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SOME NEW CHARACTERIZATIONS OF QUASI-FROBENIUS RINGS BY USING PURE-INJECTIVITY

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Abstract. A ring R is called right pure-injective if it is injective with respect to pure exact sequences. According to a well known result of L. Melkersson, every commutative Artinian ring is pure-injective, but the converse is not true, even if R is a commutative Noetherian local ring. In this paper, a series of conditions under which right pure-injective rings are either right Artinian rings or quasi-Frobenius rings are given. Also, some of our results extend previously known results for quasi-Frobenius rings.

1. Introduction

According to a well known result of L. Melkersson [11, Corollary 4.2], every commutative Artinian ring is pure-injective. But, Example 2.2 shows that the converse is not true, even if R is a commutative Noetherian local ring. In this paper, we determine several classes of rings over which every right pure-injective ring is right Artinian (see Propositions 2.3 and 2.9 and Theorems 2.11 and 2.14). Also, it is well known that over a commutative ring, every Noetherian module with essential socle is Artinian. But a right Noetherian ring with essential right socle need not be right Artinian as shown by an example due to Faith and Menal [6]. However a result of Ginn and Moss [8, Theorem] asserts that if R is a two-sided Noetherian ring such that the right socle of R is either left or right essential, then R is right and left Artinian. Recently, Chen, Ding and Yousif [1] obtained a onesided version of this theorem by showing that if R is a right Noetherian ring whose right socle is essential as a right ideal and is contained in the left socle, then R is right Artinian. In this paper, it is shown that every right pure-injective right Noetherian ring R is a right Artinian left Kasch ring if its left socle is an essential right ideal in R (see Theorem 2.11).

Recall that a ring R is called *right Johns* if R is right Noetherian and every right ideal is an annihilator. Faith and Menal [6] gave a counter example

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to show that right Johns rings need not be right artinian. Later in [7] they defined *strongly right Johns* ring (i.e., the matrix ring $M_n(R)$ is right Johns for all $n \geq 1$) and characterized such rings as right Noetherian and left FP-injective rings. In this direction, there is an open conjecture due to Faith-Menal [7]: "*every strongly right Johns ring is quasi-Frobenius*". It is shown that the Faith-Menal conjecture is true when R is right pure-injective (see Corollary 2.6). In fact, we show that a ring R is quasi-Frobenius if and only if R is a right pure-injective right Noetherian left 2-injective ring (see Theorem 2.5).

Also, we give some new characterizations of quasi-Frobenius rings and semisimple rings by using pure-injectivity and RD-injectivity, respectively. For instance, it is shown that a ring R is quasi-Frobenius if and only if R is a left and right Johns left and right pure-injective ring, if and only if, R is a right Noetherian right P-injective left mininjective ring, if and only if, R is a right semi-artinian right Kasch *IF* ring, if and only if, R is a right pure-injective right Johns left min-CS ring (i.e., if every minimal left ideal is essential in a direct summand of ${}_RR$). In commutative case, it is shown that a ring R is quasi-Frobenius if and only if R is a Noetherian RD-injective ring with T-nilpotent Jacobson, if and only if, R is a Noetherian pure-injective ring with T-nilpotent Jacobson radical and $E(R)$ is an RD-projective R-module, if and only if, every projective R-module is RD-injective and either R is coherent or $J(R)$ is finitely generated (see Theorem 2.7, Proposition 2.13 and Corollary 2.15). Finally, it is shown that a ring R is semisimple if and only if R is a right pure-injective right hereditary right p-injective ring (see Corollary 2.21). In commutative case, it is shown that a ring R is semisimple if and only if R is Noetherian pure-injective with T-nilpotent projective Jacobson radical (see Corollary 2.8).

Throughout the paper, R will denote an arbitrary ring with identity, $J(R)$ will denote its Jacobson radical and all modules will be assumed to be unitary. For a ring R, we denote by $\mathrm{Soc}({}_RR)$ and $\mathrm{Soc}(R_R)$ for the left socle and the right socle, respectively. For any nonempty set X, the right annihilator of X in R will be denoted by $\mathrm{Ann}_r(X)$. We use $K \subseteq^{ess} N$ to indicate that K is an essential submodule of N. Also, a ring R is *semilocal* if $R/J(R)$ is a semisimple Artinian ring. A ring R is *right perfect* if R is semilocal and $J(R)$ is right T-nilpotent. A ring R is said to be *right CS ring* if every right ideal of R is essential in a direct summand of R_R. A ring R is called *left Kasch* if every simple left R-module can be embedded in ${}_RR$. A ring R is called *right coherent* if every finitely generated right ideal of R is finitely presented.

2. Results

An exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules is said to be *pure exact* (resp. *RD-exact*) if the induced homomorphism $\mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C)$ is surjective for any finitely presented (resp. cyclically presented) right R-module M. A right R-module M is said to be *pure-injective* (resp. *RD-injective*) if it is injective with respect to pure exact (resp. RD-exact) sequences. Also, a right R-module M is said to be *pure-projective* (resp. *RD-projective*) if it is projective with respect to pure exact (resp. RD-exact) sequences. In model theory, pure-injective modules are more useful than the injective modules. Also, a left R-module is called *left FP-injective* (or *left absolutely pure*) if it is pure in every module that contains it.

Recall that a ring R is *semiperfect* if R is semilocal and idempotents of $R/J(R)$ can be lifted to R.

Lemma 2.1. *For a ring R, the following conditions hold*:

(a) *Every right pure-injective semilocal ring is semiperfect.*
(b) *Every right pure-injective right Noetherian ring is semiperfect.*
(c) *Every right pure-injective ring is semiperfect whenever is a direct sum of indecomposable right ideals.*

Proof. (a) follows from [17, Theorem 9(1)].
(c) follows from [17, Theorem 9(3)].
(b) follows from (c). □

By [11, Corollary 4.2], we know that every commutative Artinian ring is pure-injective. But the following example shows that the converse is not true, in general.

Example 2.2. Let R be a commutative Noetherian local domain with maximal ideal $\mathbf{m}$, complete in its $\mathbf{m}$-adic topology. By [16], R is a pure-injective R-module, but R is not an Artinian ring.

Recall that a ring R is called *left n-injective* (*mininjective*) if, for any n-generated (minimal) left ideal I of R, every R-homomorphism from I to R extends to an R-homomorphism from R to R. Left 1-injective rings are called *left P-injective*. Now, we determine several classes of rings over which every right pure-injective ring is right Artinian (Propositions 2.3 and 2.9, and Theorems 2.11 and 2.14).

Proposition 2.3. *For a ring R, the following conditions hold*:

(a) *Every right pure-injective right Noetherian ring with right or left T-nilpotent Jacobson radical is a right Artinian ring.*
(b) *Every right pure-injective right Noetherian right P-injective ring is a two-sided Artinian ring.*
(c) *Every right pure-injective right Noetherian left P-injective ring is a right Artinian ring.*

Proof. (a) Assume that R is a right pure-injective right Noetherian ring with right or left T-nilpotent Jacobson radical. Thus, by Lemma 2.1, R is a right or left perfect ring and so R is a right Artinian ring.

(b) Assume that R is a right pure-injective right Noetherian right P-injective ring. Thus, by [13, Proposition 5.15], R is left Artinian and so by (a), R is also a right Artinian ring.

(c) Assume that R is a right pure-injective right Noetherian left P-injective ring. Thus, by [13, Lemma 8.6], $J(R)$ is nilpotent and so by (a), R is a right Artinian ring. □

Recall that a right R-module M is called $\sum$-*pure-injective* if all direct sums of copies of M are pure-injective.

Corollary 2.4. *For a commutative ring R, the following statements are equivalent*:

(1) *R is an Artinian ring.*
(2) *R is a Noetherian pure-injective ring with T-nilpotent Jacobson radical.*
(3) *R is a $\sum$-pure-injective ring and $J(R)$ is finitely generated.*

Proof. (1) ⇒ (2) is clear by [11, Corollary 4.2].

(2) ⇒ (3) follows from Proposition 2.3(a) and [3, Example 1.41].

(3) ⇒ (1) Assume that R is a $\sum$-pure-injective ring and $J(R)$ is finitely generated. We know that any $\sum$-pure-injective ring is semiprimary (i.e., $R/J(R)$ is semisimple and $J(R)$ is nilpotent) and so R is semiprimary. Also, $J(R)^n$ is finitely generated for each $n \in \mathbb{N}$. Hence, the finitely generated semisimple R-module $J(R)^n/J(R)^{n+1}$ is Artinian for each $n \in \mathbb{N}$. Also, $J(R)$ is nilpotent and so it follows that R is an Artinian ring. □

A famous conjectures on quasi-Frobenius rings is the Faith-Menal conjecture: "*every strongly right Johns ring is quasi-Frobenius?*"

Faith and Menal [6] gave a counter example to show that right Johns rings may not be right artinian. They characterized strongly right Johns rings as right noetherian and left FP-injective rings (see [7, Theorem 1.1]). Now, we apply Rutter's result ([15, Corollary 3]) to show that Faith-Menal conjecture is true when R is right pure-injective.

Theorem 2.5. *A ring R is quasi-Frobenius if and only if R is a right pure-injective right Noetherian left 2-injective ring.*

Proof. Assume that R is a left 2-injective, right Noetherian and right pure-injective ring. Thus, by Proposition 2.3(c), R is a right Artinian ring and so R has ascending chain condition (ACC) on left annihilators. Also, Rutter in [15, Corollary 3] proved that a left 2-injective ring with ACC on left annihilators is quasi-Frobenius. Therefore, R is a quasi-Frobenius ring. The converse is clear. □

Since every left FP-injective ring is left 2-injective, the following result follows from Theorem 2.5.

Corollary 2.6. *A ring R is quasi-Frobenius if and only if R is a right pure-injective strongly right Johns ring.*

The following theorem generalizes the result of Couchot [2, Proposition I.2].

Theorem 2.7. *For a commutative ring R, the following statements are equivalent*:

(1) *R is a quasi-Frobenius ring.*
(2) *R is a Noetherian RD-injective ring with T-nilpotent Jacobson radical.*
(3) *R is a Noetherian pure-injective ring with T-nilpotent Jacobson radical and $E(R)$ is an RD-projective R-module.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1) Assume that R is a Noetherian RD-injective ring with T-nilpotent Jacobson. Thus, by Corollary 2.4, R is Artinian, since every RD-injective module is pure-injective. Therefore, [2, Proposition I.2] allows us to conclude.

(3) $\Rightarrow$ (1) Assume that R is a Noetherian pure-injective ring with T-nilpotent Jacobson radical and $E(R)$ is an RD-projective R-module. Thus, by Corollary 2.4, R is Artinian. Without loss of generality, we can assume that R is a local ring. So, by [16, Corollary 2], $E(R)$ is a direct sum of cyclically presented R-modules. Since R is Artinian, $E(R)$ is a finite direct sum of indecomposable cyclic R-modules. Assume that Rx is an indecomposable cyclic direct summand of $E(R)$ where $x \in E(R)$. So, Rx has a simple R-submodule, since Rx is an Artinian R-module. Also, since any indecomposable injective module is the injective envelope of each its submodule, we conclude that $Rx \cong E(R/\mathscr{M})$ where $\mathscr{M}$ is the maximal ideal of R. Now, $\mathrm{Ann}(x) = 0$, since $E(R/\mathscr{M})$ is faithful. Thus, $Rx \cong R$ and so R is self-injective. Therefore, R is a quasi-Frobenius ring. □

Recall that a ring R is said to be *right hereditary* if every right ideal of R is projective.

Corollary 2.8. *For a commutative ring R, the following statements are equivalent*:

(1) *R is a semisimple ring.*
(2) *R is Noetherian pure-injective with T-nilpotent projective Jacobson radical.*
(3) *R is hereditary Noetherian pure-injective with T-nilpotent Jacobson radical.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) follows from Corollary 2.4 and Auslander's Theorem [10, Theorem 5.72].

(3) $\Rightarrow$ (1) Assume that R is a hereditary Noetherian pure-injective ring with T-nilpotent Jacobson radical. Thus, by Corollary 2.4, R is Artinian and so $E(R)$ is finitely generated by [10, Theorem 3.64]. So, $E(R)$ is finitely presented and so it is pure-projective. Thus, by [12, Proposition 2.11], $E(R)$ is RD-projective and so Theorem 2.7 allows us to conclude. □

Faith and Menal [6] gave a counter example to show that right Johns rings need not be right Artinian. Also, there is a two-sided Artinian right Johns ring which is not quasi-Frobenius as shown by Rutter [15, Example 1]. The following result shows that every right Johns right pure-injective ring is right Artinian.

Proposition 2.9. *Every right Johns right pure-injective ring is a right Artinian right CS right CF ring.*

Proof. Assume that R is a right Johns right pure-injective ring. Thus, by Proposition 2.3(c) and since every right Johns ring is left P-injective, R is a right Artinian ring. Also, by [13, Theorem 8.9], R is a right CS right CF ring. $\square$

From Proposition 2.9 and [13, Theorem 7.1], we have:

Corollary 2.10. *A ring R is quasi-Frobenius if and only if R is a left and right Johns left and right pure-injective ring.*

A right Noetherian ring with essential right socle need not be right Artinian as shown by Faith-Menal example [6]. The following theorem shows that the pure-injectivity of R is strong enough to force a right Noetherian ring with right essential left socle to be right Artinian. Also, it may be viewed as a onesided version of a result of Ginn and Moss on two-sided Noetherian rings with essential socle [8].

Theorem 2.11. *If R is a right pure-injective right Noetherian ring and the left socle of R is right essential, then R is a right Artinian left Kasch ring.*

Proof. Assume that R is a right pure-injective right Noetherian ring and $\mathrm{Soc}({}_RR) \subseteq^{ess} R_R$. First, we show that R is a right Artinian ring. To prove, by Proposition 2.3(a), it is enough to show that $J(R)$ is right T-nilpotent. Suppose that $a_1, a_2, a_3, \dots$ is a sequence in $J(R)$. Now consider

$$\mathrm{Ann}_r(a_1) \subseteq \mathrm{Ann}_r(a_2a_1) \subseteq \mathrm{Ann}_r(a_3a_2a_1) \subseteq \cdots.$$

Since R is right Noetherian, there exists $n \in \mathbb{N}$ such that

$$\mathrm{Ann}_r(a_na_{n-1}\cdots a_1) = \mathrm{Ann}_r(a_{n+1}a_n\cdots a_1).$$

This implies that

$$(a_na_{n-1}\cdots a_1)R \cap \mathrm{Ann}_r(a_{n+1}) = 0.$$

Also, since $J(R)\mathrm{Soc}({}_RR) = 0$, we have $a_{n+1}\mathrm{Soc}({}_RR) = 0$. Thus $\mathrm{Soc}({}_RR) \subseteq \mathrm{Ann}_r(a_{n+1})$ and so $\mathrm{Ann}_r(a_{n+1}) \subseteq^{ess} R_R$, since $\mathrm{Soc}({}_RR) \subseteq^{ess} R_R$. Therefore, $a_na_{n-1}\cdots a_1 = 0$, as required. Moreover, by [13, Lemma 1.48], R is a left Kasch ring. $\square$

The following example shows that there exist two-sided Artinian, left and right CS rings R that are not quasi-Frobenius.

Example 2.12. Assume that $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where F is a field. Then R is a left and right artinian, left and right CS ring. But R is not quasi-Frobenius, because the R-module e_2R is not injective.

We know that a ring R is quasi-Frobenius if and only if every projective R-module is injective. We have the following result, which generalizes this fact in the cases R is a commutative coherent ring or R is a commutative ring with finitely generated Jacobson radical.

Proposition 2.13. *Let R be a ring. Consider the following conditions*:

(1) *R is a quasi-Frobenius ring.*
(2) *R is a right Noetherian right P-injective left mininjective ring.*
(3) *R is a right pure-injective right Johns left min-CS ring.*
(4) *R is a right Noetherian left P-injective right pure-injective and left min-CS ring with* $\mathrm{Soc}({}_RR) \subseteq^{ess} R_R$.
(5) *R is a right Noetherian left P-injective right pure-injective and left min-CS ring with* $\mathrm{Soc}(R_R) \subseteq \mathrm{Soc}({}_RR)$.
(6) *R is a coherent ring such that every projective R-module is RD-injective.*
(7) *$J(R)$ is finitely generated and every projective R-module is RD-injective.*

Then:

(a) *Conditions* (1)-(5) *are equivalent and imply conditions* (6) *and* (7).
(b) *When R is a commutative ring, the seven conditions are equivalent.*

Proof. Clearly (1) implies the conditions (2)-(7).

(2) ⇒ (1) Assume that R is a right Noetherian right P-injective left mininjective ring. Thus, by [13, Proposition 5.15], R is a left Artinian ring. Also, since R is left and right mininjective, $\mathrm{Soc}({}_RR) = \mathrm{Soc}(R_R)$ [13, Theorem 2.21]. In particular, R is semiperfect and left mininjective, so $\mathrm{Soc}({}_RR)$ is finite dimensional as a right R-module [13, Theorem 3.7]. Therefore, by [13, Lemma 3.30], R is right Artinian and so by Ikeda's theorem [13, Theorem 2.30], R is quasi-Frobenius.

(3) ⇒ (4) Assume that R is a right Johns right pure-injective left min-CS ring. Thus, by Proposition 2.9, R is right Artinian. Therefore, [13, Theorem 8.9] allows us to conclude.

(4) ⇒ (5) Given (4), so by Theorem 2.11, R is a right Artinian left Kasch ring. Thus, by [13, Lemma 4.5], $\mathrm{Soc}({}_RR) \subseteq^{ess} {}_RR$. Hence, R is a left GPF ring (i.e., R is left P-injective, semiperfect, and $\mathrm{Soc}({}_RR) \subseteq^{ess} {}_RR$) and so $\mathrm{Soc}({}_RR) = \mathrm{Soc}(R_R)$ by [13, Theorem 5.31].

(5) ⇒ (1) Given (5), so by Proposition 2.3(c), R is right Artinian. Also, by [13, Theorem 2.21], $\mathrm{Soc}({}_RR) \subseteq \mathrm{Soc}(R_R)$, since R is left P-injective and so by hypothesis $\mathrm{Soc}({}_RR) = \mathrm{Soc}(R_R) \subseteq^{ess} R_R$. Thus, [13, Lemma 4.4] implies that R is right mininjective. Since a left P-injective ring is left mininjective, R is two-sided mininjective and right artinian. Therefore, by [13, Theorem 3.31], R is a quasi-Frobenius ring.

(6) ⇒ (1) Assume that R is a commutative coherent ring such that every projective R-module is RD-injective. Thus every projective R-module is pure-injective, since every RD- injective is pure-injective. So, R is $\sum$-pure-injective,

since any direct sum of projective module is projective. Thus, R is semiprimary and so R is a perfect ring. Therefore, R is Artinian and so by Theorem 2.7, R is quasi-Frobenius.

(7) $\Rightarrow$ (1) Similar to the proof of (6) $\Rightarrow$ (1), R is $\sum$-pure-injective and so by Corollary 2.4, R is Artinian. Therefore, Theorem 2.7 allows us to conclude. $\square$

According to a famous result of Faith and Walker, a ring R is quasi-Frobenius (i.e., R is left or right Artinian and left or right self-injective) if and only if every R-module embeds in a projective module. Recall that a ring R is called *right CF* if every cyclic right R-module embeds in a free right R-module. It is still open *"whether a right CF ring is right Artinian?"*. In [9, Theorem 2.6], Gómez Pardo proved that every right (almost-)coherent right CF ring is right Artinian. Now, we obtain the following theorem that is an analogue of Gómez Pardo's theorem.

Recall that a module M_R is said to be *semi-artinian* if every nonzero factor module of M has nonzero socle, and that a ring R is right semi-artinian if R_R is a semi-artinian module.

Theorem 2.14. *Every right coherent right semi-artinian right Kasch ring is right Artinian.*

Proof. Assume that R is a right coherent right semi-artinian right Kasch ring. So, by hypothesis, every simple right R-module embeds in R_R. It follows that every simple right R-module is finitely presented, since R is right coherent [10, Corollary 4.52]. Now, suppose that $\mathscr{M}$ is a maximal right ideal of R and so $R/\mathscr{M} \cong F/K$ where K is a finitely generated submodule of finitely generated free right R-module F. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \hookrightarrow & F & \longrightarrow & F/K & \longrightarrow & 0 \\ & & & & & & \wr\downarrow & & \\ 0 & \longrightarrow & \mathscr{M} & \hookrightarrow & R & \longrightarrow & R/\mathscr{M} & \longrightarrow & 0. \end{array}$$

By Schanuel's Lemma, we have that $F \oplus \mathscr{M} \cong R \oplus K$. Thus, $\mathscr{M}$ is finitely generated, since K is finitely generated. So, every maximal right ideal of R is finitely generated. Therefore, [14, Proposition 4.8(1)] allows us to conclude. $\square$

Recall that a ring R is said to be *right IF ring* if every injective right R-module is flat. Colby and Würfel proved that a ring R is right *IF* if and only if all finitely presented right R-modules embed in a free right R-module.

Corollary 2.15. *A ring R is quasi-Frobenius if and only if it is a right semi-artinian right Kasch IF ring.*

Proof. Assume that R is a right semi-artinian right Kasch *IF* ring. So, by Theorem 2.14 and since every *IF* ring is right coherent, R is right Artinian. Also, by Bass's theorem, over a right perfect ring, any flat right R-module is projective, hence any injective right R-module is projective. Therefore, R is a quasi-Frobenius ring. The converse is clear. $\square$

A *uniform module* is a nonzero module M such that the intersection of any two nonzero submodules of M is nonzero, or, equivalently, such that every nonzero submodule of M is essential in M.

Proposition 2.16. *Let R_R be uniform and $J(R)$ left T-nilpotent. Then R is right self-injective if and only if it is right pure-injective and $E(R_R)$ is RD-projective.*

Proof. Assume that R is a right pure-injective ring and $E(R_R)$ is RD-projective. Thus, R is a indecomposable right pure-injective ring, since R_R is uniform. So, by [3, Corollary 2.27], $R \cong \mathrm{End}(R_R)$ is local and so by hypothesis R is a left perfect ring. Also, by [4, Theorem 4.6], over one-sided perfect ring, every RD-projective right (left) R-module is a direct sum of finitely presented cyclic modules. So, $E(R_R)$ is a direct sum of finitely presented cyclic modules. Also, $E(R_R)$ is indecomposable, since R_R is uniform. This implies that $E(R_R)$ is cyclic. Also, Faith [5, Lemma 2] proved every left perfect ring with cyclic right injective envelope is right self-injective. So, R is a right self-injective ring. The converse is clear. □

Now, the following result follows from Proposition 2.16 and [13, Theorems 1.50].

Corollary 2.17. *A right uniform ring R is quasi-Frobenius if and only if it is right pure-injective with left T-nilpotent Jacobson radical such that R has ACC on left or right annihilators and $E(R_R)$ is RD-projective.*

Recall that a ring R is said to be *semiprimitive* if $J(R) = 0$.

Lemma 2.18. *Every semiprimitive right pure-injective ring is a right self-injective von Neumann regular ring.*

Proof. Assume that R is a semiprimitive right pure-injective ring. So, by [17, Theorem 9], $R/J(R)$ is a von Neumann regular ring. Thus, R is a von Neumann regular ring, since R is semiprimitive. Thus, every R-module is flat and so every exact sequence is pure exact. Therefore, this implies that R is right self-injective. □

Proposition 2.19. *For a ring R with $\mathrm{Soc}(R_R) \subseteq^{ess} R_R$, the following statements are equivalent*:

(1) *R is a right self-injective von Neumann regular ring.*
(2) *R is a right pure-injective ring and $\mathrm{Soc}(R_R)$ is FP-injective.*
(3) *R is a right pure-injective ring and $\mathrm{Soc}(R_R)$ is a pure submodule of R_R.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) are clear, since over a von Neumann regular ring every submodule of a module is pure submodule.

(3) ⇒ (1) Assume that R is a right pure-injective ring in which $\mathrm{Soc}(R_R)$ is an essential pure submodule of R_R. Thus, by [16, Proposition 3], we have

$$\mathrm{Soc}(R_R)J(R) = RJ(R) \cap \mathrm{Soc}(R_R) = J(R) \cap \mathrm{Soc}(R_R).$$

Since $\mathrm{Soc}(R_R)$ is an essential submodule of R_R and $\mathrm{Soc}(R_R)J(R) = 0$, we have $J(R) = 0$. Therefore, Lemma 2.18 allows us to conclude. □

Assume that M is a right R-module. Then

$$Z(M) := \{m \in M \mid Ann_r(m) \text{ is essential in } R_R\}$$

is a singular submodule of M. A right R-module M is *singular* if $Z(M_R) = M$ and *non-singular* if $Z(M_R) = 0$. A ring R is called *right non-singular* if the right R-module R_R is non-singular.

Corollary 2.20. *A right non-singular ring R is right self-injective if and only if it is a right pure-injective right p-injective ring.*

Proof. Assume that R is a right pure-injective right p-injective ring. Thus, by [13, Theorem 5.14], $J(R) = Z(R_R)$ and so $J(R) = 0$. Therefore, Lemma 2.18 allows us to conclude. The converse is clear. □

Corollary 2.21. *A ring R is semisimple if and only if R is a right pure-injective right hereditary right p-injective ring.*

Proof. Assume that R is a right pure-injective right hereditary right p-injective ring. Thus, by Corollary 2.20, R is a right self-injective ring, since every right hereditary ring is right non-singular. Also, we know that over a right hereditary ring, every non-zero factor module of injective modules is injective. Therefore, every cyclic right R-module is injective and so R is a semisimple ring. The converse is clear. □

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