1}{16}\left[2c^6 + 6c^4t\zeta + 6c^2t^2\zeta^2 + 2t^3\zeta^3\right], \\ c_3^2 &= \frac{1}{16}\left[c^6 + 4c^4t\zeta - 2c^4t\zeta^2 + 4c^2t^2\zeta^2 - 4c^2t^2\zeta^3 + c^2t^2\zeta^4\right. \\ &\quad + 4c^3t(1 - |\zeta|^2)\eta + 8ct^2(1 - |\zeta|^2)\zeta\eta - 4ct^2(1 - |\zeta|^2)\zeta^2\eta \\ &\quad \left. + 4t^2(1 - |\zeta|^2)^2\eta^2\right], \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} c_4(c_2 - c^2) &= \frac{1}{16}\left[-c^6 - 2c^4t\zeta - (4 - 3c^2)c^2t\zeta^2 + 3c^2t^2\zeta^2 - c^4t\zeta^3\right. \\ &\quad + (4 - 3c^2)t^2\zeta^3 + c^2t^2\zeta^4 - 4c^2t(1 - |\zeta|^2)(c\eta - c\eta\zeta - \overline{\zeta}\eta^2) \\ &\quad + 4t^2(1 - |\zeta|^2)(c\zeta\eta - c\zeta^2\eta - |\zeta|^2\eta^2) \\ &\quad \left. + 4t(-c^2 + t\zeta)(1 - |\zeta|^2)(1 - |\eta|^2)\xi\right]. \end{aligned}$$

Substituting the above expressions with $t = 4 - c^2$ to (2.7) by elementary but tedious computation we get

$$\begin{aligned} H_{3,1}(f) &= \frac{1}{4}(4-c^2)\left[-4\zeta^3 + 4c(1-|\zeta|^2)\zeta\eta + (-4 + c^2 + c^2\overline{\zeta})(1-|\zeta|^2)\eta^2\right. \\ &\quad \left. + (-c^2 + (4-c^2)\zeta)(1-|\zeta|^2)(1-|\eta|^2)\xi\right]. \end{aligned} \tag{2.10}$$

Let $x := |\zeta| \in [0,1]$ and $y := |\eta| \in [0,1]$. Taking into account that $|\xi| \le 1$, from (2.10) we obtain

$$|H_{3,1}(f)| \le \frac{1}{4} F(c,x,y), \tag{2.11}$$

where

$$F(c,x,y) := (4-c^2)\left[2(2-c^2)(1-x^2)(1-x)y^2 + 4c(1-x^2)xy \right.$$
$$\left. +(c^2+(4-c^2)x)(1-x^2)+4x^3\right].$$

We will show that for $c \in [0,2],\ x \in [0,1]$ and $y \in [0,1]$,

$$F(c,x,y) \le 16. \tag{2.12}$$

I. For $c = \sqrt{2}$ we have

$$\frac{\partial F}{\partial y}(\sqrt{2},x,y) = 8\sqrt{2}(1-x^2)x \ne 0, \quad x,y \in (0,1). \tag{2.13}$$

For $c \ne \sqrt{2}$ we have

$$\frac{\partial F}{\partial y} = 4(4-c^2)(1-x^2)\left[(2-c^2)(1-x)y + cx\right] = 0$$

only for

$$y = -\frac{cx}{(2-c^2)(1-x)} =: y_0 \in (0,1),$$

which holds for $c \in (\sqrt{2},2)$.

Let $c \in (\sqrt{2},2)$. For $x \in (0,1)$ we have

$$\frac{\partial F}{\partial x}(c,x,y_0) = (4-c^2)\left[-2(2-c^2)(3x+1)(1-x)y_0^2\right.$$
$$\left. +4c(1-3x^2)y_0 + 12x^2 + (4-c^2)(1-3x^2) - 2c^2x\right] = 0$$

if and only if

$$-\frac{2c^2(3x+1)x^2}{(2-c^2)(1-x)} - \frac{4c^2(1-3x^2)x}{(2-c^2)(1-x)} + 4 - c^2 - 2c^2x + 3c^2x^2 = 0,$$

which is equivalent to

$$-3c^4x^2 - 2c^2(4-c^2)x + (4-c^2)(2-c^2) = 0. \tag{2.14}$$

Since $\Delta := 8c^4(4-c^2)(5-2c^2) < 0$ for $c \in (\sqrt{5/2},2)$, so then the equation (2.14) has no solution. Because all coefficients of (2.14) are negative from Vieta's formulae it follows that for $c \in (\sqrt{2},\sqrt{5/2})$ both solutions of (2.14) are negative. Clearly, the equation (2.14) has no solution for $c = \sqrt{5/2}$. Thus the equation (2.14) has no solution and therefore taking into account (2.13) the function F has no critical point in $(0,2)\times(0,1)\times(0,1)$.

II. We consider all faces. On the face $c = 0$,

$$q_1(x,y) := F(0,x,y) = 16\left((1-x^2)(1-x)y^2 + x\right), \quad x,y \in (0,1).$$

Since q_1 is an increasing function of $y \in (0,1)$, so it has no critical point in $(0,1) \times (0,1)$.

On the face $c = 2$,

$$F(2,x,y) = 0, \quad x, y \in (0,1).$$

On the face $x = 0$,

$$q_2(c,y) := F(c,0,y) = (4-c^2)\left[2(2-c^2)y^2 + c^2\right], \quad c \in (0,2),\ y \in (0,1).$$

Since

$$\frac{\partial q_2}{\partial y} = 4(4-c^2)(2-c^2)y = 0, \quad c \in (0,2),\ y \in (0,1),$$

only for $c = \sqrt{2}$ and

$$\frac{\partial q_2}{\partial c}(\sqrt{2}, y) = -8\sqrt{2}y^2 \neq 0, \quad y \in (0,1),$$

so q_2 has no critical point in $(0,2) \times (0,1)$.

On the face $x = 1$, $F(c,1,y)$ has no critical point for $c \in (0,2),\ y \in (0,1)$, obviously.

On the face $y = 0$,

$$\begin{aligned} q_3(c,x) &:= F(c,x,0) \\ &= (4-c^2)\left(c^2 + (4-c^2)x - c^2x^2 + c^2x^3\right), \quad c \in (0,2),\ x \in (0,1). \end{aligned}$$

We have

$$\frac{\partial q_3}{\partial x} = (4-c^2)(4 + c^2(3x^2 - 2x - 1)) = 0, \quad c \in (0,2),\ x \in (0,1),$$

only for

$$c = \frac{2}{\sqrt{(1-x)(1+3x)}} =: c_0 \in (0,2),$$

which holds for $x \in (0, 2/3)$. Moreover

$$\frac{\partial q_3}{\partial c}(c_0, x) = 0$$

if and only if

$$c_0^2(1-x^2)(1-x) = 2(1 - 2x - x^2 + x^3).$$

Since the last equation equivalently written as

$$3x^4 - 2x^3 - 5x^2 + x - 1 = 0, \quad x \in (0, 2/3),$$

has no zero (all real zeros are $x_1 \approx -1.18,\ x_2 \approx 1,64$), so q_3 has no critical point in $(0,2) \times (0,1)$.

On the face $y = 1$ for $c \in (0,2)$ and $x \in (0,1)$,

$$\begin{aligned} q_4(c,x) &:= F(c,x,1) \\ &= (4-c^2)\left[4 - c^2 + (4c + c^2)x - (4-c^2)x^2 + (4 - 4c - c^2)x^3\right]. \end{aligned}$$

A numerical computation shows that the system of equations $\partial q_4/\partial x = 0$ and $\partial q_4/\partial c = 0$ equivalent to

$$\begin{cases} 3(4-4c-c^2)x^2 - 2(4-c^2)x + 4c + c^2 = 0, \\ (-4-4c+3c^2+c^3)x^3 + (4c-c^3)x^2 + (4+2c-3c^2-c^3)x - 4c + c^3 = 0, \end{cases}$$

has a unique solution $c =: c_0 \approx 0.42524$ and $x = x_0 \approx 0.85612$, i.e., (c_0, x_0) is a unique critical point of q_4. Since clearly,

$$\frac{\partial^2 q_4}{\partial c^2}(c_0, x_0) = 12(1-x_0^2)c_0\left[(1-x_0)c_0 - 2x_0\right] - 16(1-x_0)(1-x_0^2) - 8x_0 < 0$$

and

$$\frac{\partial^2 q_4}{\partial x^2}(c_0, x_0) = (4-c_0^2)\left[6(4-4c_0-c_0^2)x_0 - 2(4-c_0^2)\right] > 0,$$

it follows that in (c_0, x_0) is a saddle point of q_4.

III. On the edges we have:
$F(c,0,0) = 4c^2 - c^4 \le 4,\ c \in [0,2]$;
$F(c,1,0) = F(c,1,1) = 4(4-c^2) \le 16,\ c \in [0,2]$;
$F(0,x,0) = 16x \le 16,\ x \in [0,1]$;
$F(2,x,0) = F(2,x,1) = 0,\ x \in [0,1]$;
$F(c,0,1) = (4-c^2)^2 \le 16,\ c \in [0,2]$;
$F(0,x,1) = 16(x^3 - x^2 + 1) \le 16,\ x \in [0,1]$;
$F(0,0,y) = 16y^2 \le 16,\ y \in [0,1]$;
$F(0,1,y) = 16,\ y \in [0,1]$;
$F(2,0,y) = F(2,1,y) = 0,\ y \in [0,1]$.

Summarizing, from the cases I-III we state that the inequality (2.12) is true. Thus from (2.11) it follows that $|H_{3,1}(f)| \le 4$. For the function f given by (2.5) which is in $\mathcal{T}$, we have $a_2 = a_4 = 0$ and $a_3 = a_5 = 2$. Hence and by (1.2) we get $H_{3,1}(f) = -4$ which ends the proof of (2.4). □

We will now found the bound of $H_{3,1}(f)$ for $f \in \mathcal{T}(1/2)$.

Theorem 2.3.

$$\max\{|H_{3,1}(f)| : f \in \mathcal{T}(1/2)\} = 1 \tag{2.15}$$

with the extremal function

$$f(z) = \frac{z}{1-z^3}, \quad z \in \mathbb{D}. \tag{2.16}$$

Proof. Let $f \in \mathcal{T}(1/2)$ be of the form (1.1). Then by (1.4) we have

$$f(z) = \frac{1}{2}z(p(z)+1), \quad z \in \mathbb{D}, \tag{2.17}$$

for some function $p \in \mathcal{P}$ of the form (1.3). As in the proof of Theorem 2.2 we may assume that $c := c_1 \in [0,2]$. Substituting the series (1.1) and (1.3) into (2.17) and equating the coefficients we get $a_2 = c/2$, $a_3 = c_2/2$, $a_4 = c_3/2$ and $a_5 = c_4/2$. Hence, and by (1.2) we have

$$H_{3,1}(f) = \frac{1}{8}\left(2cc_2c_3 - c_2^3 - 2c_3^2 + 2c_2c_4 - c^2c_4\right). \tag{2.18}$$

Using (2.8) with $t := 4 - c^2$, we get

$$2c_2c_4 - c^2c_4 = \frac{1}{8}\left[c^4t\zeta + 3c^2t^2\zeta^2 + (4-3c^2)t^2\zeta^3 + c^2t^2\zeta^4 \right.$$
$$\left. +4t^2(1-|\zeta|^2)(c\zeta - c\zeta^2 - |\zeta|^2\eta)\eta + 4t^2(1-|\zeta|^2)(1-|\eta|^2)\zeta\xi\right].$$

Substituting the above expression and (2.9) with $t = 4 - c^2$ to (2.18) we get

$$H_{3,1}(f) = \frac{1}{16}(4-c^2)^2(1-|\xi|^2)\left[-\eta^2 + (1-|\eta|^2)\zeta\xi\right].$$

Hence and by the fact that $|\zeta| \le 1$ and $|\xi| \le 1$ we have

$$|H_{3,1}(f)| \le \frac{1}{16}(4-c^2)^2(1-|\xi|^2)\left[|\eta|^2 + 1 - |\eta|^2\right]$$
$$\le \frac{1}{16}(4-c^2)^2(1-|\xi|^2) \le 1.$$

Since $a_2 = a_3 = a_5 = 0$ and $a_4 = 1$ for the function (2.16) which is in $\mathcal{T}(1/2)$, so $H_{3,1}(f) = -1$. This makes equality in (2.15). □

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