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# INFINITE HORIZON OPTIMAL CONTROL PROBLEMS OF BACKWARD STOCHASTIC DELAY DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. This paper investigates infinite horizon optimal control problems driven by a class of backward stochastic delay differential equations in Hilbert spaces. We first obtain a prior estimate for the solutions of state equations, by which the existence and uniqueness results are proved. Meanwhile, necessary and sufficient conditions for optimal control problems on an infinite horizon are derived by introducing time-advanced stochastic differential equations as adjoint equations. Finally, the theoretical results are applied to a linear-quadratic control problem.

## 1. Introduction

Peng and Pardoux [11] first proposed the following backward stochastic differential equations (BSDEs) in 1990:

$$y(t) = -\int_0^T G(s, y(s), z(s))ds + \int_0^T z(s)dW(s), \quad y(T) = \xi,$$

and proved the existence and uniqueness of solutions. Since BSDEs have been widely applied to control, finance, insurance, operations research and other fields, researchers systematically studied the theory of BSDEs in [3, 7]. Mao [10] established the existence and uniqueness theorem of adapt solutions of BSDEs under non-lipschitz conditions.

The BSDEs theory in Hilbert spaces is regarded as a natural extension of finite dimensional BSDEs theory and have been extensively studied. Bensoussan [1] obtained the solution of BSDEs in Hilbert spaces with special linear case by approximation method. Hu and Peng [6] solved the solution of BSDEs in Hilbert spaces in general linear case by functional analysis method.

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Cylindrical Wiener process appears in various models of infinite dimensional space in the form of random noise or random disturbance [12]. Fuhrman and Tessitore [5] studied the BSDEs driven by cylindrical Wiener process with values in Hilbert spaces in the following form:

$$y(t) = \int_t^\infty G(s, y(s), z(s))ds + \lambda \int_t^\infty y(s)ds + \int_t^\infty z(s)dW(s) - \int_t^\infty f(s)ds,$$

and established the existence and uniqueness theorem of solutions.

Levy process also covers basic mathematics, statistics, economics, engineering and other fields. In recent years, its relevant theories have developed rapidly as an important branch of modern probability theory. Tang and Li [13] introduced Poisson random measure independent of Brownian motion in BSDEs and obtained the existence and uniqueness of solutions. Yin and Mao [15] studied a class of BSDEs with Poisson jump and random termination time and obtained the existence and uniqueness of solutions. Meanwhile, BSDEs-related control problems have also received great attention. Li [9] studied the stochastic optimal control problem with jumps by using the backward stochastic theory; Yu [16] investigated coupled forward-backward stochastic differential equations and related linear-quadratic problems.

Stochastic systems with time-delay characteristics are common in the fields of epidemiology, engineering and risk management. Comparing with general stochastic control problems, the development of systems with time-delay depend not only on their current state, but also on previous information. Since the initial research of Kolmanovskii and Maizenberg [8], the control problems of random systems with time-delay have attracted the attention of many researchers. In 2010, [4] introduced a class of BSDEs with time-delay generator, which proved the existence and uniqueness of the solutions for a small enough time range or for a generator satisfying a small enough Lipschitz constant. Chen and Huang [2] obtained the maximal principle of BSDEs with recursive time-delay by introducing the relevant time-advanced stochastic differential equations (ASDEs) as adjoint equation. Recently, [14] established a stochastic maximum principle of the optimal control problems of forward-backward delay systems involving impulse controls.

In this paper, we consider the following recursive delayed BSDEs driven by both Cylindrical Wiener processes and Poisson processes on infinite horizon:

$$\begin{cases} y(T)-y(t) = -\displaystyle\int_t^T G\Big(s, y(s), \int_{s-\delta}^s \phi(s,t)y(t)\alpha(dt), z(s), r(s,\cdot)\Big)ds \\ \qquad\qquad + \lambda \displaystyle\int_t^T y(s)ds + \int_t^T z(s)dW(s) + \int_t^T\int_{\mathcal{E}} r(s,e)\tilde{N}(ds,de) \\ \qquad\qquad - \displaystyle\int_t^T f(s)ds, \ t \in [0,T], \\ y(s) = \varphi(s), z(s) = \psi(s), s \in [-\delta, 0), \end{cases} \tag{1.1}$$

where the notations and mappings will be given in Section 2 and Section 3. We adopt the model with cylindrical Wiener process and Poisson jump process which characterize practical phenomena more accurate than others. By means of conditioning to time parameters and equation coefficients, we generalize the theory started with the article by Chen and Huang [2] to an infinite dimensional framework. We will show that under our assumptions, for sufficiently large values of $\lambda$, (1.1) exists a unique solution. Then we state controlled backward stochastic delayed system and introduce ASDEs as adjoint equations by duality between them. Then the maximal principle are derived.

This paper is organized as follows. In Section 2, we give some necessary notations and state some preliminary results about backward stochastic delay differential equations (BSDDEs). The existence and uniqueness of solution to (1.1) is proved. In Section 3, we establish necessary and sufficient conditions of optimality. In Section 4, we apply the results obtained in Section 3 to study a linear-quadratic optimal control problem.

## 2. Preliminaries

Let $\Xi$, $K$ and $H$ denote real separable Hilbert spaces, with scalar products $\langle\cdot,\cdot\rangle_\Xi$, $\langle\cdot,\cdot\rangle_K$ and $\langle\cdot,\cdot\rangle_H$, respectively. We use the symbol $|\cdot|$ to denote the norm in various spaces, with a subscript if necessary. $L(\Xi, K)$ denotes the space of all bounded linear operators from $\Xi$ into $K$, endowed with the usual operator norm. The space of the Hilbert Schmidt operators from $\Xi$ to $K$ is $L_2(\Xi, K)$, which is given the Hilbert-Schmidt norm, making it a separable Hilbert space.

Let $(\Omega, \mathcal{F}, P)$ be a complete space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfies the usual condition, i.e., $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous increasing family of sub $\sigma$-algebra of $\mathcal{F}$ and $\mathcal{F}_0$ contains all $P$-null sets of $\mathcal{F}$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by two mutually independent stochastic sources. One is cylindrical Wiener process $\{W(t), t \geq 0\}$, and the other is Poisson measure $\{k(t), t \geq 0\}$ with compensators defined on a measurable space $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ . Here, $\pi(e)$ is a given $\sigma$-finite measure on the measurable space $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ satisfying $\int_{\mathcal{E}}(1 \wedge |e|^2)\pi(de) < \infty$. We use $N(dt, de)$ to represent the Poisson counting measure induced by $k(t)$. The compensators of $N$ is $\bar{N}(dt, de) = N(dt, de) - \pi(de)dt$ for any $A \in \mathcal{B}(\mathcal{E})$ satisfying $\pi(A) < \infty$ such that $\{\tilde{N}((0, t] \times A) = (N - \bar{N})((0, t] \times A), 0 < t < \infty\}$ is a martingale. All the concepts of measurability (e.g., predictability, etc.) for stochastic processes refer to this filtration. By $\mathcal{P}$ we denote the predictable $\sigma$-algebra generated by predictable processes. $\mathcal{B}(\Omega)$ denotes the Borel $\sigma$-algebra of any topological space $\Omega$.

We introduce some spaces:

Expression $L^2_{\mathcal{P}}(\Omega \times \mathbb{R}_+; K)$ denotes the space of equivalence classes of processes $y \in L^2(\Omega \times \mathbb{R}_+; K)$, admitting a predictable version. It is endowed with the norm

$$|y|^2_{L^2_{\mathcal{P}}(\Omega\times\mathbb{R}_+;K)} = \mathbb{E}\int_0^\infty |y(t)|^2_K dt.$$

Expression $L^p_{\mathcal{P}}(\Omega; L^2_\beta(K))$, defined for $\beta \in \mathbb{R}$ and $p, q \in [1, \infty)$, denotes the space of equivalence classes of processes $\{y(t), t \geq 0\}$, with values in $K$, such that the norm

$$|y|^p_{L^q_\beta(K)} = \mathbb{E}\left(\int_0^\infty e^{q\beta t}|\hat{y}(t)|^q_K dt\right)^{\frac{p}{q}}$$

is finite, and $y(t)$ admits a predictable version.

Expression $L^2(\mathcal{E} \times R_+, \mathcal{B}(\mathcal{E}), \pi; K)$ denotes the space of equivalence classes of $\pi$-measurable processes $r(\cdot)$ ($\mathbb{P}$-a.s.), with values in $K$, such that

$$\|r(\cdot)\| = \left(\int_{\mathcal{E}} |r(\cdot)|^2_K \pi d(e)\right)^{\frac{1}{2}} < \infty.$$

Expression $L^p_{\mathcal{P}}(\mathcal{E}; L^2_\beta(K))$, defined for $\beta \in \mathbb{R}$, denotes the space of equivalence classes of $\{r(t, \cdot), t \geq 0\}$, with values in $K$, such that the norm

$$\begin{aligned}|r|^p_{L^p_{\mathcal{P}}(\mathcal{E}; L^2_\beta(K))} &= \mathbb{E}\left\{\int_0^\infty e^{q\beta t}\left|\int_{\mathcal{E}} |r(\cdot)|^2_K \pi d(e)\right|^{\frac{q}{2}} dt\right\}^{\frac{p}{q}} \\ &= \mathbb{E}\left(\int_0^\infty e^{q\beta t}\|r(t, \cdot)\|^q dt\right)^{\frac{p}{q}}\end{aligned}$$

is finite, and $r(t, \cdot)$ admits a predictable version.

Now, let us consider a kind of infinite horizon BSDDE as follows:

$$\begin{cases} y(T)-y(t) = -\displaystyle\int_t^T G\left(s, y(s), \int_{s-\delta}^s \phi(s,t)y(t)\alpha(dt), z(s), r(s,\cdot)\right)ds \\ \qquad + \lambda \displaystyle\int_t^T y(s)ds + \int_t^T z(s)dW(s) + \int_t^T\int_{\mathcal{E}} r(s,e)\tilde{N}(ds,de) \\ \qquad - \displaystyle\int_t^T f(s)ds, \ t \in [0,T], \\ y(s) = \varphi(s), s \in [-\delta, 0). \end{cases} \tag{2.1}$$

Where $\lambda$ is a given real parameter, $\delta$ is a time delay parameter, $\alpha$ is a $\sigma$-finite measure, and $\phi(\cdot,\cdot)$ is a locally bounded precess. The function $G : \Omega \times [0,\infty) \times K \times L_2(\Xi, K) \times L^2(\mathcal{E} \times \mathbb{R}_+, \mathcal{B}(\mathcal{E}), \pi; K)$ is measurable with respect to $\mathcal{P} \bigotimes \mathcal{B}(K) \bigotimes \mathcal{B}(L_2(\Xi, K)) \bigotimes \mathcal{B}(L^2(\mathcal{E} \times \mathbb{R}_+, \mathcal{B}(\mathcal{E}), \pi; K))$ and $\mathcal{B}(K)$. $f : \Omega \times [0,\infty) \to K$ is a predictable process with integrable paths.

As a matter of convenience, we set

$$y_\delta(s) = \int_{s-\delta}^s \phi(s,t)y(t)\alpha(dt).$$

We assume the following assumptions:

**(H2.1)** $G$ is Lipschitz continuous with respect to $(y, y_\delta, z, r(\cdot))$, i.e., there exist nonnegative constants $L_y$,$L_\delta$,$L_z$ and $L_r$, such that for any $s \in [0,\infty)$,

$y^1, y^2, y^1_\delta, y^2_\delta \in K, z^1, z^2 \in L_2(\Xi, K)$, and $r^1(\cdot), r^2(\cdot) \in L^2(\mathcal{E} \times \mathbb{R}_+, \mathcal{B}(\mathcal{E}), \pi; K)$,

$$\begin{aligned}&|G(s, y^1, y^1_\delta, z^1, r^1(\cdot)) - G(s, y^2, y^2_\delta, z^2, r^2(\cdot))| \\ &\leq L_y|y^1 - y^2| + L_\delta|y^1_\delta - y^2_\delta| + L_z|z^1 - z^2| + L_r|r^1 - r^2|.\end{aligned}$$

**(H2.2)** $G$ satisfies following monotonicity condition, i.e., there exists a non-negative constant $\mu$ such that for any $s \in [0, \infty)$, $y^1, y^2, y_\delta \in K, z \in L_2(\Xi, K)$, and $r(\cdot) \in L^2(\mathcal{E} \times \mathbb{R}_+, \mathcal{B}(\mathcal{E}), \pi; K)$,

$$\langle G(s, y^1, y_\delta, z, r(\cdot)) - G(s, y^2, y_\delta, z, r(\cdot)), y^1 - y^2\rangle \leq -\mu|y^1 - y^2|^2.$$

**(H2.3)** For any $s \in [0, \infty)$, there exists $p \in [2, \infty)$ such that

$$\mathbb{E}\bigg(\int_0^\infty e^{2\beta s}|G(s, 0, 0, 0, 0)|ds\bigg)^{\frac{p}{2}} < \infty.$$

**(H2.4)** $|\phi(s, t)| \leq M$ for any $t, s \in [-\delta, T]$ and some $M > 0$, and the following condition holds:

$$M^2\delta^2 \int_{-\delta}^0 e^{-2\beta r}\alpha(dr) < 1.$$

We start by proving a prior estimate for the solution of (2.1).

**Theorem 2.1.** *Suppose that hypothesis* (H2.1)$\sim$(H2.4) *hold for some* $p \in [2, \infty)$. *We further assume that there exist processes* $(y^i, z^i, r^i(\cdot)) \in L^p_{\mathcal{P}}(\Omega; L^2_\beta(K)) \times L^p_{\mathcal{P}}(\Omega; L^2_\beta(L_2(\Xi, K))) \times L^p_{\mathcal{P}}(\Omega; L^2_\beta(L_2(\mathcal{E} \times \mathbb{R}_+, \mathcal{B}(\mathcal{E}), \pi; K)))$ *for some* $\beta \in \mathbb{R}, \lambda \in \mathbb{R}, (i = 1, 2)$,

(2.2)
$$\begin{cases} y^i(T) - y^i(t) = -\displaystyle\int_t^T G(s, y^i, y^i_\delta, z^i, r^i(\cdot))ds + \lambda\int_t^T y^i(s)ds - \int_t^T f^i(s)ds \\ \qquad\qquad + \displaystyle\int_t^T z^i(s)dW(s) + \int_t^T\int_{\mathcal{E}} r^i(s, e)\tilde{N}(ds, de),\ t \in [0, T], \\ y^i(s) = \varphi(s),\ s \in [-\delta, 0). \end{cases}$$

*Then for every* $\bar\lambda > \frac{M^2\delta}{2}\int_{-\delta}^0 e^{-2\beta r}\alpha(dr) - \beta - \mu + \frac{L^2_\delta + L^2_z + L^2_r}{2}$, *there exists* $C > 0$ *such that for*$\lambda > \bar\lambda$,

$$\begin{aligned}&(\lambda - \bar\lambda)|y_1(t) - y_2(t)|_{L^p_{\mathcal{P}}(\Omega; L^2_\beta(K))} + (\lambda - \bar\lambda)^{\frac{1}{2}}|z_1(t) - z_2(t)|_{L^p_{\mathcal{P}}(\Omega; L^2_\beta(L_2(\Xi, K)))} \\ &+ (\lambda - \bar\lambda)^{\frac{1}{2}}|r_1(t, \cdot) - r_2(t, \cdot)|_{L^p_{\mathcal{P}}(\Omega; L^2_\beta(L_2(\mathcal{E}\times R_+, \mathcal{B})(\mathcal{E}), \pi; K))} \\ &+ (\lambda - \bar\lambda)^{\frac{1}{2}}\left(\mathbb{E}\sup_{t\geq 0} e^{\beta tp}|y_1(t) - y_2(t)|^p\right)^{\frac{1}{p}} \\ \leq\ & C|f_1(t) - f_2(t)|_{L^p_{\mathcal{P}}(\Omega; L^2_\beta(K))}.\end{aligned}$$

*Proof.* We denote

$$\hat{y}(s) := y^1(s) - y^2(s),\quad \hat{y_\delta}(s) := y^1_\delta(s) - y^2_\delta(s),\quad \hat{z}(s) := z^1(s) - z^2(s)$$
$$\hat{f}(s) := f^1(s) - f^2(s),\quad \hat{r}(s, e) := r^1(s, e) - r^2(s, e),$$

$$\hat{G}(s) := G(s, y^1(s), y^1_\delta(s), z^1(s), r^1(t,\cdot)) - G(s, y^2(s), y^2_\delta(s), z^2(s), r^2(s,\cdot)).$$

Applying the Itô formula to the process $e^{2\beta t}|\hat{y}(t)|^2$, we obtain

$$\begin{aligned}
&e^{2\beta t}|\hat{y}(t)|^2 - e^{2\beta T}|\hat{y}(T)|^2 \\
&\quad + \int_t^T e^{2\beta s}\left[2(\beta+\lambda)|\hat{y}(s)|^2 + |\hat{z}(s)|^2 + \|\hat{r}(s,\cdot)\|^2\right] ds \\
= &\int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{G}(s)\rangle ds - \int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{z}(s)dW(s)\rangle \\
&\quad + \int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{f}(s)\rangle ds - 2\int_t^T e^{2\beta s}\langle \hat{y}(s), \int_{\mathcal{E}} \hat{r}(s,e)\tilde{N}(ds,de)\rangle.
\end{aligned} \tag{2.3}$$

By assumptions (H2.1) and (H2.2), we obtain

$$\begin{aligned}
&2\langle \hat{y}(s), \hat{G}(s)\rangle \\
\le\ & -2\mu|\hat{y}(s)|^2 + 2L_z|\hat{y}(s)||\hat{z}(s)| + 2L_\delta|\hat{y}(s)||\hat{y}_\delta(s)| + 2L_r|\hat{y}(s)|\|\hat{r}(s,e)\| \\
\le\ & \left(-2\mu + \frac{L_\delta^2 + L_z^2 + L_r^2}{\rho}\right)|\hat{y}(s)|^2 + \rho\left(|\hat{y}_\delta(s)|^2 + |\hat{z}(s)|^2 + \|\hat{r}(s,\cdot)\|^2\right),
\end{aligned}$$

where $\rho$ is an arbitrary constant in $(0,1]$, which we will discuss later. Then by (2.3), we obtain

$$\begin{aligned}
&e^{2\beta t}|\hat{y}(t)|^2 - e^{2\beta T}|\hat{y}(T)|^2 \\
&\quad + \int_t^T e^{2\beta s}\left(2\beta + 2\lambda + 2\mu - \frac{L_\delta^2 + L_z^2 + L_r^2}{\rho}\right)|\hat{y}(s)|^2 ds \\
&\quad + \int_t^T e^{2\beta s}\left[(1-\rho)(|\hat{z}(s)|^2 + \|\hat{r}(s,\cdot)\|)^2 - \rho|\hat{y}_\delta(s)|^2\right] ds \\
\le\ & -\int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{z}(s)dW(s)\rangle + \int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{f}(s)\rangle ds \\
&\quad - 2\int_t^T e^{2\beta s}\langle \hat{y}(s), \int_{\mathcal{E}} \hat{r}(s,e)\tilde{N}(ds,de)\rangle.
\end{aligned} \tag{2.4}$$

Noting that

$$\begin{aligned}
\int_t^T e^{2\beta s}\rho\,|\hat{y}_\delta(s)|^2\,ds &= \int_t^T e^{2\beta s}\rho\left|\int_{-\delta}^0 \phi(s,s+r)\hat{y}(s+r)\alpha(dr)\right|^2 ds \\
&\le M^2\delta\rho\int_t^T e^{2\beta s}\int_{-\delta}^0 |\hat{y}(s+r)|^2\,\alpha(dr)ds \\
&\le M^2\delta\rho\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_0^T |\hat{y}(s)|^2 e^{2\beta s}ds,
\end{aligned}$$

we have

(2.5)
$$\begin{aligned}
& e^{2\beta t}|\hat{y}(t)|^2 - e^{2\beta T}|\hat{y}(T)|^2 + \int_t^T e^{2\beta s}[(1-\rho)(|\hat{z}(s)|^2 + \|\hat{r}(s,\cdot)\|)^2]ds \\
& + \int_t^T e^{2\beta s}\left[2\beta+2\lambda+2\mu-\frac{L_\delta^2+L_z^2+L_r^2}{\rho}-M^2\delta\rho\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\right]|\hat{y}(s)|^2 ds \\
& - M^2\delta\rho\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_{t-\delta}^t e^{2\beta s}|\hat{y}(s)|^2 ds \\
\leq & -\int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{z}(s)dW(s)\rangle + \int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{f}(s)\rangle ds \\
& - 2\int_t^T e^{2\beta s}\langle \hat{y}(s), \int_{\mathcal{E}} \hat{r}(s,e)\tilde{N}(ds,de)\rangle.
\end{aligned}$$

For any $\varepsilon \geq 0$, by inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2$, we get

$$\int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{f}(s)\rangle ds \leq \varepsilon\int_t^T e^{2\beta s}|\hat{y}(s)|^2 ds + \frac{1}{\varepsilon}\int_t^T e^{2\beta s}|\hat{f}(s)|^2 ds.$$

Let $\rho = 1$ in the equation (2.5), then

$$\begin{aligned}
& e^{2\beta t}|\hat{y}(t)|^2 - e^{2\beta T}|\hat{y}(T)|^2 + 2\int_0^T e^{2\beta s}\langle \hat{y}(s), \int_{\mathcal{E}} \hat{r}(s,e)\tilde{N}(ds,de)\rangle \\
& + \int_t^T e^{2\beta s}\left[2\beta+2\lambda+2\mu-L_\delta^2-L_z^2-L_r^2-M^2\delta\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)-\varepsilon\right]|\hat{y}(s)|^2 ds \\
& - M^2\delta\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_{t-\delta}^t e^{2\beta s}|\hat{y}(s)|^2 ds \\
\leq & -\int_0^T 2e^{2\beta s}\langle \hat{y}(s), \hat{z}(s)dW(s)\rangle + \frac{1}{\varepsilon}\int_0^T e^{2\beta s}|\hat{f}(s)|^2 ds.
\end{aligned}$$

Let $\varepsilon$ adequate small, we obtain

(2.6)
$$\begin{aligned}
& e^{2\beta t}|\hat{y}(t)|^2 - e^{2\beta T}|\hat{y}(T)|^2 - M^2\delta\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_{t-\delta}^t e^{2\beta s}|\hat{y}(s)|^2 ds \\
\leq & -\int_t^T 2e^{2\beta s}\langle \hat{y}(s), \hat{z}(s)dW(s)\rangle + \frac{1}{\varepsilon}\int_t^T 2e^{2\beta s}|\hat{f}(s)|^2 ds \\
& - 2\int_t^T e^{2\beta s}\langle \hat{y}(s), \int_{\mathcal{E}} \hat{r}(s,e)\tilde{N}(ds,de)\rangle.
\end{aligned}$$

Consider the quadratic variation of stochastic integrals in (2.6), we have

$$\left(\int_t^T e^{4\beta s}|\hat{y}(s)|^2|\hat{z}(s)|^2 ds\right)^{\frac{1}{2}} \leq e^{2|\beta|T}\sup_{\sigma\in[t,T]}|\hat{y}(\sigma)|\left(\int_t^T |\hat{z}(s)|^2 ds\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}e^{2|\beta|T}\sup_{\sigma\in[t,T]}|\hat{y}(\sigma)|^2+\frac{1}{2}e^{2|\beta|T}\int_t^T|\hat{z}(s)|^2ds,$$

and

$$\left(\int_t^T e^{4\beta s}|\hat{y}(s)|^2\|\hat{r}(s,\cdot)\|^2ds\right)^{\frac{1}{2}} \leq e^{2|\beta|T}\sup_{\sigma\in[t,T]}|\hat{y}(\sigma)|\left(\int_t^T\|\hat{r}(s,\cdot)\|^2ds\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2}e^{2|\beta|T}\sup_{\sigma\in[t,T]}|\hat{y}(\sigma)|^2+\frac{1}{2}e^{2|\beta|T}\int_t^T\|\hat{r}(s,\cdot)\|^2ds.$$

By our assumptions and (2.2) it easily get $\mathbb{E}\sup_{\sigma\in[t,T]}e^{2\beta\sigma}|\hat{y}(\sigma)|^2<\infty$, then both right sides of above inequalities are integrable random variables. Thus, stochastic integrals in (2.6) are integrable random variables. Condition both sides of (2.6) to $\mathcal{F}_t$, we get

$$\begin{aligned}e^{2\beta t}|\hat{y}(t)|^2-M^2\delta\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_{t-\delta}^t e^{2\beta s}|\hat{y}(s)|^2ds\\ \leq e^{2\beta T}\mathbb{E}^{\mathcal{F}_t}|\hat{y}(T)|^2+\frac{1}{\varepsilon}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{2\beta s}|\hat{f}(s)|^2ds.\end{aligned}\tag{2.7}$$

Let us recall the assumption $\int_0^\infty e^{2\beta s}|y_i(s)|^2ds<\infty$, then we can find a sequence of $T_n\to\infty$ such that $\mathbb{E}e^{2\beta T_n}|\hat{y}(T_n)|\to 0$. Setting $T=T_n$ and letting $n\to\infty$, we obtain

$$\begin{aligned}&e^{2\beta t}|\hat{y}(t)|^2-M^2\delta\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_{t-\delta}^t e^{2\beta s}|\hat{y}(s)|^2ds\\ \leq\ &\frac{1}{\varepsilon}\mathbb{E}^{\mathcal{F}_t}\int_t^\infty e^{2\beta s}|\hat{f}(s)|^2ds\\ \leq\ &\frac{1}{\varepsilon}\mathbb{E}^{\mathcal{F}_t}\int_0^\infty e^{2\beta s}|\hat{f}(s)|^2ds:=X(t).\end{aligned}$$

Note that

$$\begin{aligned}&-M^2\delta\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_{t-\delta}^t e^{2\beta s}|\hat{y}(s)|^2ds\\ \geq\ &-M^2\delta^2\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\sup_{t\in[0,T]}e^{2\beta t}|\hat{y}(t)|^2.\end{aligned}$$

Therefore we have

$$e^{2\beta t}|\hat{y}(t)|^2-M^2\delta^2\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\sup_{t\in[0,T]}e^{2\beta t}|\hat{y}(t)|^2\leq\frac{1}{\varepsilon}\mathbb{E}^{\mathcal{F}_t}\int_0^\infty e^{2\beta s}|\hat{f}(s)|^2ds.$$

Since $X$ is a martingale, for all $p>2$, by assumption (H2.4) and Doob and Jensen inequalities, there exists a $c_p>0$ such that

$$\mathbb{E}\sup_{t\in[0,T]}e^{\beta tp}|\hat{y}(t)|^p\leq c_p\mathbb{E}\left(X(T)\right)^{\frac{p}{2}}\leq\frac{c_p}{\varepsilon^{\frac{p}{2}}}\mathbb{E}\left(\int_0^\infty e^{2\beta s}|\hat{f}(s)|^2ds\right)^{\frac{p}{2}}.$$

Setting $T \to \infty$, we arrive at

$$\mathbb{E} \sup_{t \geq 0} e^{\beta t p} |\hat{y}(t)|^p \leq \frac{c_p}{\varepsilon^{\frac{p}{2}}} \mathbb{E} \left( \int_0^\infty e^{2\beta s} |\hat{f}(s)|^2 ds \right)^{\frac{p}{2}} < \infty. \tag{2.8}$$

We have

$$\begin{aligned} & \mathbb{E} \left( \int_0^\infty e^{4\beta s} |\hat{y}(s)|^2 |\hat{z}(s)|^2 ds \right)^{\frac{p}{4}} \\ & \leq \mathbb{E} \left[ \sup_{t \geq 0} e^{\frac{\beta t p}{2}} |\hat{y}(t)|^{\frac{p}{2}} \left( \int_0^\infty e^{2\beta s} |\hat{z}(s)|^2 ds \right)^{\frac{p}{4}} \right] \\ & \leq \left\{ \mathbb{E} \sup_{t \geq 0} e^{\beta t p} |\hat{y}(t)|^p \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left( \int_0^\infty e^{2\beta s} |\hat{z}(s)|^2 ds \right)^{\frac{p}{2}} \right\}^{\frac{1}{2}}. \end{aligned}$$

By (2.8), the right side of the above inequality is finite. It follows that the limit of stochastic integral $\int_t^T 2e^{2\beta s} \langle \hat{y}(s), \hat{z}(s) dW(s) \rangle$ for $T \to \infty$ exists in $L^{\frac{p}{2}}(\Omega, \mathbb{R})$ and for some $c_p$,

$$\begin{aligned} & \mathbb{E} \left| \int_0^\infty 2e^{2\beta s} \langle \hat{y}(s), \hat{z}(s) dW(s) \rangle \right|^{\frac{p}{2}} \\ & \leq c_p \left\{ \mathbb{E} \sup_{t \geq 0} e^{\beta t p} |\hat{y}(t)|^p \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left( \int_0^\infty e^{2\beta s} |\hat{z}(s)|^2 ds \right)^{\frac{p}{2}} \right\}^{\frac{1}{2}}. \end{aligned} \tag{2.9}$$

Similarly, we have

$$\begin{aligned} & \mathbb{E} \left( \int_0^\infty e^{4\beta s} |\hat{y}(s)|^2 \|\hat{r}(s, \cdot)\|^2 ds \right)^{\frac{p}{4}} \\ & \leq \mathbb{E} \left[ \sup_{t \geq 0} e^{\frac{\beta t p}{2}} |\hat{y}(t)|^{\frac{p}{2}} \left( \int_0^\infty e^{2\beta s} \|\hat{r}(s, \cdot)\|^2 ds \right)^{\frac{p}{4}} \right] \\ & \leq \left\{ \mathbb{E} \sup_{t \geq 0} e^{\beta t p} |\hat{y}(t)|^p \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left( \int_0^\infty e^{2\beta s} \|\hat{r}(s, \cdot)\|^2 ds \right)^{\frac{p}{2}} \right\}^{\frac{1}{2}}, \end{aligned}$$

and there exists a $\bar{c}_p$, such that

$$\begin{aligned} & \mathbb{E} \left| \int_0^\infty 2e^{2\beta s} \langle \hat{y}(s), \int_{\mathcal{E}} \hat{r}(s, e) \tilde{N}(ds, de) \rangle \right|^{\frac{p}{2}} \\ & \leq c_p \left\{ \mathbb{E} \sup_{t \geq 0} e^{\beta t p} |\hat{y}(t)|^p \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left( \int_0^\infty e^{2\beta s} \|\hat{r}(s, \cdot)\|^2 ds \right)^{\frac{p}{2}} \right\}^{\frac{1}{2}} < \infty. \end{aligned} \tag{2.10}$$

Choose a sequence $T_n \to \infty$ such that $e^{2\beta T_n}\mathbb{E}|\hat{y}(T_n)|^2 \to 0$. Setting $T = T_n$ in (2.5) and letting $n \to \infty$ we obtain

(2.11)

$$\begin{aligned}
&e^{2\beta t}|\hat{y}(t)|^2 + \int_t^\infty e^{2\beta s}[(1-\rho)(|\hat{z}(s)|^2 + \|\hat{r}(s,\cdot)\|)^2]ds \\
&+ \int_t^\infty e^{2\beta s}\left[2\beta+2\lambda+2\mu-\frac{L_\delta^2+L_z^2+L_r^2}{\rho}-M^2\delta\rho\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\right]|\hat{y}(s)|^2 ds \\
&- M^2\delta\rho\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_{t-\delta}^t e^{2\beta s}|\hat{y}(s)|^2 ds \\
\leq\ & -\int_t^\infty 2e^{2\beta s}\langle\hat{y}(s), \hat{z}(s)dW(s)\rangle \\
&+ \int_t^\infty 2e^{2\beta s}\langle\hat{y}(s), \hat{f}(s)\rangle ds - 2\int_t^\infty e^{2\beta s}\langle\hat{y}(s), \int_{\mathcal{E}}\hat{r}(s,e)\tilde{N}(ds,de)\rangle.
\end{aligned}$$

Now, for simplicity, we denote

$$\begin{aligned}
&|\hat{z}|_{L^2_\beta(L_2(\Xi,K))} = \left(\int_0^\infty e^{2\beta s}|\hat{z}(s)|^2_{L_2(\Xi,K)}ds\right)^{\frac{1}{2}}, \\
&|\hat{y}|_{L^2_\beta(K)} = \left(\int_0^\infty e^{2\beta s}|\hat{y}(s)|^2_K ds\right)^{\frac{1}{2}}, \\
&|\hat{r}(\cdot)|_{L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K))} = \left(\int_0^\infty e^{2\beta s}\|\hat{r}(s,\cdot)\|^2 ds\right)^{\frac{1}{2}}, \\
&|\hat{f}|_{L^2_\beta(K)} = \left(\int_0^\infty e^{2\beta s}|\hat{f}(s)|^2_K ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Conditioning both sides of (2.11) to $\mathcal{F}_t$ and choosing $\rho < 1$ so close to 1 such that $2\beta + 2\lambda + 2\mu - \frac{L_\delta^2+L_z^2+L_r^2}{\rho} - M^2\delta\rho\int_{-\delta}^0 e^{-2\beta r}\alpha(dr) > 0$, we have

$$\begin{aligned}
(2.12)\qquad & e^{2\beta t}|\hat{y}(t)|^2 - M^2\delta\rho\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)\int_{t-\delta}^t e^{2\beta s}|\hat{y}(s)|^2 ds \\
&\leq \mathbb{E}^{\mathcal{F}_t}\int_t^\infty 2e^{2\beta s}\langle\hat{y}(s), \hat{f}(s)\rangle ds \\
&\leq \mathbb{E}^{\mathcal{F}_t}\int_0^\infty 2e^{2\beta s}\langle\hat{y}(s), \hat{f}(s)\rangle ds \\
&\leq \mathbb{E}^{\mathcal{F}_t}\left(|\hat{y}|^{\frac{p}{2}}_{L^2_\beta(K)}|\hat{f}|^{\frac{p}{2}}_{L^2_\beta(K)}\right),
\end{aligned}$$

and by Doob and Jensen inequalities, there exists a $c_p > 0$ such that

$$(2.13)\qquad \mathbb{E}\sup_{t\geq 0} e^{\beta tp}|\hat{y}(t)|^p \leq \mathbb{E}\left(|\hat{y}|^{\frac{p}{2}}_{L^2_\beta(K)}|\hat{f}|^{\frac{p}{2}}_{L^2_\beta(K)}\right).$$

It also follows from (2.11) that

$$\begin{aligned}&2(\lambda-\bar{\lambda})|\hat{y}|^2_{L^2_\beta(K)}+c\left(|\hat{z}|^2_{L^2_\beta(L_2(\Xi,K))}+|\hat{r}(\cdot)|^2_{L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K))}\right)\\ &\le|\hat{y}|_{L^2_\beta(K)}|\hat{f}|_{L^2_\beta(K)}+\left|\int_0^\infty 2e^{2\beta s}\langle\hat{y}(s),\hat{z}(s)dW(s)\rangle\right|\\ &\quad+\left|\int_0^\infty 2e^{2\beta s}\langle\hat{y}(s),\int_{\mathcal{E}}\hat{r}(s,e)\tilde{N}(ds,de)\rangle\right|.\end{aligned}$$

Raising to the power $\frac{p}{2}$, taking expectation, and recalling (2.9), (2.10) and (2.13), we obtain, for suitable constants $c_i$ and any $\varepsilon$,

$$\begin{aligned}&(\lambda-\bar{\lambda})^{\frac{p}{2}}\mathbb{E}|\hat{y}|^p_{L^2_\beta(K)}+c_1\mathbb{E}|\hat{z}|^p_{L^2_\beta(L_2(\Xi,K))}+c_2\mathbb{E}|\hat{r}(\cdot)|^p_{L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K))}\\ &\le c_3\mathbb{E}\left(|\hat{y}|^{\frac{p}{2}}_{L^2_\beta(K)}|\hat{f}|^{\frac{p}{2}}_{L^2_\beta(K)}\right)+c_4\left\{\mathbb{E}|\hat{z}|^p_{L^2_\beta(L_2(\Xi,K))}\right\}^{\frac{1}{2}}\left\{\mathbb{E}\left(|\hat{y}|^{\frac{p}{2}}_{L^2_\beta(K)}|\hat{f}|^{\frac{p}{2}}_{L^2_\beta(K)}\right)\right\}^{\frac{1}{2}}\\ &\quad+c_5\left\{\mathbb{E}|\hat{r}(\cdot)|^p_{L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K))}\right\}^{\frac{1}{2}}\left\{\mathbb{E}\left(|\hat{y}|^{\frac{p}{2}}_{L^2_\beta(K)}|\hat{f}|^{\frac{p}{2}}_{L^2_\beta(K)}\right)\right\}^{\frac{1}{2}}\\ &\le c_3\mathbb{E}\left(|\hat{y}|^{\frac{p}{2}}_{L^2_\beta(K)}|\hat{f}|^{\frac{p}{2}}_{L^2_\beta(K)}\right)+\varepsilon\mathbb{E}|\hat{z}(t)|^p_{L^2_\beta(L_2(\Xi,K))}+\left(\frac{c_4^2+c_5^2}{4\varepsilon}\right)\mathbb{E}\left(|\hat{y}|^{\frac{p}{2}}_{L^2_\beta(K)}|\hat{f}|^{\frac{p}{2}}_{L^2_\beta(K)}\right)\\ &\quad+\varepsilon\mathbb{E}|\hat{r}(\cdot)|^p_{L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K))}.\end{aligned}$$

Choosing sufficiently small $\varepsilon$ and using Cauchy-Schwarz inequality, we obtain, for some $c$,

$$\begin{aligned}&(\lambda-\bar{\lambda})^{\frac{p}{2}}\mathbb{E}|\hat{y}|^p_{L^2_\beta(K)}+c_1\mathbb{E}|\hat{z}|^p_{L^2_\beta(L_2(\Xi,K))}+c_2\mathbb{E}|\hat{r}(\cdot)|^p_{L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K))}\\ &\le c\left\{\mathbb{E}|\hat{y}|^p_{L^2_\beta(K)}\right\}^{\frac{1}{2}}\left\{\mathbb{E}|\hat{f}|^p_{L^2_\beta(K)}\right\}^{\frac{1}{2}}.\end{aligned}$$

Taking into account (2.13) once more, we reach a conclusion

$$\begin{aligned}&(\lambda-\bar{\lambda})|y^1-y^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))}+(\lambda-\bar{\lambda})^{\frac{1}{2}}|z^1-z^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\Xi,K)))}\\ &\quad+(\lambda-\bar{\lambda})^{\frac{1}{2}}|r^1(\cdot)-r^2(\cdot)|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K)))}\\ &\quad+(\lambda-\bar{\lambda})^{\frac{1}{2}}\left(\mathbb{E}\sup_{t\ge0}e^{\beta tp}|y^1(t)-y^2(t)|^p\right)^{\frac{1}{p}}\\ &\le C|f^1-f^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))}.\end{aligned}$$

□

Now, we prove the existence and uniqueness of the solution of equation (2.1) under assumptions (H2.1)∼(H2.4).

**Theorem 2.2.** *Suppose that hypotheses* (H2.1)∼(H2.4) *hold for some* $p\in[2,\infty)$ *and assume that there exist processes* $f\in L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))$ *for some* $\beta\in\mathbb{R}$. *Then for* $\lambda>\frac{M^2\delta}{2}\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)-\beta-\mu+\frac{L_\delta^2+L_z^2+L_r^2}{2}$, *the equation* (2.1) *has a unique solution* $(y(t),z(t),r(t,\cdot))$ *such that*

$$y(t)\in L^p_{\mathcal{P}}(\Omega;L^2_\beta(K)),\ z(t)\in L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\Xi,K))),$$

$$r(t,\cdot)\in L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\mathcal{E}\times\mathbb{R}_+,\mathcal{B}(\mathcal{E}),\pi;K))).$$

*Proof.* Consider the space

$$\mathcal{D}:=L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))\times L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\Xi,K)))\times L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\mathcal{E}\times\mathbb{R}_+,\mathcal{B}(\mathcal{E}),\pi;K))).$$

Let $\mathcal{D}$ endow with the norm

$$|y|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))}+|z|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\Xi,K)))}+|r(\cdot)|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\mathcal{E}\times\mathbb{R}_+,\mathcal{B}(\mathcal{E}),\pi;K)))}.$$

For every $\lambda$, we define a mapping $\Gamma:\mathcal{D}\to\mathcal{D}$, setting $(y(t),z(t),r(t,\cdot))=\Gamma(O(t),P(t),Q(t,\cdot))$ if $(y(t),z(t),r(t,\cdot))$ is the solution of the equation, $\mathbb{P}$-a.s.,

$$\begin{cases} y(T)-y(t)= -\displaystyle\int_t^T G(s,O(s),\int_{s-\delta}^s\phi(s,t)O(t)\alpha(dt),P(s),Q(s,\cdot))ds \\ \qquad\qquad +\lambda\displaystyle\int_t^T y(s)ds+\int_t^T z(s)dW(s)+\int_t^T\int_{\mathcal{E}} r(s,e)\tilde{N}(ds,de) \\ \qquad\qquad -\displaystyle\int_t^T f(s)ds,\ t\in[0,T], \\ y(s)=\varphi(s), s\in[-\delta,0). \end{cases}$$

By Theorem 2.1, for $\lambda>\frac{M^2\delta}{2}\int_{-\delta}^0 e^{-2\beta r}\alpha(dr)-\beta-\mu+\frac{L_\delta^2+L_z^2+L_r^2}{2}$,

$$\begin{aligned}&(\lambda-\bar\lambda)|y^1-y^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))}+(\lambda-\bar\lambda)^{\frac12}|z^1-z^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\Xi,K)))}\\ &+(\lambda-\bar\lambda)^{\frac12}|r^1(\cdot)-r^2(\cdot)|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K)))}\\ \le{}& C\left\{\mathbb{E}\Big(\int_0^\infty e^{2\beta s}|G(s,O^1(s),O^1_\delta,P^1(s),Q^1(s,\cdot))\right.\\ &\left.\qquad -G(s,O^2(s),O^2_\delta,P^2(s),Q^2(s,\cdot))|^2ds\Big)\right\}^{\frac1p}.\end{aligned}$$

Here $O^i_\delta=\int_{s-\delta}^s\phi(s,t)O^i(t)\alpha(dt), i=1,2$. By the Lipschitz condition on $G$, we have, for some constant $C>0$,

$$\begin{aligned}&(\lambda-\bar\lambda)|y^1-y^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))}+(\lambda-\bar\lambda)^{\frac12}|z^1-z^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\Xi,K)))}\\ &+(\lambda-\bar\lambda)^{\frac12}|r^1(\cdot)-r^2(\cdot)|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K)))}\\ \le{}& c(\lambda-\bar\lambda)|O^1-O^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))}+(\lambda-\bar\lambda)^{\frac12}|P^1-P^2|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\Xi,K)))}\\ &+(\lambda-\bar\lambda)^{\frac12}|Q^1(\cdot)-Q^2(\cdot)|_{L^p_{\mathcal{P}}(\Omega;L^2_\beta(L_2(\mathcal{E}\times R_+,\mathcal{B}(\mathcal{E}),\pi;K)))}.\end{aligned}$$

This shows that $\Gamma$ is a contraction in $\mathcal{D}$ for all $\lambda$ sufficiently large. Its unique fixed point is the required solution.

Next, we define $Q$ as the set of those real numbers $\lambda>\bar\lambda$ such that for every $f\in L^p_{\mathcal{P}}(\Omega;L^2_\beta(K))$, there exists a unique solution $(y(t),z(t),r(t,\cdot))\in\mathcal{D}$ corresponding to $\lambda$ and $f$. Then, we can immediately get $Q$ coincides with $(\bar\lambda,\infty)$ from Theory 3.7 of [5]. $\square$

## 3. Optimal control problem for backward

In this section, we aim at deriving maximal principal for an infinite horizon optimal control problem described by the following BSDDEs:

(3.1)
$$\begin{cases} y(T)-y(t)= -\int_t^T G(s,y(s),\int_{s-\delta}^s \phi(s,t)y(t)\alpha(dt),z(s),r(s,\cdot),v(t))ds \\ \qquad\qquad +\int_t^T z(s)dW(s)+\int_t^T\int_{\mathcal{E}} r(s,e)\tilde{N}(ds,de)-\int_t^T f(s)ds,\ t\in[0,T], \\ y(s)=\varphi(s),\ s\in[-\delta,0). \end{cases}$$

Here, let $U$ be a nonempty convex subset of $\mathbb{R}^k$, we denote by $\mathcal{U}$ the set of all $\mathcal{F}_t$-adapted admissible control processes.

The objective is to maximize the following performance functional over $\mathcal{U}$:

$$J(v(\cdot))=\int_0^\infty l(s,y(s),\int_{s-\delta}^s \phi(s,t)y(t)\alpha(dt),z(s),r(s,\cdot),v(s))ds+\gamma(y(0)),$$

where $l:\Omega\times[0,\infty)\times K\times L_2(\Xi,K)\times L^2(\mathcal{E}\times\mathbb{R}_+,\mathcal{B}(\mathcal{E}),\pi;K)\times\mathcal{U}\to K$ is adapted with respect to $\{\mathcal{F}\}_{t\geq 0}$ and satisfies for each $v(\cdot)\in\mathcal{U}$,

$$\mathbb{E}\left[\int_0^\infty |l(s,y(s),\int_{s-\delta}^s \phi(s,t)y(t)\alpha(dt),z(s),r(s,\cdot),v(s))|ds\right]<\infty.$$

Next, we give the following assumptions:

**(H3.1)** $G$ is continuously differentiable in $(y,y_\delta,z,r,v)$. The partial derivatives $G_y,G_{y_\delta},G_z,G_r,G_v$ and $G$ with respect to $(y,y_\delta,z,r,v)$ are uniformly bounded.

**(H3.2)** $l$ and $\gamma$ are differentiable with respect to $(y,y_\delta,z,r,v)$ and $y$ respectively for each $v(\cdot)\in\mathcal{U}$, and all the derivatives are bounded.

*Remark* 3.1. If $v(\cdot)$ is admissible control and our above assumptions hold, then by Theorem 2.2, (3.1) has a unique solution $(y^v(\cdot),z^v(\cdot),r^v(\cdot,\cdot))\in\mathcal{D}$.

Now, let $(u(\cdot),y^u(\cdot),z^u(\cdot),r^u(\cdot,\cdot))$ be an optimal solution of our problem. Take an arbitrary $v(\cdot)$ in $\mathcal{U}$, then, for each $0\leq\rho\leq 1$, let $v^\rho(\cdot)=u(\cdot)+\rho(v(\cdot)-u(\cdot))\in\mathcal{U}$. Let $(y^\rho(\cdot),z^\rho(\cdot),r^\rho(\cdot,\cdot))$ be the state processes of system (3.1) with $v^\rho(\cdot)$.

To derive a first-order necessary condition in terms of small $\rho$, we let $(\tilde{y}(\cdot)$, $\tilde{z}(\cdot)$, $\tilde{r}(\cdot,\cdot))$ be the solution of the following BSDDEs which called variational equations:

(3.2)
$$\begin{cases} d\tilde{y}(s)=\ -[G_y^u(s)\tilde{y}(s)+G_{y_\delta}^u(s)\tilde{y_\delta}(s)+G_z^u(s)\tilde{z}(s)+G_r^u(s,\cdot)\tilde{r}(s,\cdot) \\ \qquad\qquad +G_v^u(s)\tilde{v}(s)]ds+\tilde{z}(s)dW(s)+\int_{\mathcal{E}}\tilde{r}(s,\cdot)\tilde{N}(ds,de),\ s\in[0,T], \\ \tilde{y}(s)=0,\quad s\in[-\delta,0), \end{cases}$$

where $G_k^u(s)=G_k(s,y^u(s),y_\delta^u(s),z^u(s),r^u(s,\cdot),u(s)),k=y,y_\delta,z,r(\cdot),v$, and $\tilde{v}(s)=v^\rho(s)-u(s)$.

Set

$$\bar{y}^\rho(s)=\rho^{-1}[y^\rho(s)-y^u(s)]-\tilde{y}(s),\ \bar{z}^\rho(s)=\rho^{-1}[z^\rho(s)-z^u(s)]-\tilde{z}(s).$$

We assume:

**(H3.3)**
$$\begin{aligned}\mathbb{E}\Big[\int_0^\infty &\big(|\tilde{y}(s)l_y^u(s)|+|l_{y_\delta}^u(s)\tilde{y}_\delta(s)|+|\tilde{z}(s)l_z^u(s)|\\ &+|\tilde{r}(s)l_r^u(s,e)|+|\tilde{v}(s)l_v^u(s)|\big)\,ds\Big]<\infty.\end{aligned}$$

**Lemma 3.1.** *Assume* (H3.1)$\sim$(H3.3) *hold. Then we have*

$$\lim_{\rho\to 0}\mathbb{E}[\sup_{0\le s\le T}|\bar{y}^\rho(s)|^2ds]=0,\ \lim_{\rho\to 0}\mathbb{E}[\int_0^T|\bar{z}^\rho(s)|^2ds]=0.$$

Since $u(\cdot)$ is an optimal control of our problem, the following inequality holds for any $v(\cdot)\in\mathcal{U}$ and the corresponding $v^\rho(\cdot)$:

$$\rho^{-1}[J(v^\rho(\cdot))-J(u(\cdot))]\le 0.$$

According to the above lemma, we have following result.

**Lemma 3.2.** *If* (H3.1)$\sim$(H3.3) *hold, for all* $v(\cdot)\in\mathcal{U}$*, we have*

$$\begin{aligned}\mathbb{E}\gamma_y(y(0))\tilde{y}(0)+\mathbb{E}\Big[\int_0^\infty &\big(\tilde{y}(s)l_y^u(s)+\tilde{y}_\delta(s)l_{y_\delta}^u(s)+\tilde{z}(s)l_z^u(s)\\ &+\tilde{r}(s)l_r^u(s,e)+\tilde{v}(s)l_v^u(s)\big)\,ds\Big]\le 0.\end{aligned}$$

*Remark* 3.2. Using Lemma 2.1, Lebesgue dominated convergence theorem and Taylor expansion, it is very simple to prove Lemmas 3.1 and 3.2.

Next, we introduce the dual equation of the variational equation (3.2) as follows:

(3.3)
$$\begin{cases} dp(s)= \left\{\mathbb{E}^{\mathcal{F}_s}\left[\int_s^{s+\delta}(G_{y_\delta}^u(t)p(t)-l_{y_\delta}^u(t))\phi(t,s)\chi_{[0,T]}(t)dt\right]\dfrac{\alpha(ds)}{ds}\right\}ds\\ \qquad\quad +[-l_y^u(s)+G_y^u(s)p(s)]ds+[-l_z^u(s)+G_z^u(s)p(s)]dW(s)\\ \qquad\quad +\displaystyle\int_{\mathcal{E}}[-l_r^u(s,e)+G_r^u(s,e)p(s)]\tilde{N}(ds,de),s\in[0,T],\\ p(0)=-\gamma_y(y^u(0)).\end{cases}$$

The Hamiltonian function $H:[0,\infty)$ is defined by

$$H(s,y,y_\delta,z,r(\cdot),v,p):=l(s,y,y_\delta,z,r(\cdot),v)-\langle G(s,y,y_\delta,z,r(\cdot),v),p\rangle.$$

The associated adjoint equation as

(3.4)

$$\begin{cases} dp(s) = \Big\{ - H_y(s, y(s), y_\delta(s), z(s), r(s,\cdot), u(s), p(s)) \\ \qquad - \mathbb{E}^{\mathcal{F}_s}\Big[\int_s^{s+\delta} H_{y_\delta}(t, y(t), y_\delta(t), z(t), r(t,\cdot), u(t), p(t))\phi(t,s)\chi_{[0,T]}(t)dt\Big] \\ \qquad\qquad \dfrac{\alpha(ds)}{ds}\Big\}ds \\ \qquad - H_z(s, y(s), y_\delta(s), z(s), r(s,\cdot), u(s), p(s))dW(s) \\ \qquad - \displaystyle\int_\varepsilon H_r(s, y(s), y_\delta(s), z(s), r(s,\cdot), u(s), p(s))\tilde{N}(ds, de), s \in [0,T], \\ p(0) = -\gamma_y(y(0)), \end{cases}$$

where $\frac{\alpha(ds)}{ds}$ is the Radon-Nikodym derivative.

*Remark* 3.3. For a given admissible control $v(\cdot)$, (3.3) and (3.4) are ASDEs and admit a unique solution under conditions (H3.1) and (H3.2) by [2].

Now we can give the first main result of this paper.

**Theorem 3.1** (Necessary conditions of optimality)**.** *Let* (H2.1)$\sim$(H2.4) *and* (H3.1)$\sim$(H3.3) *hold. Suppose that $u(\cdot)$ is an optimal control of our problem and $(y^u(\cdot), z^u(\cdot), r^u(\cdot,\cdot))$ is the corresponding optimal state trajectory. We also assume the following holds*:

**(H3.4)** *The following growth condition holds*:

$$\mathbb{E}\left[\int_0^T p^2(t)(\tilde{y}^2(t) + G_v^u(s)\tilde{v}(s))ds\right] < \infty.$$

*Then we have*

$$\langle H_v(s, y(s), y_\delta(s), z(s), r(s,\cdot), u(s), p(s)), v - u(s)\rangle \le 0 \tag{3.5}$$

*for any $v \in U$, a.e., where $p(\cdot)$ is the solution of the adjoint equation* (3.4).

*Proof.* Applying Itô formula to $\langle p(s), \widetilde{y}(s)\rangle$, we have

$$\begin{aligned} &\mathbb{E}p(T)\tilde{y}(T) - p(0)\tilde{y}(0) \\ = &\mathbb{E}\int_0^T p(s)[-G_{y_\delta}^u(s)\tilde{y}_\delta(s) - G_v^u(s)\tilde{v}(s)]ds - \mathbb{E}\int_0^T \tilde{y}(s)l_y^u(s)ds \\ &+ \mathbb{E}\int_0^T \left\langle \tilde{y}(s), \mathbb{E}^{\mathcal{F}_s}\left[\int_s^{s+\delta}(G_{y_\delta}^u(t)p(t) - l_{y_\delta}^u(t))\phi(t,s)\chi_{[0,T]}(t)dt\right]\frac{\alpha(ds)}{ds}\right\rangle ds \\ &- \mathbb{E}\int_0^T \tilde{z}(s)l_z^u(s)ds - \mathbb{E}\int_0^T \tilde{r}(s)l_r^u(s,e)ds. \end{aligned}$$

Since the following results hold:

$$
\begin{aligned}
&\mathbb{E}\int_0^T \left\langle p(s), G^u_{y_\delta}(s)\tilde{y_\delta}(s)\right\rangle ds\\
=\ &\mathbb{E}\int_0^T \left\langle \mathbb{E}^{\mathcal{F}_r}\left[\int_r^{r+\delta} G^u_{y_\delta}(s)p(s)\phi(s,r)\chi_{[0,T]}(s)ds\right], \tilde{y}(r)\right\rangle \alpha(dr)\\
=\ &\mathbb{E}\int_0^T \left\langle \mathbb{E}^{\mathcal{F}_s}\left[\int_s^{s+\delta} G^u_{y_\delta}(t)p(t)\phi(t,s)\chi_{[0,T]}(t)dt\right]\frac{\alpha(ds)}{ds}, \tilde{y}(s)\right\rangle ds,
\end{aligned}
$$

and

$$
\mathbb{E}\int_0^T l^u_{y_\delta}(s)\tilde{y}_\delta(s)ds = \mathbb{E}\int_0^T \langle \mathbb{E}^{\mathcal{F}_s}\left[\int_s^{s+\delta} l^u_{y_\delta}(t)\phi(t,s)\chi_{[0,T]}(t)dt\right]\frac{\alpha(ds)}{ds}, \tilde{y}(s)\rangle ds,
$$

let $T\to\infty$, by assumption (H3.4), we have

$$
\begin{aligned}
&\gamma_y(y(0))\tilde{y}(0)\\
=\ &-\mathbb{E}\int_0^\infty [p(s)G^u_v(s)\tilde{v}(s)]ds - \mathbb{E}\int_0^\infty \tilde{y}(s)l^u_y(s)ds - \mathbb{E}\int_0^\infty \tilde{y}_\delta(s)l^u_{y_\delta}(s)ds\\
&-\mathbb{E}\int_0^\infty \tilde{z}(s)l^u_z(s)ds - \mathbb{E}\int_0^\infty \tilde{r}(s)l^u_r(s,e)ds.
\end{aligned}
$$

By Lemma 3.2, we have

$$
\mathbb{E}\int_0^\infty [\tilde{v}(s)l^u_v(s) - p(s)G^u_v(s)\tilde{v}(s)]ds \le 0,
$$

i.e.,

$$
\mathbb{E}\left[\int_0^\infty \langle H_v(s,y(s),y_\delta(s),z(s),r(s,\cdot),u(s),p(s)), \tilde{v}(s)\rangle ds\right] \le 0.
$$

Consulting the proof of Theorem 1.5 in [14], for $v\in U$, we get (3.5) to be true a.s. The left side of (3.5) is equivalent to

$$
\begin{aligned}
&\langle H_v(s,y(s),y_\delta(s),z(s),r(s,\cdot),u(s),p(s)), u(s)\rangle\\
=\ &\max_{v\in U}\langle H_v(s,y(s),y_\delta(s),r(s,\cdot),u(s),p(s)), v\rangle.
\end{aligned}
$$

□

**Theorem 3.2** (Sufficient conditions of optimality)**.** *Let* (H2.1)∼(H2.4) *and* (H3.1)∼(H3.4) *hold. Suppose that for $u(\cdot)\in\mathcal{U}$, $(y(\cdot),z(\cdot),r(\cdot,\cdot))$ is the corresponding optimal state trajectory and $p(\cdot)$ is the corresponding solution of the adjoint equation* (3.4)*. If* (3.5) *holds, $H(s,y(s),y_\delta(s),z(s),r(s,\cdot),u(s),p(s))$ is a concave function of $(y,y_\delta,z,r(\cdot),v)$ and $\gamma$ is concave in $y$, then $u(\cdot)$ is an optimal control for our problem.*

*Proof.* Choose a $v(\cdot)\in\mathcal{U}$ and let $(y(\cdot),z(\cdot),r(\cdot,\cdot))$ be the corresponding solution of (3.1). Let

$$
I_1 := \mathbb{E}\Big[\int_0^\infty \{l(s,y(s),y_\delta(s),z(s),r(s,\cdot),u(s))
$$

$$- l(s, y^v(s), y^v_\delta(s), z(s), r(s,\cdot), v(s))\}ds\Big],$$

$$I_2 := [\gamma_y(y(0)) - \gamma_y(y^v(0))].$$

To simplify, we use notations

$$\Theta(s) = (y(s), y_\delta(s), z(s), r(s,\cdot)), \Theta^v(s) = (y^v(s), y^v_\delta(s), z(s), r(s,\cdot)).$$

We want to prove that

$$J(u(\cdot)) - J(v(\cdot)) = I_1 + I_2 \geq 0. \tag{3.6}$$

Since $\gamma$ is concave on $y$, $I_2 \geq \gamma_y(y(0))^\top(y(0)-y^v(0)) = -(p(0))^\top(y(0)-y^v(0))$. Applying Itô formula to $\langle p(\cdot), y(\cdot) - y^v(\cdot)\rangle$ and taking expectation, we get

$$\begin{aligned}
I_2 \geq & -(p(0))^\top(y(0) - y^v(0)) \\
= & -\mathbb{E}\int_0^\infty \langle p(s), G(s,\Theta(s),u(s)) - G(s,\Theta^v(s),v(s))\rangle ds \\
& -\mathbb{E}\int_0^\infty \Big\langle y(s) - y^v(s), \Big\{H_y(s,\Theta(s),u(s),p(s)) \\
& + \mathbb{E}^{\mathcal{F}_s}\left[\int_s^{s+\delta} H_{y_\delta}(t,\Theta(s),u(t),p(t))\phi(t,s)\chi_{[0,T]}(t)dt\right]\frac{\alpha(ds)}{ds}\Big\}\Big\rangle ds \\
& -\mathbb{E}\int_0^\infty \langle z(s) - z^v(s), H_z(s,\Theta(s),u(s),p(s))\rangle \\
& -\mathbb{E}\int_0^\infty \langle r(s,\cdot) - r^v(s,\cdot), H_r(s,\Theta(s),u(s),p(s))\rangle.
\end{aligned} \tag{3.7}$$

On the other hand, $I_1$ can be rewritten as

$$\begin{aligned}
I_1 = & \mathbb{E}\int_0^\infty [H(s,\Theta(s),u(s),p(s)) - H(s,\Theta^v(s),v(s),p(s))] \\
& + \mathbb{E}\int_0^\infty \langle p(s), G(s,\Theta(s),u(s)) - G(s,\Theta^v(s),v(s))\rangle ds.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\begin{aligned}
& J(u(\cdot)) - J(v(\cdot)) \\
= & I_1 + I_2 \\
\geq & \mathbb{E}\int_0^\infty [H(s,\Theta(s),u(s),p(s)) - H(s,\Theta^v(s),v(s),p(s)) \\
& -\mathbb{E}\int_0^\infty \Big\langle y(s) - y^v(s), \Big\{H_y(s,\Theta(s),u(s),p(s)) \\
& + \mathbb{E}^{\mathcal{F}_s}\left[\int_s^{s+\delta} H_{y_\delta}(t,\Theta(s),u(t),p(t))\phi(t,s)\chi_{[0,T]}(t)dt\right]\frac{\alpha(ds)}{ds}\Big\}\Big\rangle ds \\
& -\mathbb{E}\int_0^\infty \langle z(s) - z^v(s), H_z(s,\Theta(s),u(s),p(s))\rangle ds
\end{aligned} \tag{3.9}$$

$$- \mathbb{E}\int_0^\infty \langle r(s,\cdot) - r^v(s,\cdot), H_r(s,\Theta(s),u(s),p(s))\rangle ds.$$

Noting that

$$\begin{aligned}
(3.10)\quad & \mathbb{E}\int_0^\infty \Big\langle y(s) - y^v(s), \\
&\qquad \mathbb{E}^{\mathcal{F}_s}\left[\int_s^{s+\delta} H_{y_\delta}(t,\Theta(s),u(t),p(t))\phi(t,s)\chi_{[0,T]}(t)dt\right]\frac{\alpha(ds)}{ds}\Big\rangle ds \\
&= \mathbb{E}\int_0^\infty \langle y_\delta(s) - y^v_\delta(s), H_{y_\delta}(s,\Theta(s),u(s),p(s))\rangle\, ds,
\end{aligned}$$

and $\Theta(s) \to H(s,\Theta(s),u(s),p(s))$ is a concave, we have

$$(3.11)\qquad J(u(\cdot)) - J(v(\cdot)) \geq \mathbb{E}\big[\int_0^\infty \langle H_v(s,\Theta(s),u(s),p(s)), \tilde{v}(s)\rangle ds\big] = 0.$$

It follows that $u(\cdot)$ is an optimal control. □

## 4. Linear-quadratic problem

In this section, we will study a linear-quadratic problem of BSDDEs. Applying the theoretical result in Section 3, we aim to give optimal control. We consider the following linear-quadratic BSDDEs:

$$(4.1)\quad \begin{cases} y(T) - y(t) = -\displaystyle\int_t^T [A(s)y(s) + B(s)y(s-\delta) + C(s)z(s) \\ \qquad\qquad + D(s)r(s,\cdot) + G(s)v(s)]ds + \displaystyle\int_t^T z(s)dW(s) \\ \qquad + \displaystyle\int_t^T\int_{\mathcal{E}} r(s,e)\tilde{N}(ds,de),\ t\in[0,T], \\ y(s) = \varphi(s),\ s\in[-\delta,0), \end{cases}$$

where $A(s), B(s), C(s), D(s)$ and $G(s)$ are deterministic functions.

The object is to maximize the following functional over $\mathcal{U}$,

$$J(v(\cdot)) = -\frac{1}{2}\mathbb{E}\left[\int_0^\infty R(s)v^2(s)ds\right] + Ky(0)$$

for some constant $K$ and nonnegative function $R(s)$ defined on $[0,\infty]$. By Theorem 3.1, the Hamiltonian function of our optimization problem becomes

$$\begin{aligned} H(t,y,y_\delta,z,r(\cdot),v,p) := &-\frac{1}{2}R(t)v^2(t) - [A(t)y(t) + B(t)y(t-\delta) \\ &+ C(t)z(t) + D(t)r(t,\cdot) + G(t)v(t)]p(t), \end{aligned}$$

and the adjoint equation becomes

$$\begin{cases} dp(t) = \{A(t)p(t) + B(t)\mathbb{E}^{\mathcal{F}_t}[p(t+\delta)]\}dt \\ \qquad\qquad + C(t)p(t)dW(t) + \int_{\mathcal{E}} D(t)p(t)\tilde{N}(dt,de), \ t \in [0,T], \\ p(0) = K, p(t) = 0, \ T \le t \le T+\delta. \end{cases} \tag{4.2}$$

It is obvious that when $u(t) = R^{-1}(t)G(t)p(t)$, we have

$$\begin{aligned} &\langle H_v(s, y(s), y_\delta(s), z(s), r(s,\cdot), u(s), p(s)), u(s)\rangle \\ = &\max_{v\in\mathcal{U}} \langle H_v(s, y(s), y_\delta(s), r(s,\cdot), u(s), p(s)), v\rangle. \end{aligned}$$

We substitute $u(t) = R^{-1}(t)G(t)p(t)$ into (4.1) and consider following equation:

$$\begin{cases} dy(t) = -[A(t)y(t) + B(t)y(t-\delta) + C(t)z(t) + D(t)r(t,\cdot) \\ \qquad\qquad + G^2(t)R^{-1}(t)p(t)]ds + z(t)dW(t) \\ \qquad\qquad + \int_{\mathcal{E}} r(t,e)\tilde{N}(dt,de), \ t \in [0,T], \\ dp(t) = \{A(t)p(t) + B(t)\mathbb{E}^{\mathcal{F}_t}[p(t+\delta)]\}dt \\ \qquad\qquad + C(t)p(t)dW(t) + \int_{\mathcal{E}} D(t)p(t)\tilde{N}(dt,de), \ t \in [0,T], \\ y(t) = \varphi(t), \ t \in [-\delta, 0); \\ p(0) = K, p(t) = 0, \ T \le t \le T+\delta. \end{cases} \tag{4.3}$$

*Remark* 4.1. If (4.3) admits a unique solution, then we get an explicit expression of optimal control $u(\cdot)$.

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