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ON DISCONTINUOUS ELLIPTIC PROBLEMS INVOLVING THE FRACTIONAL p-LAPLACIAN IN $\mathbb{R}^N$

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Abstract. We are concerned with the following fractional p-Laplacian inclusion:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u \in \lambda[\underline{f}(x, u(x)), \overline{f}(x, u(x))] \quad \text{in} \quad \mathbb{R}^N,$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator, $0 < s < 1 < p < +\infty$, $sp < N$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to each variable separately. We show that our problem with the discontinuous nonlinearity f admits at least one or two nontrivial weak solutions. In order to do this, the main tool is the Berkovits-Tienari degree theory for weakly upper semicontinuous set-valued operators. In addition, our main assertions continue to hold when $(-\Delta)_p^s u$ is replaced by any non-local integro-differential operator.

1. Introduction

In the present paper, we consider the existence of nontrivial weak solution to the fractional p-Laplacian inclusion:

$$(P) \qquad (-\Delta)_p^s u + V(x)|u|^{p-2}u \in \lambda[\underline{f}(x, u(x)), \overline{f}(x, u(x))] \quad \text{in} \quad \mathbb{R}^N,$$

where the fractional p-Laplacian operator $(-\Delta)_p^s$ is defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \, dy, \quad x \in \mathbb{R}^N.$$

Here, $0 < s < 1 < p < +\infty$, $sp < N$, $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x - y| < \varepsilon\}$ and the measurable functions $\underline{f}$, $\overline{f}$ are induced by a possibly discontinuous function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ at the second variable. For the motivations that lead to the study on boundary value problems of some partial differential equations with discontinuous nonlinearities, we refer the reader to the contribution [16, 17] of Chang. Afterward, many efforts have been devoted to the study of variational

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problems for elliptic equation with this nonlinearity; see for example [2, 4, 5, 8, 11, 12, 23, 34] and the references therein.

In the last years, a great deal of attention has been paid to the study of nonlinear equations involving fractional and nonlocal operators of elliptic type in the description of the mathematical theory to concrete some phenomena such as social sciences, fractional quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process, game theory and Levy processes; see for instance [9, 10, 15, 21, 25, 29, 30] and so on. Especially, in contrast with the usual elliptic partial differential equations which are governed by local differential operators like the p-Laplace operator, discriminated characterization of the fractional operator in (P) is the nonlocality, in the sense that this operator takes care of the behavior of the solution in the whole space. In this respect, increasing research of elliptic equations involving the fractional Laplacian type problems has been interesting to many people; see [6, 13, 22, 27, 32, 37, 41] and the references therein. Meanwhile, most of results have been obtained by a critical point theory, initially introduced by Ambrosetti and Rabinowitz in [3], which is one of the main tools for finding solutions to elliptic equations of variational type; see for example [6, 18, 22, 26, 27, 33, 36, 37, 41]. Especially, the authors in [22] have been investigated the existence and multiplicity results for the fractional p-Laplacian type problems:

$$\begin{cases} (-\Delta)_p^s u = \lambda f(x,u) \quad \text{in} \;\; \Omega, \\ u = 0 \quad \text{on} \;\; \mathbb{R}^N \backslash \Omega, \end{cases}$$

where f satisfies a Carathéodory condition; see [33] for $p = 2$.

The first purpose of this paper, as observing new approach deeply different from that studied in previous related works [6, 18, 22, 26, 27, 33, 36, 37, 41], is to obtain the existence of at least one nontrivial weak solution for the problem (P) without employing the variational method, in particular, mountain pass theorem in [3]. More precisely, by using the Berkovits-Tienari degree theory in [8] for upper semicontinuous set-valued operators of class (S_+) which is a generalization of the single-valued version due to Browder [14], we show that the corresponding integral operator equation to the given problem has a critical point. To do this, we analyze some properties of an operator associated with the discontinuous nonlinear term which plays an important role in applying the degree theory of this type. In particular, we investigate that this operator is a locally Lipschitz functional with compact gradient. However the main difficulty in showing this assertion in $\mathbb{R}^N$ arises from the lack of the compactness of the Sobolev embedding. To overcome the difficulty of the noncompact embedding, we utilize a new compact embedding result under suitable conditions on the potential $V(x)$ which is originally introduced by Bartsch and Wang [7].

The second aim of this paper is to obtain the existence of at least two nontrivial weak solutions for the problem (P). However, in our first main result, no additional information is given on the existence of at least two solutions.

We point out that the strategy for overcoming this difficulty is to utilize the existence of a nontrivial global minimizer for the energy functional to find another solution. As far as we are aware, there were no such existence results for fractional p-Laplacian problems, and we are only aware of the paper [24] in this situation.

Lastly, our main assertions continue to hold when $(-\Delta)_p^s u$ in (P) is replaced by any non-local integro-differential operator $\mathcal{L}_K$, defined pointwise by

$$\mathcal{L}_K u(x) = 2\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))K(x-y)dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ will be specified later.

This paper is structured as follows. First we recall briefly the Berkovits-Tienari degree theory for weakly upper semicontinuous set-valued operators of class (S_+) and some basic properties for locally Lipschitz continuous functionals in reflexive Banach spaces. Next, using the fact that every critical point of the energy functional associated with (P) is a weak solution for our problem, we obtain the existence of at least one or two nontrivial weak solutions for problem (P). Finally we present that the existence and multiplicity results for the integro differential operator which is a generalization of the fractional p-Laplacian are still satisfied.

2. Preliminaries and main result

We first give the Berkovits-Tienari degree theory for weakly upper semicontinuous set-valued operators of class (S_+) which is based on the papers [8, 23]. Let X be a real Banach space and X^* be its dual space with dual pairing $\langle \cdot, \cdot \rangle$. Given a nonempty subset Ω of X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. Let $B_r(x)$ denote the open ball in X of radius $r > 0$ centered at x. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

Definition 2.1. Let U be an open set in X and let Y be another real Banach space. A set-valued operator $h : \overline{U} \to 2^Y$ is said to be

(1) *upper semicontinuous* if the set $h^{-1}(A) = \{u \in \overline{U} \,|\, h(u) \cap A \neq \emptyset\}$ is closed for all closed sets A in Y.
(2) *weakly upper semicontinuous* if $h^{-1}(A)$ is closed for all weakly closed sets A in Y.
(3) *compact* if it is upper semicontinuous and the image of any bounded set is relatively compact.
(4) *bounded* if h maps bounded sets into bounded sets.
(5) *locally bounded* if for each $u \in \overline{U}$ there exists a neighborhood V of u such that the set $h(V) = \bigcup_{u \in V} h(u)$ is bounded.

Definition 2.2. Let U be an open set in X. A set-valued operator $h : \overline{U} \to 2^{X^*} \setminus \emptyset$ is said to be

(1) *pseudomonotone* if for any sequence $\{u_n\}$ in $\overline{U}$ and for any sequence $\{w_n\}$ in X^* with $w_n \in h(u_n)$ such that $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to\infty} \langle w_n, u_n - u\rangle \le 0,$$

we have $\lim_{n\to\infty} \langle w_n, u_n - u\rangle = 0$ and if $u \in \overline{U}$ and $w_j \rightharpoonup w$ in X^* for some subsequence $\{w_j\}$ of $\{w_n\}$, then $w \in h(u)$.

(2) *quasimonotone* if for any sequence $\{u_n\}$ in $\overline{U}$ and for any sequence $\{w_n\}$ in X^* with $w_n \in h(u_n)$ such that $u_n \rightharpoonup u$ in X, we have

$$\liminf_{n\to\infty} \langle w_n, u_n - u\rangle \ge 0.$$

The following lemma means that operators of class (S_+) are invariant under quasimonotone perturbations.

Lemma 2.3 ([23]). *Let U be an open set in a real reflexive Banach space X. Suppose that $h : \overline{U} \to 2^{X^*}$ is a set-valued operator with nonempty values and $S : \overline{U} \to X^*$ is a single-valued operator. If h is quasimonotone and S is of class (S_+), then the sum $h + S$ is of class (S_+).*

Remark 2.4. It is known that in this case the duality operator $j : X \to X^*$ is injective, bounded, continuous, and of class (S_+), and such that $\langle jx, x\rangle = \|x\|^2$ and $\|jx\| = \|x\|$ for $x \in X$; see *e.g.*, [40].

As a crucial tool for obtaining our result, we present the Berkovits-Tienari degree theory for weakly upper semicontinuous set-valued operators of class (S_+) in reflexive Banach spaces.

Lemma 2.5 ([4]). *Let G be any bounded open subset of X and let $h : \overline{G} \to 2^{X^*}$ be of class (S_+), locally bounded, and weakly upper semicontinuous such that $h(u)$ is nonempty, closed, and convex for each $u \in \overline{G}$. If $w \notin h(\partial G)$, then an integer $d(h, G, w)$ is defined, called the degree of h on G over w, and the degree has the following properties*:

(a) (*Existence*) *If $d(h, G, w) \neq 0$, then $w \in h(G)$.*

(b) (*Homotopy Invariance*) *Suppose that $H : [0, 1] \times \overline{G} \to 2^{X^*}$ is of class (S_+), locally bounded, and weakly upper semicontinuous such that $H(t, u)$ is nonempty, closed, and convex for each $(t, u) \in [0, 1] \times \overline{G}$. If $c : [0, 1] \to X^*$ is a continuous path in X^* such that $c(t) \notin H(t, u)$ for all $(t, u) \in [0, 1] \times \partial G$, then $d(H(t, \cdot), G, c(t))$ is constant for all $t \in [0, 1]$.*

(c) (*Normalization*) *If $w \in J(G)$, then we have $d(J, G, w) = 1$. In particular, if $0 \in G$, then $d(\varepsilon J, G, 0) = 1$ for every $\varepsilon > 0$.*

Here, a homotopy $H : [0, 1] \times \overline{G} \to 2^{X^*}$ is of class (S_+) in the sense that for any sequence $\{t_n, u_n\}$ in $[0, 1] \times \overline{G}$ and for any sequence $\{w_n\}$ in X^* with $w_n \in H(t_n, u_n)$ such that $t_n \to t$, $u_n \rightharpoonup u$ in X, and

$$\limsup_{n\to\infty} \langle w_n, u_n - u\rangle \le 0,$$

we have $u_n \to u$ in X.

Next we briefly introduce the following definitions and some properties for locally Lipschitz continuous functionals. For a real Banach space $(X, \|\cdot\|_X)$, we say that a functional $h : X \to \mathbb{R}$ is called locally Lipschitz when, for every $u \in X$, there corresponds a neighborhood U of u and a constant $\mathcal{L} \geq 0$ such that

$$|h(v) - h(w)| \leq \mathcal{L}\|v - w\|_X \quad \text{for all} \quad v, w \in U. \tag{2.1}$$

Let $u, v \in X$. The symbol $h^\circ(u; v)$ indicates the generalized directional derivative of h at point u along direction v, namely

$$h^\circ(u; v) := \limsup_{w \to u, t \to 0^+} \frac{h(w + tv) - h(w)}{t}.$$

The generalized gradient of the function h at u, denoted by $\partial h(u)$, is the set

$$\partial h(u) := \big\{u^* \in X^* : \langle u^*, v\rangle \leq h^\circ(u; v) \quad \text{for all} \;\; v \in X\big\}.$$

A functional $h : X \to \mathbb{R}$ is called Gâteaux differentiable at $u \in X$ if there is $\varphi \in X^*$(denoted by $h'(u)$) such that

$$\lim_{t \to 0^+} \frac{h(u + tv) - h(u)}{t} = h'(u)(v)$$

for all $v \in X$. It is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the function $u \to h'(u)$ is a continuous map from X to its dual X^*. We recall that if h is continuously Gâteaux differentiable, then it is locally Lipschitz and one has $h^\circ(u; v) = h'(u)(v)$ for all $u, v \in X$. If $h : X \to \mathbb{R}$ is a locally Lipschitz functional and $x \in X$, then we say that x is a critical point of h if it satisfies the inequality

$$h^\circ(x; y) \geq 0$$

for all $y \in X$ or, equivalently, $0 \in \partial h(x)$.

We give some properties of the locally Lipschitz functional which will be used later.

Proposition 2.6 ([19])**.** *Let $h : X \to \mathbb{R}$ be locally Lipschitz functional. Then*

(i) $(-h)^\circ(u; z) = h^\circ(u; -z)$ *for all* $u, z \in X$;

(ii) $h^\circ(u; z) = \max\{\langle u^*, z\rangle_X : u^* \in \partial h(u)\} \leq \mathcal{L}\|z\|$ *with* $\mathcal{L}$ *as in* (2.1) *for all* $u, z \in X$;

(iii) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $(h + j)^\circ(u; z) = h^\circ(u; z) + \langle j'(u), z\rangle_X$ *for all* $u, z \in X$;

(iv) (*Lebourg's mean value theorem*) *Let* u *and* v *be two points in* X. *Then, there exist a point* w *in the open segment between* u *and* v, *and a* $u^*_\omega \in \partial h(\omega)$ *such that*

$$h(u) - h(v) = \langle u^*_\omega, u - v\rangle_X;$$

(v) *Let* Y *be a Banach space and* $j : Y \to X$ *a continuously differentiable function. Then* $h \circ j$ *is locally Lipschitz and*

$$\partial(h \circ j)(u) \subseteq \partial h(j(y)) \circ j'(y) \quad \text{for all} \quad y \in Y;$$

(vi) *If* $h_1, h_2 : X \to \mathbb{R}$ *are locally Lipschitz, then*

$$\partial(h_1 + h_2)(u) \subseteq \partial h_1(u) + \partial h_2(u);$$

(vii) $\partial h(u)$ *is convex and weakly* compact and the set-valued mapping* $\partial h : X \to 2^{X^*}$ *is weakly* u.s.c.*;

(viii) $\partial(\lambda h)(u) = \lambda \partial h(u)$ *for every* $\lambda \in \mathbb{R}$.

Lemma 2.7 ([38])**.** *Let* $h : X \to \mathbb{R}$ *be a locally Lipschitz functional with compact gradient. Then,* h *is sequentially weakly continuous, i.e., if* $\{u_n\}$ *is a sequence in* X *such that* $u_n \rightharpoonup u$ *in* X *as* $n \to \infty$*, then* $h(u_n) \to h(u)$ *in* X *as* $n \to \infty$.

From now on we study the problem (P) with discontinuous nonlinearity which involves the fractional p-Laplacian. Let $s \in (0, 1)$ and $p \in (1, +\infty)$. We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ as follows:

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dxdy < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + |u|^p_{W^{s,p}(\mathbb{R}^N)} \right)^{\frac{1}{p}},$$

where

$$\|u\|^p_{L^p(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |u|^p \, dx \quad \text{and} \quad |u|^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dxdy.$$

Let $s \in (0, 1)$ and $1 < p < +\infty$. Then $W^{s,p}(\mathbb{R}^N)$ is a separable and reflexive Banach space. Also, the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, that is $W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$ (see e.g. [1,31]).

Lemma 2.8 ([28, 31])**.** *Let* $0 < s < 1 < p < +\infty$ *with* $ps < N$*. Then the space* $W^{s,p}(\mathbb{R}^N)$ *is continuously embedded in* $L^q(\mathbb{R}^N)$ *for any* $q \in [p, p_s^*]$ *where* $p_s^* = Np/(N - sp)$ *is the fractional critical exponent.*

Let the potential $V \in C(\mathbb{R}^N)$ be a continuous function. Suppose that

(V) $V \in C(\mathbb{R}^N)$, $\inf_{x \in \mathbb{R}^N} V(x) > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$ for all $M \in \mathbb{R}$.

When V satisfies (V), the basic space

$$X_s(\mathbb{R}^N) := \{u \in W^{s,p}(\mathbb{R}^N) : V|u|^p \in L^1(\mathbb{R}^N)\}$$

denote the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{X_s(\mathbb{R}^N)} := \left(|u|^p_{W^{s,p}(\mathbb{R}^N)} + \|V^{\frac{1}{p}} u\|^p_{L^p(\mathbb{R}^N)} \right)^{\frac{1}{p}}.$$

Lemma 2.9 ([36]). *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Then there is a compact embedding $X_s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $q \in [p, p_s^*)$.*

Let us define a functional $J : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dxdy + \frac{1}{p} \int_{\mathbb{R}^N} V(x) \, |u(x)|^p \, dx.$$

Then the functional J is well defined on $X_s(\mathbb{R}^N)$, $J \in C^1(X_s(\mathbb{R}^N), \mathbb{R})$ and its Fréchet derivative is given by

$$\langle J'(u), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}} \, dxdy + \int_{\mathbb{R}^N} V(x) \, |u(x)|^{p-2} \, uv \, dx$$

for any $v \in X_s(\mathbb{R}^N)$ where $\langle \cdot, \cdot \rangle$ denotes the pairing of $X_s(\mathbb{R}^N)$ and its dual $(X_s(\mathbb{R}^N))^*$; see [32].

Lemma 2.10 ([32]). *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Let $0 < s < 1 < p < +\infty$. Then the functional $J' : X_s(\mathbb{R}^N) \to (X_s(\mathbb{R}^N))^*$ is of type (S_+), i.e., if $u_n \rightharpoonup u$ in $X_s(\mathbb{R}^N)$ and $\limsup_{n\to\infty} \langle J'(u_n) - J'(u), u_n - u \rangle \le 0$, then $u_n \to u$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$.*

Corollary 2.11 ([32]). *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Then the functional J' is strictly monotone, coercive and hemicontinuous on $X_s(\mathbb{R}^N)$. Furthermore, the functional J' is a bounded homeomorphism onto $(X_s(\mathbb{R}^N))^*$.*

Proof. It is obvious that the operator J' is strictly monotone, coercive and hemicontinuous on X. By the Browder-Minty theorem, the inverse operator $(J')^{-1}$ exists; see Theorem 26.A in [39]. If we apply Lemma 2.10, then the proof of continuity of the inverse operator $(J')^{-1}$ is analogous to that in the case of the p-Laplacian and is omitted. □

We assume that

(F1) f is measurable with respect to each variable separately.

(F2) There exist nonnegative functions $\rho \in L^{p'}(\mathbb{R}^N)$ and $\sigma \in L^\infty(\mathbb{R}^N)$ such that

$$|f(x, t)| \le \rho(x) + \sigma(x) \, |t|^{p-1}$$

for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

Moreover, we denote by $\mathcal{G}$ the family of all locally bounded functions $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(m_1) $f(\cdot, z)$ is measurable for every $z \in \mathbb{R}$;

(m_2) there exists a set $\Omega_0 \subseteq \mathbb{R}^N$ with $m(\Omega_0) = 0$ such that the set

$$D_f := \bigcup_{x \in \mathbb{R}^N \setminus \Omega_0} \{z \in \mathbb{R} : f(x, \cdot) \text{ is discontinuous at } z\}$$

has measure zero.

(m_3) for $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, the functions

$$\underline{f}(x, z) := \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|\xi - z| < \delta} f(x, \xi) \quad \text{and} \quad \overline{f}(x, z) := \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|\xi - z| < \delta} f(x, \xi)$$

are superpositionally measurable, that is, when $\underline{f}(\cdot, u(\cdot))$ and $\overline{f}(\cdot, u(\cdot))$ are measurable on $\mathbb{R}^N$ for any measurable function $u : \mathbb{R}^N \to \mathbb{R}$.

Clearly, if $f \in \mathcal{G}$, then f satisfies (F1). For fixed $x \in \mathbb{R}^N$, as the function of t, the function F is defined by

$$F(x, t) := \int_0^t f(x, \xi)\, d\xi \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

The generalized gradients of the function t do exist, that is,

$$\partial F(x, t) := \partial_t F(x, t) = \partial_z F^\circ(x, t; \theta),$$

where

$$F^\circ(x, t; z) = \limsup_{h \to 0, \eta \downarrow 0} \frac{f(x, t + h + \eta z) - f(x, t + h)}{h}.$$

Define the functional $\Psi : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N} F(x, u)\, dx.$$

Next we define a functional $I_\lambda : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$I_\lambda(u) = J(u) - \lambda \Psi(u).$$

Proposition 2.12 ([17])**.** *If $f \in \mathcal{G}$ satisfies* (F2)*, then $F : X \to \mathbb{R}$ is a locally Lipschitz functional in $L^p(\mathbb{R}^N)$ and*

$$\partial \Psi(u) \subseteq \partial F(x, u) = [\underline{f}(x, u(x)), \overline{f}(x, u(x))]$$

for almost all $x \in \mathbb{R}^N$.

Definition 2.13. Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption (V) holds. We say that $u \in X_s(\mathbb{R}^N)$ is a weak solution of the problem (P) if there exists a function $w \in \partial F(x, u)$ for almost all $x \in \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}}\, dxdy + \int_{\mathbb{R}^N} V(x)\, |u(x)|^{p-2}\, uv\, dx = \lambda \int_{\mathbb{R}^N} wv\, dx$$

for all $v \in X_s(\mathbb{R}^N)$.

In view of Proposition 2.12, this equation corresponds to the following operator problem

$$0 \in (J' - \lambda \partial \Psi)(u).$$

Lemma 2.14. *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. If $f \in \mathcal{G}$ satisfies* (F2)*, then $\Psi : X_s(\mathbb{R}^N) \to \mathbb{R}$ is a locally Lipschitz functional with compact gradient.*

Proof. We firstly prove that Ψ is locally Lipschitz functional. We denote by κ_p the constant of the embedding $X_s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$. Let $u, v \in X_s(\mathbb{R}^N)$. Apply Lemma 2.9 and the Hölder inequality to obtain

$$\begin{aligned}
&|\Psi(u) - \Psi(v)| \\
&\le \int_{\mathbb{R}^N} |F(x,u) - F(x,v)| \, dx \\
&\le \int_{\mathbb{R}^N} (\rho(x) + \sigma(x)\, |u(x)|^{p-1} + \rho(x) + \sigma(x)\, |v(x)|^{p-1})\, |u(x) - v(x)| \, dx \\
&\le 2\|\rho\|_{L^{p'}(\mathbb{R}^N)} \|u - v\|_{L^p(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)} (\|u\|^{p-1}_{L^p(\mathbb{R}^N)} + \|v\|^{p-1}_{L^p(\mathbb{R}^N)}) \|u - v\|_{L^p(\mathbb{R}^N)} \\
&\le 2\kappa_p \|\rho\|_{L^{p'}(\mathbb{R}^N)} \|u - v\|_{X_s(\mathbb{R}^N)} + \kappa_p \|\sigma\|_{L^\infty(\mathbb{R}^N)} (\|u\|^{p-1}_{X_s(\mathbb{R}^N)} \\
&\quad + \|v\|^{p-1}_{X_s(\mathbb{R}^N)}) \|u - v\|_{X_s(\mathbb{R}^N)}.
\end{aligned}$$

From the above computation, it is obvious that Ψ is locally Lipschitz functional.

Now, we prove that $\partial \Psi(u)$ is compact. Apply Lebourg's mean value theorem and Proposition 2.6(ii), for every $v \in X_s(\mathbb{R}^N)$ we choose an element u in $X_s(\mathbb{R}^N)$ and $u^* \in \partial \Psi(u)$ such that

$$\langle u^*, v \rangle \le \Psi^\circ(u; v) \tag{2.2}$$

and $\Psi^\circ(u; \cdot) : L^p(\mathbb{R}^N) \to \mathbb{R}$ is a subadditive function; see Proposition 2.6. Furthermore, $u^* \in (X_s(\mathbb{R}^N))^*$ is also continuous with respect to the topology induced on $X_s(\mathbb{R}^N)$ by the norm $\| \cdot \|_{L^p(\mathbb{R}^N)}$. In fact, if we set a Lipschitz constant $\mathcal{L} > 0$ for Ψ in a neighborhood of u, then it follows from Proposition 2.6(ii) that for all $z \in X_s(\mathbb{R}^N)$ we obtain

$$\langle u^*, z \rangle \le \mathcal{L} \|z\|_{L^p(\mathbb{R}^N)}, \quad \langle u^*, -z \rangle \le \mathcal{L} \|-z\|_{L^p(\mathbb{R}^N)},$$

and thus

$$\langle u^*, z \rangle \le \mathcal{L} \|z\|_{L^p(\mathbb{R}^N)}.$$

Hence, from the Hahn-Banach Theorem, u^* can be extended to an element of the dual $L^p(\mathbb{R}^N)$ (complying with (2.2)) for every $v \in L^p(\mathbb{R}^N)$, this means that u^* can be regarded as an element of $L^{p'}(\mathbb{R}^N)$ and write for every $v \in L^p(\mathbb{R}^N)$

$$\langle u^*, v \rangle = \int_{\mathbb{R}^N} u^*(x) v(x) \, dx. \tag{2.3}$$

Let $\{u_n\}$ be a sequence in $X_s(\mathbb{R}^N)$ such that $\|u_n\|_{X_s(\mathbb{R}^N)} \le M$ for all $n \in \mathbb{N}$ and for some positive constant M, and take $u^*_{F_n} \in \partial \Psi(u_n)$ for all $n \in \mathbb{N}$. From

(F2) and (2.3) we have

$$\begin{aligned}\langle u_{F_n}^*, v\rangle &= \int_{\mathbb{R}^N} u_{F_n}^* v(x)\,dx \le \int_{\mathbb{R}^N} |u_{F_n}^*||v(x)|\,dx \\ &\le \int_{\mathbb{R}^N} (\rho(x)+\sigma(x)\,|u_n(x)|^{p-1})|v(x)|\,dx \\ &\le \|\rho\|_{L^{p'}(\mathbb{R}^N)}\|v\|_{L^p(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{L^p(\mathbb{R}^N)}^{p-1}\|v\|_{L^p(\mathbb{R}^N)} \\ &\le (1+\kappa_p)^p(\|\rho\|_{L^{p'}(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}M^{p-1})\|v\|_{X_s(\mathbb{R}^N)}\end{aligned}$$

for all $n \in \mathbb{N}$ and for all $u \in X_s(\mathbb{R}^N)$. Hence

$$\|u_{F_n}^*\|_{(X_s(\mathbb{R}^N))^*} \le (1+\kappa_p)^p(\|\rho\|_{L^{p'}(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}M^{p-1}),$$

namely, the sequence $\{u_{F_n}^*\}$ is bounded. So, passing to a subsequence, we have that the sequence $\{u_{F_n}^*\}$ weakly converges to u_F^* in $(X_s(\mathbb{R}^N))^*$ as $n\to\infty$. We will prove that $\{u_{F_n}^*\}$ has a strong convergence in $(X_s(\mathbb{R}^N))^*$. Suppose to the contrary that there exists some $\varepsilon_0 > 0$ such that

$$\|u_{F_n}^* - u_F^*\|_{(X_s(\mathbb{R}^N))^*} > \varepsilon_0$$

for all $n \in \mathbb{N}$. Then there is a $v_n \in B_1(0)$ such that

$$\langle u_{F_n}^* - u_F^*, v_n\rangle > \varepsilon_0. \tag{2.4}$$

Noting that $\{v_n\}$ is a bounded sequence and passing to a subsequence, one has

$$v_n \rightharpoonup v \quad \text{in} \quad X_s(\mathbb{R}^N), \qquad \|v_n - v\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as} \quad n\to\infty$$

by Lemma 2.9. So, for n large enough, we have

$$\big|\langle u_{F_n}^* - u_F^*, v\rangle\big| < \frac{\varepsilon_0}{3}, \quad |\langle u_F^*, v_n - v\rangle| < \frac{\varepsilon_0}{3},$$

$$\|v_n - v\|_{L^p(\mathbb{R}^N)} < \frac{\varepsilon_0}{3(\|\rho\|_{L^{p'}(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}{\kappa_p}^{p-1}M^{p-1})}.$$

Then

$$\begin{aligned}\langle u_{F_n}^* - u_F^*, v_n\rangle &= \langle u_{F_n}^* - u_F^*, v\rangle + \langle u_{F_n}^*, v_n - v\rangle - \langle u_F^*, v_n - v\rangle \\ &< \frac{2\varepsilon_0}{3} + \int_{\mathbb{R}^N} |u_{F_n}^*||v_n(x) - v(x)|\,dx \\ &\le \frac{2\varepsilon_0}{3} + \int_{\mathbb{R}^N} (\rho(x)+\sigma(x)\,|u_n(x)|^{p-1})|v_n(x)-v(x)|\,dx \\ &\le \frac{2\varepsilon_0}{3} + \|\rho\|_{L^{p'}(\mathbb{R}^N)}\|v_n - v\|_{L^p(\mathbb{R}^N)} \\ &\quad + \|\sigma\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{L^p(\mathbb{R}^N)}^{p-1}\|v_n - v\|_{L^p(\mathbb{R}^N)} \\ &\le \frac{2\varepsilon_0}{3} + (\|\rho\|_{L^{p'}(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}{\kappa_p}^{p-1}M^{p-1})\|v_n - v\|_{L^p(\mathbb{R}^N)} \\ &< \varepsilon_0,\end{aligned}$$

which contradicts to (2.4). □

Now we give that every critical point of the functional I_λ is weak solution for our problem. The basic idea of the proof of the following consequence comes from [12]; see also [5].

Lemma 2.15. *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Assume that $f \in \mathcal{G}$ satisfies* (F2). *Then the critical points of the functional I_λ are weak solutions for the problem (P).*

Proof. If $u_0 \in X_s(\mathbb{R}^N)$ is a critical point of I_λ, then one has

$$I_\lambda{}^\circ(u_0; v) = (J - \lambda\Psi)^\circ(u_0; v) \geq 0 \quad \text{for all} \quad v \in X_s(\mathbb{R}^N).$$

Since $J \in C^1(X_s(\mathbb{R}^N), \mathbb{R})$, we have, by Proposition 2.6, in particular

$$0 \leq (J - \lambda\Psi)^\circ(u_0; v) = J'(u_0; v) + (-\lambda\Psi)^\circ(u_0; v) = \langle J'(u_0), v\rangle + (-\lambda\Psi)^\circ(u_0; v),$$

whence

$$-\langle J'(u_0), v\rangle \leq (-\lambda\Psi)^\circ(u_0; v) \quad \text{for all} \quad v \in X_s(\mathbb{R}^N).$$

This means

$$-J'(u_0) \in \partial(-\lambda\Psi)(u_0),$$

namely

$$J'(u_0) \in \partial(\lambda\Psi)(u_0). \tag{2.5}$$

Since $X_s(\mathbb{R}^N)$ is compactly embedded and dense in $L^p(\mathbb{R}^N)$, from Theorem 2.2 in [17] one has

$$\partial(-\lambda\Psi)(u_0) \subseteq \partial(-\lambda\Psi_{|L^p(\mathbb{R}^N)})(u_0).$$

From (F2) and because $-\lambda f \in \mathcal{G}$, we deduce that $\overline{f}$, $\underline{f}$, $-\lambda\overline{f}$, and $-\lambda\underline{f}$ satisfy all the assumptions in Theorem 2.1 of [17]. Thus

$$\partial(\Psi_{|L^p(\mathbb{R}^N)})(u_0) \subseteq [\underline{f}(x, u_0(x)),\, \overline{f}(x, u_0(x))]_{p'},$$

where

$$\begin{aligned} &[\underline{f}(\cdot, u_0(\cdot)),\, \overline{f}(\cdot, u_0(\cdot))]_{p'} \\ &= \{\omega \in L^{p'}(\mathbb{R}^N) : \omega(x) \in [\underline{f}(x, u_0(x)),\, \overline{f}(x, u_0(x))] \text{ a.e. in } \mathbb{R}^N\}. \end{aligned}$$

Using (2.5),

$$J'(u_0) \in \lambda[\underline{f}(\cdot, u_0(\cdot)),\, \overline{f}(\cdot, u_0(\cdot))]_{p'}$$

and then we have

$$(-\Delta)_p^s u_0 + V(x)\,|u|^{p-2}\,u \in \lambda[\underline{f}(\cdot, u_0(\cdot)),\, \overline{f}(\cdot, u_0(\cdot))]_{p'}.$$

Thus, Radon-Nikodym theorem implies that there exists a unique

$$\omega_0(\cdot) \in \lambda[\underline{f}(\cdot, u_0(\cdot)),\, \overline{f}(\cdot, u_0(\cdot))]_{p'}$$

such that

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p-2}(u_0(x) - u_0(y))(v(x) - v(y))}{|x - y|^{N+ps}}\,dxdy \\ &\qquad + \int_{\mathbb{R}^N} V(x)\,|u_0(x)|^{p-2}\,u(x)v(x)\,dx = \int_{\mathbb{R}^N} \omega_0(x)v(x)\,dx \end{aligned}$$

for each $v \in X_s(\mathbb{R}^N)$. This means that u_0 is a weak solution of the problem

$$(-\Delta)_p^s u + V(x)\,|u|^{p-2}\,u = \omega_0(x) \quad \text{in} \quad \mathbb{R}^N.$$

This completes the proof. $\square$

Now we obtain the positivity of the infimum of all eigenvalues for the problem

$$(E) \qquad (-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N.$$

Although the idea of the proof is completely the same as in that of Lemma 3.1 in [20], for the sake of convenience, we give the proof of the following proposition.

Proposition 2.16. *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Then the eigenvalue problem* (E) *has a pair* (λ_1, u_1) *of a principal eigenvalue λ_1 and an eigenfunction u_1 with $\lambda_1 > 0$ and $0 < u_1 \in X_s(\mathbb{R}^N)$.*

Proof. Set

$$\lambda_1 = \inf\left\{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)\,|v|^p\,dx\right\}$$

the infimum being taken over all v such that $\int_{\mathbb{R}^N} |v|^p dx = 1$. We shall prove that λ_1 is the least eigenvalue of (E). The expression for λ_1 presented above will be referred to as its variational characterization. Obviously $\lambda_1 \geq 0$. Let $\{v_n\}_{n=1}^\infty$ be the minimizing sequence for λ_1, *i.e.*,

$$(2.6) \qquad \begin{aligned} &\int_{\mathbb{R}^N} |v_n|^p\,dx = 1 \text{ and} \\ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx = \lambda_1 + \delta_n \end{aligned}$$

with $\delta_n \to 0^+$ for $n \to \infty$. It follows from (2.6) that $\|v_n\|_{X_s(\mathbb{R}^N)} \leq c$ for some constant $c > 0$. The reflexivity of $X_s(\mathbb{R}^N)$ yields the weak convergence $v_n \rightharpoonup u_1$ in $X_s(\mathbb{R}^N)$ for some u_1 (at least for some subsequence of $\{v_n\}$). The compact embedding $X_s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ implies the strong convergence $v_n \to u_1$ in $L^p(\mathbb{R}^N)$. It follows from (2.6) and the Minkowski inequality that

$$\begin{aligned} 1 &= \lim_{n\to\infty}\left(\int_{\mathbb{R}^N} |v_n|^p\,dx\right)^{\frac{1}{p}} \\ &\leq \lim_{n\to\infty}\left(\int_{\mathbb{R}^N} |v_n - u_1|^p\,dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} |u_1|^p\,dx\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^N} |u_1|^p\,dx\right)^{\frac{1}{p}}, \end{aligned}$$

and analogously

$$\left(\int_{\mathbb{R}^N} |u_1|^p \, dx\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^N} |u_1 - v_n|^p \, dx\right)^{\frac{1}{p}} + \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |v_n|^p \, dx\right)^{\frac{1}{p}} = 1.$$

Hence

$$\int_{\mathbb{R}^N} |u_1|^p \, dx = 1.$$

In particular, $u_1 \not\equiv 0$. The weak lower semicontinuity of the norm in $X_s(\mathbb{R}^N)$ yields

$$\begin{aligned}
\lambda_1 &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_1(x) - u_1(y)|^p}{|x-y|^{N+ps}} \, dxdy + \int_{\mathbb{R}^N} V(x) \, |u_1|^p \, dx = \|u_1\|_X^p \\
&\leq \liminf_{n\to\infty} \|v_n\|_X^p \\
&= \liminf_{n\to\infty} \left\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+ps}} \, dxdy + \int_{\mathbb{R}^N} V(x) \, |v_n|^p \, dx \right\} \\
&= \liminf_{n\to\infty} (\lambda_1 + \delta_n) = \lambda_1,
\end{aligned}$$

i.e.,

$$\lambda_1 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_1(x) - u_1(y)|^p}{|x-y|^{N+ps}} \, dxdy + \int_{\mathbb{R}^N} V(x) \, |u_1|^p \, dx. \tag{2.7}$$

It follows from (2.7) that $\lambda_1 > 0$ and it is easy to see that λ_1 is the least eigenvalue of (E) with the corresponding eigenfunction u_1.

Moreover, if u is an eigenfunction corresponding to λ_1, then $|u|$ is also an eigenfunction corresponding to λ_1. Hence we can suppose that $u_1 > 0$ a.e. in $\mathbb{R}^N$. □

Now, we are ready to state the main result of this paper. We investigate the solvability of nonlinear elliptic equations involving the fractional p-Laplacian, by using the Berkovits-Tienari degree theory for set-valued operators of class (S_+).

Theorem 2.17. *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Assume that $f \in \mathcal{G}$ satisfies* (F2). *If $\lambda\lambda_1 \|\sigma\|_{L^\infty(\mathbb{R}^N)} < 1$, then the problem (P) admits at least one nontrivial weak solution in $X_s(\mathbb{R}^N)$.*

Proof. Note by Corollary 2.11, Lemmas 2.7, 2.10, and 2.14 that the bounded continuous operator J' is of class (S_+) and $-\partial\Psi$ is quasimonotone. Hence, taking into account Lemma 2.3 the sum $J' - \partial(\lambda\Psi)$ is bounded, upper semicontinuous, and of class (S_+) or pseudomonotone. For each $u \in X_s(\mathbb{R}^N)$, we can express as

$$\langle w, u \rangle = \int_{\mathbb{R}^N} wu \, dx$$

for some $w \in \partial\Psi(u)$. Applying Hölder's inequality, we assert that

$$\begin{aligned}
&-\int_{\mathbb{R}^N} wu\,dx \\
&\geq -\int_{\mathbb{R}^N} (\rho(x)+\sigma(x)\,|u(x)|^{p-1})u\,dx \\
&\geq -\|\sigma\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}|u|^p\,dx - \left(\int_{\mathbb{R}^N}|\rho(x)|^{p'}\,dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^N}|u|^p\,dx\right)^{\frac{1}{p}} \\
&\geq -\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)}\|u\|^p_{X_s(\mathbb{R}^N)} - \lambda_1^{\frac{1}{p}}\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u\|_{X_s(\mathbb{R}^N)}
\end{aligned}$$

and hence

$$\begin{aligned}
\langle J'(u)-\lambda w, u\rangle &= \|u\|^p_{X_s(\mathbb{R}^N)} - \lambda\int_{\mathbb{R}^N} wu\,dx \\
&\geq \left(1-\lambda\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)}\right)\|u\|^p_{X_s(\mathbb{R}^N)} - \lambda\lambda_1^{\frac{1}{p}}\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u\|_{X_s(\mathbb{R}^N)}.
\end{aligned}$$

This implies, owing to $\lambda\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)} < 1$ and $p > 1$, that there exists a positive constant R such that

$$\langle J'(u)-\lambda w, u\rangle > 0 \quad \text{for all } u \in X_s(\mathbb{R}^N) \text{ with } \|u\|_{X_s(\mathbb{R}^N)} \geq R \text{ and } w \in \partial\Psi(u).$$

Observe by Remark 2.4 that the duality operator $j : X_s(\mathbb{R}^N) \to (X_s(\mathbb{R}^N))^*$ is injective, bounded, continuous, and of class (S_+), and such that $\langle jx, x\rangle = \|x\|^2_{X_s(\mathbb{R}^N)}$ and $\|jx\|_{(X_s(\mathbb{R}^N))^*} = \|x\|_{X_s(\mathbb{R}^N)}$ for $x \in X_s(\mathbb{R}^N)$. Let $\varepsilon > 0$ be arbitrary but fixed. We consider a homotopy $H : [0,1]\times\overline{B_R(0)} \to 2^{(X_s(\mathbb{R}^N))^*}$ defined by

$$H(t,u) := (1-t)(J'-\partial(\lambda\Psi))(u) + \varepsilon ju \quad \text{for } (t,u) \in [0,1]\times\overline{B_R(0)}.$$

Then the operators $(J'-\partial(\lambda\Psi))+\varepsilon j$ and j are of class (S_+), the affine homotopy H is also of class (S_+). Moreover, we have $0 \notin H(t,u)$ for all $(t,u) \in [0,1]\times\partial B_R(0)$. The homotopy invariance and normalization properties of the degree d imply that

$$d\left((J'-\partial(\lambda\Psi))+\varepsilon j, B_R(0), 0\right) = d\left(\varepsilon j, B_R(0), 0\right) = 1.$$

Put $\varepsilon = 1/n$ for each $n \in \mathbb{N}$. The existence property of the degree in Lemma 2.5 yields that there exist points $u_n \in B_R(0)$ and $w_n \in (J'-\partial(\lambda\Psi))(u_n)$ such that

$$w_n + \frac{1}{n}ju_n = 0.$$

Passing to a subsequence, if necessary, we may suppose that $u_n \rightharpoonup u$ in $X_s(\mathbb{R}^N)$ for some $u \in X_s(\mathbb{R}^N)$. Then it follows from the boundedness of $\{ju_n\}$ that $w_n \to 0$ in $(X_s(\mathbb{R}^N))^*$ and hence

$$\lim_{n\to\infty}\langle w_n, u_n-u\rangle = 0.$$

Note that the weak limit u belongs to the closed convex hull of the open ball $B_R(0)$ and so $u \in \overline{B_R(0)} \subset X_s(\mathbb{R}^N)$. Since $J'-\partial(\lambda\Psi)$ is pseudomonotone, we assert that the inclusion $0 \in (J' - \partial(\lambda\Psi))(u)$ has a solution in $X_s(\mathbb{R}^N)$ and so u is a critical point of I_λ. In view of Lemma 2.15, the conclusion holds. This completes the proof. $\square$

Next we consider the existence of two distinct nontrivial weak solutions for problem (P). To do this, we need that the following additional condition on f:

(F3) There exists $\delta > 0$ such that

$$f(x,t) \geq s(x)t^{\gamma_0 - 1}$$

for almost all $x \in \mathbb{R}^N$ and $0 < t \leq \delta$, where $s \geq 0, s \in C(\mathbb{R}^N, \mathbb{R})$ and $1 < \gamma_0 < p$.

Theorem 2.18. *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Assume that $f \in \mathcal{G}$ satisfies* (F2)–(F3)*. Then there exists a positive constant λ^* such that problem (P) admits two nontrivial weak solutions in $X_s(\mathbb{R}^N)$ in which one has negative energy and another has positive energy for any $\lambda \in (0, \lambda^*)$.*

Proof. We first claim that there exists $\varphi \in X_s(\mathbb{R}^N)$ such that $\varphi \geq 0, \varphi \neq 0$ and $I_\lambda(\eta\varphi) < 0$ for $\eta > 0$ small enough. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^N$; $\varphi(x) \equiv 1$ for all $x \in B_R(x_0)$; $\varphi(x) \equiv 0$ for all $x \in \mathbb{R}^N \setminus B_{2R}(x_0)$, where $B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| \leq R\}$. Then it is obvious that $\varphi \in X_s(\mathbb{R}^N)$. Also it follows from (F3) that for any $\eta \in (0,1)$,

$$\begin{aligned}
I_\lambda(\eta\varphi) &= J(\eta\varphi) - \lambda\Psi(\eta\varphi) \\
&= \frac{1}{p}\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|\eta\varphi(x) - \eta\varphi(y)|^p}{|x-y|^{N+ps}}\,dxdy \\
&\quad + \frac{1}{p}\int_{B_{2R}(x_0)} V(x)\,|\eta\varphi|^p\,dx - \lambda\int_{B_{2R}(x_0)} F(x,\eta\varphi)\,dx \\
&\leq \frac{\eta^p}{p}\left(\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{N+ps}}\,dxdy + \int_{B_{2R}(x_0)} V(x)\,|\varphi|^p\,dx\right) \\
&\quad - \lambda\int_{B_{2R}(x_0)} \frac{\eta^{\gamma_0}}{\gamma_0}\,|s(x)|\,|\varphi|^{\gamma_0}\,dx \\
&\leq \frac{\eta^p}{p}\left(\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{N+ps}}\,dxdy + \int_{B_{2R}(x_0)} V(x)\,|\varphi|^p\,dx\right) \\
&\quad - \frac{\lambda\eta^{\gamma_0}}{\gamma_0}\int_{B_{2R}(x_0)} |s(x)|\,dx.
\end{aligned}$$

Choose a positive constant δ such that

$$0 < \delta < \min\left\{1, \frac{\frac{\lambda p}{\gamma_0}\int_{B_{2R}(x_0)} |s(x)|\,dx}{\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\,dxdy + \int_{B_{2R}(x_0)} V(x)\,|\varphi|^p\,dx}\right\},$$

then $\eta < \delta^{1/(p-\gamma_0)}$ implies that

$$I_\lambda(\eta\varphi) < 0,$$

as claimed.

Let us define the quantity

$$\lambda^* = \min\left\{ \varrho^{p-1}\left(\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)}\varrho^{p-1} + p\lambda_1^{\frac{1}{p}}\|\rho\|_{L^{p'}(\mathbb{R}^N)}\right)^{-1}, \left(\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)}\right)^{-1}\right\},$$

where ϱ will be chosen later. Then for any $u \in X_s(\mathbb{R}^N)$ and $\lambda \in (0, \lambda^*)$ it follows from the assumption (F2) that

$$\begin{aligned}
I_\lambda(u) &= J(u) - \lambda\Psi(u) \\
&= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy - \lambda\int_{\mathbb{R}^N}F(x,u)\,dx \\
&\geq \frac{1}{p}\left(\|u\|^p_{X_s(\mathbb{R}^N)} - \lambda\|\sigma\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}|u|^p\,dx\right) \\
&\quad - \left(\int_{\mathbb{R}^N}|\rho(x)|^{p'}\,dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^N}|u|^p\,dx\right)^{\frac{1}{p}} \\
&\geq \frac{1}{p}\left(1 - \lambda\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)}\right)\|u\|^p_{X_s(\mathbb{R}^N)} - \lambda\lambda_1^{\frac{1}{p}}\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u\|_{X_s(\mathbb{R}^N)}.
\end{aligned} \tag{2.8}$$

Since $\lambda\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)} < 1$ and $p > 1$, we conclude that

$$I_\lambda(u) \to +\infty \text{ as } \|u\|_{X_s(\mathbb{R}^N)} \to +\infty \text{ for all } u \in X_s(\mathbb{R}^N) \text{ and } \lambda \in (0, \lambda^*).$$

This means that I_λ is coercive for all $\lambda \in (0, \lambda^*)$. By the coercivity of the functional I_λ, we get that there exists a global minimizer $u_1 \in X_s(\mathbb{R}^N)$ of I_λ (Theorem 1.2 in [35]). This together with the above claim yields that

$$I_\lambda(u_1) = \inf_{u\in X_s(\mathbb{R}^N)\setminus\{0\}} I_\lambda(u) < 0.$$

Hence we deduce that u_1 is a nontrivial global minimizer of the functional I_λ in $X_s(\mathbb{R}^N)$ for any $\lambda \in (0, \lambda^*)$.

Finally, we will establish that our problem has another weak solution with positive energy. As in Theorem 2.17, we deduce that the functional I_λ has a nontrivial critical point u. Denote it by $u = u_2$ with $\|u_2\|_{X_s(\mathbb{R}^N)} = \varrho > 0$. By the inequality (2.8), we yield

$$\begin{aligned}
I_\lambda(u_2) &\geq \frac{1}{p}\left(1 - \lambda\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)}\right)\|u_2\|^p_{X_s(\mathbb{R}^N)} - \lambda\lambda_1^{\frac{1}{p}}\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u_2\|_{X_s(\mathbb{R}^N)} \\
&= \frac{1}{p}\left(1 - \lambda\lambda_1\|\sigma\|_{L^\infty(\mathbb{R}^N)}\right)\varrho^p - \lambda\lambda_1^{\frac{1}{p}}\|\rho\|_{L^{p'}(\mathbb{R}^N)}\varrho
\end{aligned}$$

and thus, owing to the definition of λ^*, we assert that $I_\lambda(u_2) > 0$ for any $\lambda \in (0, \lambda^*)$. Therefore, we conclude that u_2 is another weak solution with positive energy. This completes the proof. □

3. Appendix

In this section, we consider the existence of weak solutions for equations in $\mathbb{R}^N$, driven by a non-local integro-differential operator of elliptic type as follows:

$$(P_K) \qquad -\mathcal{L}_K u + V(x)|u|^{p-2}u \in \lambda[\underline{f}(x,u(x)),\, \overline{f}(x,u(x))] \quad \text{in } \mathbb{R}^N,$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ is a kernel function satisfying properties that

(K1) $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^p, 1\}$;
(K2) there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(N+ps)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
(K3) $K(x) = K(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

By the condition (K1), the function

$$(x,y) \mapsto (u(x)-u(y))K(x-y)^{\frac{1}{p}} \in L^p(\mathbb{R}^{2N})$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. Let us denote by $W_K^{s,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{W_K^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + |u|^p_{W_K^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}},$$

where

$$|u|^p_{W_K^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p K(x-y)\,dxdy.$$

Lemma 3.1 ([37])**.** *Let $K : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ be a kernel function satisfying the conditions* (K1)–(K3)*. Then if $v \in W_K^{s,p}(\mathbb{R}^N)$, then $v \in W^{s,p}(\mathbb{R}^N)$. Moreover*

$$\|v\|_{W^{s,p}(\mathbb{R}^N)} \leq \max\{1, \theta^{-\frac{1}{p}}\}\|v\|_{W_K^{s,p}(\mathbb{R}^N)};$$

From Lemmas 2.8 and 3.1, we can obtain the following assertion immediately.

Lemma 3.2 ([37])**.** *Let $K : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ satisfy the conditions* (K1)–(K3)*. Then there exists a positive constant $C_0 = C_0(N,p,s)$ such that for any $v \in W_K^{s,p}(\mathbb{R}^N)$ and $p \leq q \leq p_s^*$*

$$\begin{aligned}\|v\|^p_{L^q(\mathbb{R}^N)} &\leq C_0 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}dxdy\\ &\leq \frac{C_0}{\theta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p K(x-y)\,dxdy;\end{aligned}$$

In this section, the basic space

$$X_K^s(\mathbb{R}^N) := \left\{u \in W_K^{s,p}(\mathbb{R}^N) : V|u|^p \in L^1(\mathbb{R}^N)\right\}$$

denote the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{X_K^s(\mathbb{R}^N)} := \left(|u|^p_{W_K^{s,p}(\mathbb{R}^N)} + \|V^{\frac{1}{p}}u\|^p_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{p}},$$

where the function V satisfies the condition (V).

Combining with Lemmas 2.9 and 3.2, we get the following consequence.

Lemma 3.3 ([36])**.** *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that conditions* (V) *and* (K1)-(K3) *are satisfied. Then there is a compact embedding $X_K^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $q \in [p, p_s^*)$.*

Definition 3.4. Let $0 < s < 1 < p < +\infty$ with $ps < N$ and conditions (V) and (K1)-(K3) are satisfied. We say that $u \in X_K^s(\mathbb{R}^N)$ is a weak solution of the problem (P_K) if there exists a function $w \in \partial F(x,u)$ for almost all $x \in \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))K(x-y)\,dxdy \\ + \int_{\mathbb{R}^N} V(x)\,|u(x)|^{p-2}\,uv\,dx = \lambda \int_{\mathbb{R}^N} wv\,dx$$

for all $v \in X_K^s(\mathbb{R}^N)$.

Let us define a functional $J_{p,K} : X_K^s(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_{p,K}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p K(x-y)\,dxdy.$$

Then the functional $J_{p,K}$ is well defined on $X_K^s(\mathbb{R}^N)$, $J_{p,K} \in C^1(X_0,\mathbb{R})$ and its Fréchet derivative is given by

$$\langle J'_{p,K}(u), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))K(x-y)\,dxdy$$

for any $v \in X_K^s(\mathbb{R}^N)$ where $\langle\cdot,\cdot\rangle$ denotes the pairing of $X_K^s(\mathbb{R}^N)$ and its dual $(X_K^s(\mathbb{R}^N))^*$; see [32].

Let us define the quantity

$$\lambda_{1,K} = \inf\left\{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)|^p K(x-y)\,dxdy + \int_{\mathbb{R}^N} V(x)\,|v|^p\ dx : \int_{\mathbb{R}^N} |v|^p dx = 1\right\}.$$

Arguing as in Proposition 2.6, we derive that $\lambda_{1,K}$ is the least eigenvalue of the problem

$$-\mathcal{L}_K u + V(x)|u|^{p-2}u = \lambda|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N.$$

Theorem 3.5. *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that conditions* (V) *and* (K1)-(K3) *are satisfied. Assume that $f \in \mathcal{G}$ satisfies* (F2)*. If $\lambda\lambda_{1,K}\|\sigma\|_{L^\infty(\mathbb{R}^N)} < 1$, then the problem (P_K) admits at least one nontrivial weak solution in $X_K^s(\mathbb{R}^N)$.*

Proof. If we replace $X_s(\mathbb{R}^N)$ and J by $X_K^s(\mathbb{R}^N)$ and J_K, respectively, then obvious modifications of the proofs of Lemmas 2.14 and 2.15 yield that the same assertions hold. Therefore it follows upon proceeding the same way as in the proof of Theorem 2.17 that the conclusion holds. This completes the proof. □

Theorem 3.6. *Let* $0 < s < 1 < p < +\infty$ *with* $ps < N$ *and suppose that conditions* (V) *and* (K1)-(K3) *are satisfied. Assume that* $f \in \mathcal{G}$ *satisfies* (F2)–(F3). *Then there exists a positive constant* λ^* *such that problem* (P_K) *admits two nontrivial weak solutions in* $X_K^s(\mathbb{R}^N)$ *in which one has negative energy and another has positive energy for any* $\lambda \in (0, \lambda^*)$.

Proof. The idea of the proof is essentially the same as that of the proof of Theorem 2.18. □

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