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ENDPOINT ESTIMATES FOR MULTILINEAR FRACTIONAL MAXIMAL OPERATORS

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Abstract. We study the mapping property of multilinear fractional maximal operators in Lipschitz spaces. It should be pointed out that some of the techniques employed in the study of fractional integral operators do not apply to fractional maximal operators.

1. Introduction

For $0 < \alpha < mn$, the multilinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ is defined by

$$\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(x) = \sup_{B\ni x} |B|^{\alpha/n} \prod_{j=1}^{m} \frac{1}{|B|}\int_B |f_j(y_j)|dy_j.$$

In 2008, Tang [8] proved that multilinear fractional integral

$$I_{\alpha,m}(f_1,\dots,f_m)(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} dy_1\cdots dy_m$$

is bounded from product of Morrey spaces to Morrey spaces (Lipschitz spaces, BMO space). From the fact that

$$\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(x) \le CI_{\alpha,m}(|f_1|,\dots,|f_m|)(x), \tag{1.1}$$

the boundedness of $\mathcal{M}_{\alpha,m}$ on Lebesgue spaces and Morrey spaces follows directly. However, $\|\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)\|_{Lip_\beta} \le C\|I_{\alpha,m}(|f_1|,\dots,|f_m|)\|_{Lip_\beta}$ may be not true. A natural question is whether $\mathcal{M}_{\alpha,m}$ share the same boundedness as $I_{\alpha,m}$ in Lipschitz spaces. We will give an affirmative answer in this paper. It should be pointed out that some of the techniques employed in [8] do not apply to fractional maximal operators.

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The main purpose of this paper is to establish endpoint estimates for the multilinear fractional Hardy operator and multilinear fractional maximal operator. The study of the boundedness of single integral operators on Lipschitz spaces ([1], [6] and [9]) may not be as delicate as that of operators on Lebesgue spaces, Hardy spaces and Morrey spaces ([3], [4], [5], [7] and [8]) but still requires the use of certain beautiful and elegant ideas. Before stating our results, let us first introduce some notation.

We first recall the definitions of the Morrey spaces and Lipschitz spaces.

Definition 1.1 (Morrey space)**.** For $1 < q \le p < \infty$, we say that a function f belongs to Morrey space M_q^p if

$$\|f\|_{M_q^p} := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|^{1/q-1/p}} \left(\int_{B(x,r)} |f(y)|^q dy\right)^{1/q} < \infty;$$

Definition 1.2 (Lipschitz space)**.** For $0 < \alpha < 1$, the Lipschitz space Lip_α is the set of functions f such that

$$\|f\|_{\mathrm{Lip}_\alpha} := \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

For $0 < \alpha < mn$, the multilinear fractional Hardy operator $\mathcal{H}_{\alpha,m}$ is defined by

$$\begin{aligned} &\mathcal{H}_{\alpha,m}(f_1,\ldots,f_m)(x) \\ &= \frac{1}{|x|^{mn-\alpha}} \int_{|(y_1,\ldots,y_m)|<|x|} f_1(y_1)\cdots f_m(y_m)dy_1\cdots dy_m,\ x\in\mathbb{R}^n\setminus\{0\}, \end{aligned}$$

where $|(y_1,\ldots,y_m)| = \sqrt{\sum_{j=1}^m |y_j|^2}$. When $m = 1$, we also write $\mathcal{H}_\alpha$ as the line case. The classical fractional Hardy operator $\mathcal{H}_\alpha$ was first defined by Fu, et al. [2].

Let us now formulate our results as follows. The following results are new even for the case $m = 1$.

Theorem 1.1. *Let $0 < \alpha < mn$ and $1 < q_j \le p_j < \infty$ with $j = 1,\ldots,m$. If $0 < \alpha - \sum_{j=1}^m \frac{n}{p_j} < \min\{1, n-\frac{n}{p_1}\}$, $f_j \in M_{q_j}^{p_j}$ and $\mathcal{H}_{\alpha,m}(f_1,\ldots,f_m)(0) = 0$, then $\mathcal{H}_{\alpha,m}(f_1,\ldots,f_m) \in Lip_{\alpha-\sum_{j=1}^m \frac{n}{p_j}}$. Moreover, there is a constant $C > 0$ such that*

$$\|\mathcal{H}_{\alpha,m}(f_1,\ldots,f_m)\|_{Lip_{\alpha-\sum_{j=1}^m \frac{n}{p_j}}} \le C\prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}}.$$

Theorem 1.2. *Let $0 < \alpha < mn$ and $1 < q_j < p_j < \infty$ with $j = 1,2,\ldots,m$. If $0 < \alpha - \sum_{j=1}^m \frac{n}{p_j} < \min\{1, n-\frac{n}{p_1}\}$ and $f_j \in M_{q_j}^{p_j}$, then $\mathcal{M}_\alpha(f_1,\ldots,f_m) \in Lip_{\alpha-\sum_{j=1}^m \frac{n}{p_j}}$. Moreover, there is a constant $C > 0$ such that*

$$\|\mathcal{M}_{\alpha,m}(f_1,\ldots,f_m)\|_{Lip_{\alpha-\sum_{j=1}^m \frac{n}{p_j}}} \le C\prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}}.$$

2. Proofs of Theorems 1.1∼1.2

Proof of Theorem 1.1. For any fixed points $x_1, x_2 \in \mathbb{R}^n\backslash\{0\}$, we may assume that $|x_1| \geq |x_2|$. We consider the following two cases.

Case 1: $|x_1| \geq 2|x_2|$.

Using $2|x_2| \leq |x_1|$, we have $|x_1| \leq 2(|x_1| - |x_2|)$, then

$$\begin{aligned}
&\Big|\mathcal{H}_{\alpha,m}(f)(x_1) - \mathcal{H}_{\alpha,m}(f)(x_2)\Big| \\
&\leq \frac{1}{|x_1|^{mn-\alpha}} \prod_{j=1}^{m} \int_{|y_j|<|x_1|} |f_j(y_j)| dy_j + \frac{1}{|x_2|^{n-\alpha}} \prod_{j=1}^{m} \int_{|y_j|<|x_2|} |f_j(y_j)| dy_j \\
&\leq C \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}} \left(|x_1|^{\alpha - \sum_{j=1}^{m} \frac{n}{p_j}} + |x_2|^{\alpha - \sum_{j=1}^{m} \frac{n}{p_j}} \right) \\
&\leq C(|x_1| - |x_2|)^{\alpha - \sum_{j=1}^{m} \frac{n}{p_j}} \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}}.
\end{aligned}$$

Case 2: $|x_1| < 2|x_2|$.

Let $\Omega := B(0,a)\backslash B(0,b) (a > b > 0)$ and $R_1, \ldots, R_i, \ldots$ be a maximal collection of disjoint balls of radius $\frac{a-b}{2}$ centered at $\{z_0 : |z_0| = \frac{a+b}{2}\}$. It is easy to see that $\bigcup_i R_i \subset \Omega$. We claim that $\Omega \subset \bigcup_i 3R_i$. If not, taking $\tilde{z} \in \Omega$ and $\tilde{z} \notin \bigcup_i 3R_i$, and z_0 be the point on the straight line $l_{O,\tilde{z}}$ such that $|z_0| = \frac{a+b}{2}$, then $|\tilde{z} - z_0| \leq \frac{a-b}{2}$ and $|z_i - \tilde{z}| \geq \frac{3(a-b)}{2}$, where z_i is the center of R_i. Thus,

$$|z_i - z_0| \geq |z_i - \tilde{z}| - |\tilde{z} - z_0| \geq a - b,$$

which gives a contradiction since the ball $R_0 := B(z_0, \frac{a-b}{2})$ can be added into the collection $R_1, \ldots, R_i, \ldots$.

Returning to the proof of the case $|x_1| < 2|x_2|$.

$$\begin{aligned}
&\Big|\mathcal{H}_{\alpha,m}(f_1, \ldots, f_m)(x_1) - \mathcal{H}_{\alpha,m}(f_1, \ldots, f_m)(x_2)\Big| \\
&\leq \Big|\mathcal{H}_{\alpha,m}(f_1, \ldots, f_m)(x_1) - \Big(\frac{|x_2|}{|x_1|}\Big)^{mn-\alpha} \mathcal{H}_{\alpha,m}(f_1, \ldots, f_m)(x_2)\Big| \\
&\quad + \Big|\Big(\frac{|x_2|}{|x_1|}\Big)^{mn-\alpha} \mathcal{H}_{\alpha,m}(f_1, \ldots, f_m)(x_2) - \mathcal{H}_{\alpha,m}(f_1, \ldots, f_m)(x_2)\Big| \\
&:= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

For I_1, let $a := \sqrt{|x_1|^2 - \sum_{j=2}^{m} |y_j|^2}$, $b := \sqrt{|x_2|^2 - \sum_{j=2}^{m} |y_j|^2}$, and $\Omega := B(0,a)\backslash B(0,b)$. From $a + b \leq |x_1| + |x_2|$ and $a^2 - b^2 = |x_1|^2 - |x_2|^2$, it follows that $a - b \geq |x_1| - |x_2|$ and

$$|\Omega| = C(a^n - b^n) \leq C\big(a^2 - b^2\big)^{n/2} \leq C\big(|x_1|^2 - |x_2|^2\big)^{n/2} \leq C\big(|x_1| - |x_2|\big)^n.$$

The inequality above and $0 < \alpha - \sum_{j=1}^m \frac{n}{p_j} < \min\{1, n - \frac{n}{p_1}\}$ gives

$$\begin{aligned}
\mathrm{I}_1 &\le \frac{1}{|x_1|^{mn-\alpha}} \int_{|x_2|\le|(y_1,\dots,y_m)|<|x_1|} \prod_{j=1}^m |f_j(y_j)| dy_1 \cdots dy_m \\
&\le \frac{1}{|x_1|^{mn-\alpha}} \int_{|y_2|\le|x_1|} \cdots \int_{|y_m|\le|x_1|} \int_{\{y_1: y_1\in\Omega\}} \prod_{j=1}^m |f_j(y_j)| dy_1 \cdots dy_m \\
&\le \frac{1}{|x_1|^{mn-\alpha}} \int_{|y_2|\le|x_1|} \cdots \int_{|y_m|\le|x_1|} \prod_{j=2}^m |f_j(y_j)| \sum_i \int_{3R_i} |f_1(y_1)| dy_1 \cdots dy_m \\
&\le \frac{C\|f_1\|_{M_{q_1}^{p_1}}}{|x_1|^{mn-\alpha}} \int_{|y_2|\le|x_1|} \cdots \int_{|y_m|\le|x_1|} \prod_{j=2}^m |f_j(y_j)| \sum_i |R_i|^{1-1/p_m} dy_2 \cdots dy_m \\
&\le \frac{C\big(|x_1|-|x_2|\big)^{-n/p_1} \|f_1\|_{M_{q_1}^{p_1}}}{|x_1|^{mn-\alpha}} \int_{|y_2|\le|x_1|} \cdots \int_{|y_m|\le|x_1|} \prod_{j=2}^m |f_j(y_j)| \\
&\quad \sum_i |R_i| dy_2 \cdots dy_m \\
&\le \frac{C|\Omega|\big(|x_1|-|x_2|\big)^{-n/p_1} \|f_1\|_{M_{q_1}^{p_1}}}{|x_1|^{mn-\alpha}} \int_{|y_2|\le|x_1|} \cdots \int_{|y_m|\le|x_1|} \\
&\quad \prod_{j=2}^m |f_j(y_j)| dy_2 \cdots dy_m \\
&\le \frac{C\big(|x_1|-|x_2|\big)^{n-n/p_1}}{|x_1|^{n-\alpha+\sum_{j=2}^m \frac{n}{p_j}}} \prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}} \\
&\le C\big(|x_1|-|x_2|\big)^{\alpha-\sum_{j=1}^m \frac{n}{p_j}} \prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}}.
\end{aligned}$$

For I_2, by differential mean value theorem, there exists a point ξ on the straight line $l_{|x_2|,|x_1|}$ such that

$$\frac{1}{|x_2|^{mn-\alpha}} - \frac{1}{|x_1|^{mn-\alpha}} = C\frac{|x_1|-|x_2|}{\xi^{mn+1-\alpha}} \le C\frac{|x_1|-|x_2|}{|x_2|^{mn+1-\alpha}},$$

which shows that

$$\begin{aligned}
\mathrm{I}_2 &\le \frac{|x_1|-|x_2|}{|x_2|^{mn-\alpha+1}} \int_{|(y_1,\dots,y_m)|\le|x_2|} \prod_{j=1}^m |f_j(y_j)| dy_1 \cdots dy_m \\
&\le C\frac{|x_1|-|x_2|}{|x_2|^{1-\alpha+\sum_{j=1}^m \frac{n}{p_j}}} \prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}}
\end{aligned}$$

$$\leq C\big(|x_1|-|x_2|\big)^{\alpha-\sum_{j=1}^{m}\frac{n}{p_1}}\prod_{j=1}^{m}\|f_j\|_{M_{q_j}^{p_j}}.$$

Combining **Case 1** and **Case 2**, we have

$$\begin{aligned}&\Big|\mathcal{H}_{\alpha,m}(f_1,\dots,f_m)(x_1)-\mathcal{H}_{\alpha,m}(f_1,\dots,f_m)(x_2)\Big|\\ &\leq C\big(|x_1|-|x_2|\big)^{\alpha-\sum_{j=1}^{m}\frac{n}{p_1}}\prod_{j=1}^{m}\|f_j\|_{M_{q_j}^{p_j}}\\ &\leq C|x_1-x_2|^{\alpha-\sum_{j=1}^{m}\frac{n}{p_1}}\prod_{j=1}^{m}\|f_j\|_{M_{q_j}^{p_j}}.\end{aligned}\tag{2.1}$$

Now we consider the case $x_1\in\mathbb{R}^n\backslash\{0\}$ and $x_2=0$. By $\mathcal{H}_{\alpha,m}(f_1,\dots,f_m)(0)=0$, we get

$$\begin{aligned}&\Big|\mathcal{H}_{\alpha,m}(f_1,\dots,f_m)(x_1)-\mathcal{H}_{\alpha,m}(f_1,\dots,f_m)(0)\Big|\\ &=|x_1|^{\alpha-\sum_{j=1}^{m}\frac{n}{p_j}}\Big|\mathcal{H}_{\sum_{j=1}^{m}\frac{n}{p_j},m}(f_1,\dots,f_m)(x_1)\Big|\\ &\leq C|x_1|^{\alpha-\sum_{j=1}^{m}\frac{n}{p_j}}\prod_{j=1}^{m}\|f_j\|_{M_{q_j}^{p_j}}.\end{aligned}$$

Hence, the proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. For two fixed points $x,y\in\mathbb{R}^n$, we may assume that

$$\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(x)\geq\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(y).$$

It follows from Tang's results (Theorem 1.2, [8]) that

$$I_{\alpha,m}(f_1,\dots,f_m)\in Lip_{\alpha-\sum_{j=1}^{m}\frac{n}{p_j}},$$

which implies that

$$\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(x)\leq CI_{\alpha,m}(f_1,\dots,f_m)(x)<\infty.$$

Thus, for any $\epsilon>0$, there is a cube $B_1:=B(z,r)\ni x$ such that

$$|B_1|^{\alpha/n}\prod_{j=1}^{m}\frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j+\epsilon>\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(x).\tag{2.2}$$

Since $y\in B\big(z,r+|x-y|\big)=:B_2$, we deduce that

$$|B_2|^{\alpha/n}\prod_{j=1}^{m}\frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\leq\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(y).\tag{2.3}$$

By (2.2) and (2.3), we have

$$\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(x)-\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(y)$$

$$\leq |B_1|^{\alpha/n}\prod_{j=1}^m \frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j - |B_2|^{\alpha/n}\prod_{j=1}^m \frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j + \epsilon$$
$$\leq Cr^{\alpha}\Big(\prod_{j=1}^m \frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j - \prod_{j=1}^m \frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\Big) + \epsilon.$$

Case 1: $r \leq |x-y|$. Since $f_j \in M^{p_j}_{q_j}$, then

$$\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(x) - \mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(y)$$
$$\leq Cr^{\alpha}\Big(\prod_{j=1}^m \frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j - \prod_{j=1}^m \frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\Big) + \epsilon$$
$$\leq Cr^{\alpha}\prod_{j=1}^m \|f_j\|_{M^{p_j}_{q_j}}\Big(\Big(\frac{1}{|B_1|}\Big)^{\sum_{j=1}^m \frac{1}{p_j}} + \Big(\frac{1}{|B_2|}\Big)^{\sum_{j=1}^m \frac{1}{p_j}}\Big) + \epsilon$$
$$\leq C|x-y|^{\alpha-\sum_{j=1}^m \frac{n}{p_j}}\prod_{j=1}^m \|f_j\|_{M^{p_j}_{q_j}} + \epsilon.$$

Case 2: $r \geq |x-y|$. We claim that

$$\Big|\prod_{j=1}^m \frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j - \prod_{j=1}^m \frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\Big|$$
$$\leq Cr^{-\alpha}|x-y|^{\alpha-\sum_{j=1}^m \frac{n}{p_j}}\prod_{j=1}^m \|f_j\|_{M^{p_j}_{q_j}}. \tag{2.4}$$

Once we have established this we can see that

$$\mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(x) - \mathcal{M}_{\alpha,m}(f_1,\dots,f_m)(y)$$
$$\leq Cr^{\alpha}\Big(\prod_{j=1}^m \frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j - \prod_{j=1}^m \frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\Big) + \epsilon$$
$$\leq C|x-y|^{\alpha-\sum_{j=1}^m \frac{n}{p_j}}\prod_{j=1}^m \|f_j\|_{M^{p_j}_{q_j}} + \epsilon.$$

Therefore, the proof of Theorem 1.2 is completed by letting $\epsilon \to 0$.

We now give the proof of (2.4). Let $\Omega := B(z, r+|x-y|)\backslash B(z,r)$ and $R_1,\dots,R_i,\dots$ be a maximal collection of disjoint balls of radius $|x-y|/2$ centered at $\{z_0 : |z-z_0| = r+\frac{|x-y|}{2}\}$. It is easy to see that $\bigcup_i R_i \subset \Omega \subset \bigcup_i 3R_i$.

First, we come to the case $m = 1$.

$$\Big|\frac{1}{|B_1|}\int_{B_1}|f_1(y_1)|dy_1 - \frac{1}{|B_2|}\int_{B_2}|f_1(y_1)|dy_1\Big|$$
$$\leq \Big|\frac{1}{|B_1|}\int_{B_1}|f_1(y_1)|dy_1 - \frac{1}{|B_2|}\int_{B_1}|f_1(y_1)|dy_1\Big|$$

$$
\begin{aligned}
&+\left|\frac{1}{|B_2|}\int_{B_1}|f_1(y_1)|dy_1-\frac{1}{|B_2|}\int_{B_2}|f_1(y_1)|dy_1\right|\\
&\le\left(\frac{1}{|B_1|}-\frac{1}{|B_2|}\right)\int_{B_1}|f_1(y_1)|dy_1+\frac{1}{|B_2|}\int_{\Omega}|f_1(y_1)|dy_1\\
&:=\mathrm{II}_1+\mathrm{II}_2.
\end{aligned}
$$

For II_1, Hölder inequality and the condition $r>|x-y|$ give

$$
\begin{aligned}
\mathrm{II}_1&\le\frac{C|x-y|^n}{|B_2|}\left(\frac{1}{|B_1|}\int_{B_1}|f_1(y_1)|^{q_1}dy_1\right)^{1/q_1}\\
&\le C|x-y|^n r^{-n-n/p_1}\|f_1\|_{M^{p_1}_{q_1}}\\
&\le Cr^{-n}|x-y|^{n-n/p_1}\|f_1\|_{M^{p_1}_{q_1}}.
\end{aligned}
$$

For II_2,

$$
\begin{aligned}
\mathrm{II}_2&\le\sum_i\frac{1}{|B_2|}\int_{3R_i}|f_1(y_1)|dy_1\\
&\le Cr^{-n}\sum_i|R_i|^{1-1/p_1}\|f_1\|_{M^{p_1}_{q_1}}\\
&\le Cr^{-n}|x-y|^{-n/p_1}\|f_1\|_{M^{p_1}_{q_1}}\sum_i|R_i|\\
&\le Cr^{-n}|x-y|^{n-n/p_1}\|f_1\|_{M^{p_1}_{q_1}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|\frac{1}{|B_1|}\int_{B_1}|f_1(y_1)|dy_1-\frac{1}{|B_2|}\int_{B_2}|f_1(y_1)|dy_1\right|\\
&\le Cr^{-n}|x-y|^{n-n/p_1}\|f_1\|_{M^{p_1}_{q_1}}.
\end{aligned}
\tag{2.5}
$$

Thus, (2.4) is proved, when $0<\alpha<n$ and $m=1$.

When $m>1$, taking $\tilde{\alpha}:=\alpha-\frac{n}{p_m}$, then $0<\tilde{\alpha}-\sum_{j=1}^{m-1}\frac{n}{p_j}<\min\{1,n-\frac{n}{p_1}\}$. We assume that (2.4) holds for the case $m-1$; that is,

$$
\begin{aligned}
&\left|\prod_{j=1}^{m-1}\frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j-\prod_{j=1}^{m-1}\frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\right|\\
&\le Cr^{-\tilde{\alpha}}|x-y|^{\tilde{\alpha}-\sum_{j=1}^{m-1}\frac{n}{p_j}}\prod_{j=1}^{m-1}\|f_j\|_{M^{p_j}_{q_j}}.
\end{aligned}
\tag{2.6}
$$

The inequalities (2.5), (2.6) and $0<\alpha-\sum_{j=1}^{m}\frac{n}{p_j}<\min\{1,n-\frac{n}{p_1}\}$ follows that

$$
\left|\prod_{j=1}^{m}\frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j-\prod_{j=1}^{m}\frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\right|
$$

$$
\begin{aligned}
&\leq \left|\prod_{j=1}^{m}\frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j-\frac{1}{|B_2|}\int_{B_2}|f_m(y_m)|dy_m\prod_{j=1}^{m-1}\frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j\right|\\
&\quad+\left|\frac{1}{|B_2|}\int_{B_2}|f_m(y_m)|dy_m\prod_{j=1}^{m-1}\frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j-\prod_{j=1}^{m}\frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\right|\\
&\leq \left|\frac{1}{|B_1|}\int_{B_1}|f_m(y_m)|dy_m-\frac{1}{|B_2|}\int_{B_2}|f_m(y_m)|dy_m\right|\cdot\prod_{j=1}^{m-1}\frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j\\
&\quad+\frac{1}{|B_2|}\int_{B_2}|f_m(y_m)|dy_m\left|\prod_{j=1}^{m-1}\frac{1}{|B_1|}\int_{B_1}|f_j(y_j)|dy_j-\prod_{j=1}^{m-1}\frac{1}{|B_2|}\int_{B_2}|f_j(y_j)|dy_j\right|\\
&\leq Cr^{-n}|x-y|^{n-n/p_m}r^{-\sum_{j=1}^{m-1}\frac{n}{p_j}}\prod_{j=1}^{m}\|f_j\|_{M_{p_j}^{q_j}}+Cr^{-\frac{n}{p_m}}r^{-\tilde{\alpha}}|x-y|^{\tilde{\alpha}-\sum_{j=1}^{m-1}\frac{n}{p_j}}\\
&\prod_{j=1}^{m}\|f_j\|_{M_{q_j}^{p_j}}\\
&\leq Cr^{-\alpha}|x-y|^{\alpha-\sum_{j=1}^{m}\frac{n}{p_j}}\prod_{j=1}^{m}\|f_j\|_{M_{q_j}^{p_j}},
\end{aligned}
$$

which shows that (2.4) holds. □

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