

## ASYMPTOTIC EXACTNESS OF SOME BANK–WEISER ERROR ESTIMATOR FOR QUADRATIC TRIANGULAR FINITE ELEMENT

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ABSTRACT. We analyze a posteriori error estimator for the conforming  $P2$  finite element on triangular meshes which is based on the solution of local Neumann problems. This error estimator extends the one for the conforming  $P1$  finite element proposed in [4]. We prove that it is asymptotically exact for the Poisson equation when the underlying triangulations are mildly structured and the solution is smooth enough.

### 1. Introduction

In this work we propose and analyze a posteriori error estimator of the Bank–Weiser type for the conforming  $P2$  finite element method of the Poisson equation

$$(1) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded polygon with the boundary  $\partial\Omega$  and  $(f, g) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$  are given functions. This type of an error estimator was introduced by Bank and Weiser [4] for the conforming  $P1$  finite element. It is implicit and belongs to the class of element residual methods in that local Neumann problems are solved independently on each element using higher-order correction spaces to compute the error estimator. Some extensions were made to the lowest-order Raviart–Thomas element in [3] and also to the Stokes equation in [5, 10, 12, 16].

It is well described in [2, Section 3.1] why implicit a posteriori error estimators are often preferred over explicit ones. In particular, by utilizing the superconvergence result for the  $P1$  element (see, e.g., [19]), it was shown in [8, 13]

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that the Bank–Weiser error estimator for the  $P1$  element using quadratic correction spaces (composed of quadratic edge bubbles) is asymptotically exact, i.e., the effectivity index (which is measured by the ratio of the estimated error to the true error) approaches one as the mesh size tends to zero under proper conditions on the triangulations. We mention that other types of asymptotically exact error estimators have been obtained by gradient recovery methods [14, 20] and hierarchical basis error estimation [15].

In this paper, the Bank–Weiser error estimator is extended to the conforming  $P2$  finite element while retaining the property of asymptotic exactness under the usual assumptions that the triangulations are mildly structured (in the sense of Xu and Zhang [19]) and the solution  $u$  of the problem (1) is smooth enough. It has been found that the error estimator is sensitive to the choice of correction spaces used to solve local Neumann problems (see [1, 2] for some discussion in quadrilateral cases). In [11, 17] the authors considered some error estimators of the Bank–Weiser type for the conforming  $P2$  element whose  $P3$  correction spaces exclude only vertex degrees of freedom, but it turned out that they are not as effective as for the  $P1$  element. In this paper we choose a slightly smaller correction space than those of [11, 17] and show that it leads to an asymptotically exact error estimator. The argument is similar to but more complicated than those of [8, 13], and the superconvergence result for the quadratic finite element from [9] plays a crucial role.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and define the Bank–Weiser error estimator for the conforming  $P2$  finite element. We prove some super-closeness result for the moment-based quadratic interpolation in Section 3 and then establish the asymptotic exactness of the error estimator in Section 4. Finally, in Section 5, some numerical results are provided to illustrate the theoretical result.

## 2. Finite element method and error estimator

Let  $\mathcal{T}_h = \{T\}$  be a shape-regular partition of  $\Omega$  into triangles with the mesh size  $h = \max_{T \in \mathcal{T}_h} h_T$ , where  $h_T$  is the diameter of  $T$ . For a triangle  $T$ , we denote the unit outward normal vector to  $\partial T$  by  $\mathbf{n}_T = (n_1, n_2)$  and the unit tangent vector on  $\partial T$  by  $\mathbf{t}_T = (-n_2, n_1)$ . Let  $\omega_T$  be the union of at most three triangles of  $\mathcal{T}_h$  sharing an edge with  $T$ .

We will use the notation  $\frac{\partial v}{\partial \mathbf{z}} := \nabla v \cdot \mathbf{z}$  to denote the directional derivative of a function  $v$  along the unit vector  $\mathbf{z}$ . The jump and the mean value of the normal derivative of a piecewise smooth function  $v$  across an interior edge  $e = \partial T \cap \partial T'$  are defined as

$$\left[ \left[ \frac{\partial v}{\partial \mathbf{n}} \right] \right]_e = (\nabla v|_T - \nabla v|_{T'}) \cdot \mathbf{n}, \quad \left\langle \frac{\partial v}{\partial \mathbf{n}} \right\rangle_e = \frac{1}{2}(\nabla v|_T + \nabla v|_{T'}) \cdot \mathbf{n},$$

where  $\mathbf{n} = \mathbf{n}_T|_e$  or  $\mathbf{n}_{T'}|_e$ . For a boundary edge  $e \subset \partial\Omega \cap \partial T$ , we simply set

$$\left[ \left[ \frac{\partial v}{\partial \mathbf{n}} \right] \right]_e = 0, \quad \left\langle \frac{\partial v}{\partial \mathbf{n}} \right\rangle_e = \nabla v|_T \cdot \mathbf{n}_T.$$

The subscript  $e$  will be dropped whenever no confusion is likely to arise.

Let  $\mathbb{P}_k(T)$  be the space of all polynomials of degree at most  $k$  restricted to  $T$  and define the cubic *correction space* for  $\mathbb{P}_2(T)$  by

$$\mathbb{P}_3^0(T) = \{v \in \mathbb{P}_3(T) : v \text{ vanishes at vertices and midpoints of edges of } T\}.$$

It is easy to verify that  $\mathbb{P}_3(T) = \mathbb{P}_2(T) \oplus \mathbb{P}_3^0(T)$  and  $\mathbb{P}_3^0(T)$  is 4-dimensional with the basis functions

$$\psi_0 = \lambda_1 \lambda_2 \lambda_3, \quad \psi_i = \lambda_{i+1} \lambda_{i+2} (\lambda_{i+1} - \lambda_{i+2}) \quad (1 \leq i \leq 3),$$

where  $\lambda_i$ 's are barycentric coordinates on  $T$  (cf. [9]). By the scaling argument one can obtain

$$(2) \quad \|v\|_{0,T} + h_T^{1/2} \|v\|_{0,\partial T} \lesssim h_T \|\nabla v\|_{0,T} \quad \forall v \in \mathbb{P}_3^0(T).$$

Hereafter we will frequently use the notation  $a \lesssim b$  (resp.  $a \gtrsim b$ ) in place of the inequality  $a \leq Cb$  (resp.  $a \geq Cb$ ) with the constant  $C > 0$  independent of the mesh size  $h$ .

Now let  $u_h \in H^1(\Omega)$  be the conforming  $P_2$  finite element approximation to the solution  $u$  of the problem (1). Following the idea of Bank and Weiser [4], we consider the following error estimator based on solution of local problems.

**Definition.** For every  $T \in \mathcal{T}_h$ , let  $\varepsilon_T \in \mathbb{P}_3^0(T)$  be the solution of

$$(3) \quad (\nabla \varepsilon_T, \nabla v)_T = (f + \Delta u_h, v)_T - \frac{1}{2} \int_{\partial T} \left[ \left[ \frac{\partial u_h}{\partial \mathbf{n}_T} \right] \right] v \, ds \quad \forall v \in \mathbb{P}_3^0(T)$$

and set

$$(4) \quad \eta = \left( \sum_{T \in \mathcal{T}_h} \|\nabla \varepsilon_T\|_{0,T}^2 \right)^{1/2}.$$

Note that (3) represents a  $4 \times 4$  matrix system. Like in [2, 17], we may take  $v = \varepsilon_T$  in (3) and apply (2) to obtain

$$\|\nabla \varepsilon_T\|_{0,T} \lesssim h_T \|f + \Delta u_h\|_{0,T} + h_T^{1/2} \left\| \left[ \left[ \frac{\partial u_h}{\partial \mathbf{n}_T} \right] \right] \right\|_{0,\partial T}.$$

Combined with the local lower bound for standard residuals, this shows that  $\eta$  yields a local lower bound for the true error  $\|\nabla(u - u_h)\|_{0,\Omega}$ . The global upper bound of  $\eta$  may be obtained under the saturation assumption as was done in [4] for the  $P_1$  finite element method.

*Remark 2.1.* In [11] Liao considered some cubic and quartic correction spaces which are bigger than  $\mathbb{P}_3^0(T)$ . Numerical experiments show that the resulting error estimators are not asymptotically exact even on uniform triangulations.

### 3. Super-closeness result

This section is devoted to the proof of some super-closeness result which is an analogue of Lemma 7.1 in [13]. First we need the following result.

**Lemma 3.1.** *Let  $p_I \in \mathbb{P}_2([a, b])$  be the moment-based quadratic interpolation of  $p \in C([a, b])$  such that  $p_I(a) = p(a)$ ,  $p_I(b) = p(b)$  and  $\int_a^b p_I(x) dx = \int_a^b p(x) dx$ . If  $p \in \mathbb{P}_3([a, b])$ , then we have  $p''(\frac{a+b}{2}) = p''_I$ .*

*Proof.* By Simpson's rule we obtain  $p_I(m) = p(m)$ , where  $m = \frac{a+b}{2}$ , and thus

$$p(x) - p_I(x) = c(x - a)(x - b)(x - m)$$

for some constant  $c$ . This gives

$$p''(x) - p''_I = c\{6x - 2(a + b + m)\}$$

which vanishes if and only if  $x = m$ . □

The moment-based quadratic interpolation  $w_I$  of  $w \in H^2(\Omega)$  on a triangle  $T$  is defined in the same way, i.e.,  $w_I(\mathbf{z}) = w(\mathbf{z})$  for each vertex  $\mathbf{z}$  of  $T$  and  $\int_e w_I ds = \int_e w ds$  for each edge  $e$  of  $T$ . Following [19], we also say that two adjacent triangles  $T, T' \in \mathcal{T}_h$  form an  $O(h_T^{1+\alpha})$  approximate parallelogram if the lengths of two opposite edges of  $T \cup T'$  differ by  $O(h_T^{1+\alpha})$ . In this case it holds that  $||T| - |T'|| = O(h_T^{2+\alpha})$ , where  $|S|$  denotes the area of  $S$ .

**Lemma 3.2.** *Assume that two adjacent triangles  $T, T' \in \mathcal{T}_h$  form an  $O(h_T^{1+\alpha})$  approximate parallelogram. Let  $e = \partial T \cap \partial T'$  be their common edge with the midpoint  $\mathbf{m}_e$ ,  $\mathbf{n}_e = \mathbf{n}_T|_e$  and  $\mathbf{t}_e = \mathbf{t}_T|_e$ . If  $w \in \mathbb{P}_3(T \cup T')$ , then we have*

$$\left| \frac{\partial^2 w}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(\mathbf{m}_e) - \frac{\partial}{\partial \mathbf{t}_e} \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle_e \right| \lesssim h_T^{1+\alpha} |w|_{3, \infty, T \cup T'}.$$

*Proof.* Let  $\{e_k\}_{k=1}^3$  denote the edges of  $T$  with the edge lengths  $\{l_k\}_{k=1}^3$ , the midpoints  $\{\mathbf{m}_k\}_{k=1}^3$ , the unit outward normal vectors  $\{\mathbf{n}_k\}_{k=1}^3$  and the unit tangent vectors  $\{\mathbf{t}_k\}_{k=1}^3$  in the counterclockwise orientation. Similar notation applies to  $T'$  (see Figure 1). Assume that  $e = e_1 = e'_1$ , so that we have  $\mathbf{n}_e = \mathbf{n}_1 = -\mathbf{n}'_1$  and  $\mathbf{t}_e = \mathbf{t}_1 = -\mathbf{t}'_1$ .

We will crucially use the following identity (cf. (2.9) of [9])

$$(5) \quad 4|T| \frac{\partial^2}{\partial \mathbf{t}_k \partial \mathbf{n}_k} = (l_{k+2}^2 - l_{k+1}^2) \frac{\partial^2}{\partial \mathbf{t}_k^2} + l_{k+1}^2 \frac{\partial^2}{\partial \mathbf{t}_{k+1}^2} - l_{k+2}^2 \frac{\partial^2}{\partial \mathbf{t}_{k+2}^2}.$$

First, this gives

$$(6) \quad \begin{aligned} & 4(|T| + |T'|) \frac{\partial^2 w}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(\mathbf{m}_e) \\ &= 4|T| \frac{\partial^2 w}{\partial \mathbf{t}_1 \partial \mathbf{n}_1} \Big|_T(\mathbf{m}_1) + 4|T'| \frac{\partial^2 w}{\partial \mathbf{t}'_1 \partial \mathbf{n}'_1} \Big|_{T'}(\mathbf{m}'_1) \\ &= (l_3^2 - l_2^2) \frac{\partial^2 w}{\partial \mathbf{t}_1^2}(\mathbf{m}_1) + l_2^2 \frac{\partial^2 w}{\partial \mathbf{t}_2^2}(\mathbf{m}_1) - l_3^2 \frac{\partial^2 w}{\partial \mathbf{t}_3^2}(\mathbf{m}_1) \end{aligned}$$

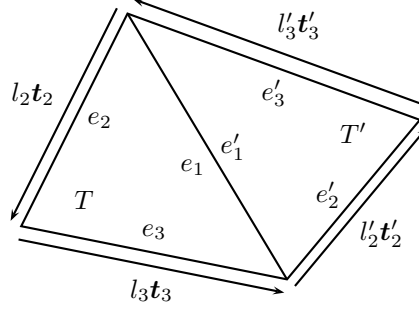


FIGURE 1. Two adjacent triangles  $T$  and  $T'$  with the common edge  $e = e_1 = e'_1$ .

$$+ (l_3'^2 - l_2'^2) \frac{\partial^2 w}{\partial \mathbf{t}_1'^2}(\mathbf{m}'_1) + l_2'^2 \frac{\partial^2 w}{\partial \mathbf{t}_2'^2}(\mathbf{m}'_1) - l_3'^2 \frac{\partial^2 w}{\partial \mathbf{t}_3'^2}(\mathbf{m}'_1).$$

Next we derive a similar formula for  $w_I$ . Applying the identity (5) again and then Lemma 3.1 on each edge  $e_i$  ( $1 \leq i \leq 3$ ) gives

$$\begin{aligned} 4|T| \left. \frac{\partial^2 w_I}{\partial \mathbf{t}_1 \partial \mathbf{n}_1} \right|_T &= (l_3^2 - l_2^2) \left. \frac{\partial^2 w_I}{\partial \mathbf{t}_1^2} \right|_T + l_2^2 \left. \frac{\partial^2 w_I}{\partial \mathbf{t}_2^2} \right|_T - l_3^2 \left. \frac{\partial^2 w_I}{\partial \mathbf{t}_3^2} \right|_T \\ &= (l_3^2 - l_2^2) \frac{\partial^2 w}{\partial \mathbf{t}_1^2}(\mathbf{m}_1) + l_2^2 \frac{\partial^2 w}{\partial \mathbf{t}_2^2}(\mathbf{m}_2) - l_3^2 \frac{\partial^2 w}{\partial \mathbf{t}_3^2}(\mathbf{m}_3). \end{aligned}$$

Similarly, we obtain

$$4|T'| \left. \frac{\partial^2 w_I}{\partial \mathbf{t}_1' \partial \mathbf{n}_1'} \right|_{T'} = (l_3'^2 - l_2'^2) \frac{\partial^2 w}{\partial \mathbf{t}_1'^2}(\mathbf{m}'_1) + l_2'^2 \frac{\partial^2 w}{\partial \mathbf{t}_2'^2}(\mathbf{m}'_2) - l_3'^2 \frac{\partial^2 w}{\partial \mathbf{t}_3'^2}(\mathbf{m}'_3).$$

With the abbreviation for the area-weighted average

$$\left\{ \frac{\partial w_I}{\partial \mathbf{n}_e} \right\}_e := \frac{|T|}{|T| + |T'|} \left. \frac{\partial w_I}{\partial \mathbf{n}_e} \right|_T + \frac{|T'|}{|T| + |T'|} \left. \frac{\partial w_I}{\partial \mathbf{n}_e} \right|_{T'},$$

it follows that

$$\begin{aligned} (7) \quad & 4(|T| + |T'|) \frac{\partial}{\partial \mathbf{t}_e} \left\{ \frac{\partial w_I}{\partial \mathbf{n}_e} \right\}_e \\ &= 4|T| \left. \frac{\partial^2 w_I}{\partial \mathbf{t}_1 \partial \mathbf{n}_1} \right|_T + 4|T'| \left. \frac{\partial^2 w_I}{\partial \mathbf{t}_1' \partial \mathbf{n}_1'} \right|_{T'} \\ &= (l_3^2 - l_2^2) \frac{\partial^2 w}{\partial \mathbf{t}_1^2}(\mathbf{m}_1) + l_2^2 \frac{\partial^2 w}{\partial \mathbf{t}_2^2}(\mathbf{m}_2) - l_3^2 \frac{\partial^2 w}{\partial \mathbf{t}_3^2}(\mathbf{m}_3) \\ &\quad + (l_3'^2 - l_2'^2) \frac{\partial^2 w}{\partial \mathbf{t}_1'^2}(\mathbf{m}'_1) + l_2'^2 \frac{\partial^2 w}{\partial \mathbf{t}_2'^2}(\mathbf{m}'_2) - l_3'^2 \frac{\partial^2 w}{\partial \mathbf{t}_3'^2}(\mathbf{m}'_3). \end{aligned}$$

Comparing (6) and (7), we can see that

$$\frac{\partial^2 w}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(\mathbf{m}_e) - \frac{\partial}{\partial \mathbf{t}_e} \left\{ \frac{\partial w_I}{\partial \mathbf{n}_e} \right\}_e = \frac{I_2 - I_3}{4(|T| + |T'|)},$$

where

$$\begin{aligned} I_2 &= l_2^2 \frac{\partial^2 w}{\partial \mathbf{t}_2^2}(\mathbf{m}_1) - l_2^2 \frac{\partial^2 w}{\partial \mathbf{t}_2^2}(\mathbf{m}_2) + l_2'^2 \frac{\partial^2 w}{\partial \mathbf{t}_2'^2}(\mathbf{m}'_1) - l_2'^2 \frac{\partial^2 w}{\partial \mathbf{t}_2'^2}(\mathbf{m}'_2), \\ I_3 &= l_3^2 \frac{\partial^2 w}{\partial \mathbf{t}_3^2}(\mathbf{m}_1) - l_3^2 \frac{\partial^2 w}{\partial \mathbf{t}_3^2}(\mathbf{m}_3) + l_3'^2 \frac{\partial^2 w}{\partial \mathbf{t}_3'^2}(\mathbf{m}'_1) - l_3'^2 \frac{\partial^2 w}{\partial \mathbf{t}_3'^2}(\mathbf{m}'_3). \end{aligned}$$

Note that

$$\begin{aligned} I_2 &= \sum_{1 \leq i, j \leq 2} \left\{ \left( \frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{m}_1) - \frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{m}_2) \right) \tilde{t}_2^i \tilde{t}_2^j \right. \\ &\quad \left. + \left( \frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{m}'_1) - \frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{m}'_2) \right) \tilde{t}_2'^i \tilde{t}_2'^j \right\}, \end{aligned}$$

where

$$\tilde{\mathbf{t}}_r = (\tilde{t}_r^1, \tilde{t}_r^2) = l_r \mathbf{t}_r, \quad \tilde{\mathbf{t}}_r' = (\tilde{t}_r'^1, \tilde{t}_r'^2) = l_r' \mathbf{t}_r'.$$

Since we have  $\frac{\partial^2 w}{\partial x_i \partial x_j} \in \mathbb{P}_1(T \cup T')$ ,  $\mathbf{m}_1 - \mathbf{m}_2 = \frac{1}{2} \tilde{\mathbf{t}}_3$  and  $\mathbf{m}'_1 - \mathbf{m}'_2 = \frac{1}{2} \tilde{\mathbf{t}}_3'$ , it follows that

$$\begin{aligned} \frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{m}_1) - \frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{m}_2) &= \frac{1}{2} \sum_{1 \leq k \leq 2} \frac{\partial^3 w}{\partial x_i \partial x_j \partial x_k} \tilde{t}_3^k, \\ \frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{m}'_1) - \frac{\partial^2 w}{\partial x_i \partial x_j}(\mathbf{m}'_2) &= \frac{1}{2} \sum_{1 \leq k \leq 2} \frac{\partial^3 w}{\partial x_i \partial x_j \partial x_k} \tilde{t}_3'^k. \end{aligned}$$

Consequently,

$$\begin{aligned} |I_2| &= \left| \frac{1}{2} \sum_{1 \leq i, j, k \leq 2} \frac{\partial^3 w}{\partial x_i \partial x_j \partial x_k} (\tilde{t}_2^i \tilde{t}_2^j \tilde{t}_3^k + \tilde{t}_2'^i \tilde{t}_2'^j \tilde{t}_3'^k) \right| \\ &\leq \frac{1}{2} |w|_{3, \infty, T \cup T'} \sum_{1 \leq i, j, k \leq 2} |\tilde{t}_2^i \tilde{t}_2^j \tilde{t}_3^k + \tilde{t}_2'^i \tilde{t}_2'^j \tilde{t}_3'^k|. \end{aligned}$$

Using the estimates  $|\tilde{\mathbf{t}}_r + \tilde{\mathbf{t}}_r'| \lesssim h_T^{1+\alpha}$  and  $|\tilde{\mathbf{t}}_r| + |\tilde{\mathbf{t}}_r'| \lesssim h_T$ , one can show that for  $1 \leq i, j, k \leq 2$ ,

$$|\tilde{t}_2^i \tilde{t}_2^j \tilde{t}_3^k + \tilde{t}_2'^i \tilde{t}_2'^j \tilde{t}_3'^k| \lesssim h_T^{3+\alpha},$$

which results in

$$|I_2| \lesssim h_T^{3+\alpha} |w|_{3, \infty, T \cup T'}.$$

Similarly, we obtain

$$|I_3| \lesssim h_T^{3+\alpha} |w|_{3, \infty, T \cup T'}.$$

These two results imply that

$$\left| \frac{\partial^2 w}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(\mathbf{m}_e) - \frac{\partial}{\partial \mathbf{t}_e} \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle_e \right| \lesssim h_T^{1+\alpha} |w|_{3,\infty,T \cup T'}.$$

Finally, following the last part of the proof of Lemma 7.1 in [13], we obtain

$$\begin{aligned} \left| \frac{\partial}{\partial \mathbf{t}_e} \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle_e - \frac{\partial}{\partial \mathbf{t}_e} \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle_e \right| &\leq \frac{||T| - |T'|}{2(|T| + |T'|)} \left| \frac{\partial^2 w_I}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \Big|_T - \frac{\partial^2 w_I}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \Big|_{T'} \right| \\ &\lesssim h_T^\alpha \cdot h_T |w|_{3,\infty,T \cup T'} = h_T^{1+\alpha} |w|_{3,\infty,T \cup T'}. \end{aligned}$$

This completes the proof. □

#### 4. Asymptotic exactness of error estimator

In this section we assume that the triangulation  $\mathcal{T}_h$  is mildly structured in the following sense (cf. [6, 9, 19]).

**Condition**  $(\alpha, \sigma)$ : There exists a partition of  $\mathcal{T}_h$  into two disjoint sets  $\mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}$  and positive constants  $\alpha, \sigma$  such that

- any two adjacent triangles  $T, T' \in \mathcal{T}_{1,h}$  form an  $O(h_T^{1+\alpha})$  approximate parallelogram;
- $\sum_{T \in \mathcal{T}_{2,h}} |T| \lesssim h^\sigma$ .

Under these conditions, the following superconvergence result was shown in [9]

$$(8) \quad \|\nabla(u_I - u_h)\|_{0,\Omega} \lesssim h^{2+\rho} (\|u\|_{4,\Omega} + |u|_{3,\infty,\Omega}),$$

where  $\rho = \min(\alpha, \sigma/2, 1/2)$  and  $u_I$  denotes the moment-based quadratic interpolation of  $u$ .

Now we adapt the proofs of [8, 13] to establish the asymptotic exactness of the error estimator  $\eta$  defined by (4) under the condition  $(\alpha, \sigma)$ . For given  $w \in H^2(\Omega)$ , we define an auxiliary function  $q_w|_T \in \mathbb{P}_3^0(T)$  (similar to  $\varepsilon_T$  defined by (3)) on each element  $T \in \mathcal{T}_h$  to be the solution of

$$(9) \quad (\nabla q_w, \nabla v)_T = -(\Delta w, v)_T + \int_{\partial T} \left\langle \frac{\partial w_I}{\partial \mathbf{n}_T} \right\rangle v \, ds - (\nabla w_I, \nabla v)_T$$

for all  $v \in \mathbb{P}_3^0(T)$ . Using integration by parts leads to

$$(10) \quad (\nabla q_w, \nabla v)_T = (\nabla(w - w_I), \nabla v)_T + \int_{\partial T} \left( \left\langle \frac{\partial w_I}{\partial \mathbf{n}_T} \right\rangle - \frac{\partial w}{\partial \mathbf{n}_T} \right) v \, ds.$$

Taking  $v = q_w$ , applying the Cauchy–Schwarz inequality and then using (2), we obtain for  $w \in H^3(\omega_T)$

$$\|\nabla q_w\|_{0,T} \lesssim \|\nabla(w - w_I)\|_{0,T} + h_T^{1/2} \left\| \left\langle \frac{\partial w_I}{\partial \mathbf{n}_T} \right\rangle - \frac{\partial w}{\partial \mathbf{n}_T} \right\|_{0,\partial T},$$

and thus

$$(11) \quad \|\nabla(w - w_I - q_w)\|_{0,T} \leq \|\nabla(w - w_I)\|_{0,T} + \|\nabla q_w\|_{0,T} \lesssim h_T^2 |w|_{3,\omega_T}$$

by the standard interpolation estimates. In the following lemma this result is strengthened when  $T$  lies in the interior of  $\Omega_{1,h} = \bigcup_{T \in \mathcal{T}_{1,h}} \bar{T}$ .

**Lemma 4.1.** *Assume that  $T \in \mathcal{T}_h$  has no boundary edges and  $\omega_T \subset \Omega_{1,h}$ . If  $u \in H^4(\omega_T)$ , then we have*

$$\|\nabla(u - u_I - q_u)\|_{0,T} \lesssim h_T^{2+\min(\alpha,1)} \|u\|_{4,\omega_T}.$$

*Proof.* First we show that

$$(12) \quad \|\nabla(w - w_I - q_w)\|_{0,T} \lesssim h_T^{2+\alpha} |w|_{3,\omega_T} \quad \forall w \in \mathbb{P}_3(\omega_T).$$

Let  $\mathbf{n}_e = \mathbf{n}_T|_e$  and  $\mathbf{t}_e = \mathbf{t}_T|_e$ , and write the equality (10) as

$$(13) \quad (\nabla(w - w_I - q_w), \nabla v)_T = \sum_{e \subset \partial T} \int_e \left( \frac{\partial w}{\partial \mathbf{n}_e} - \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle \right) v ds.$$

For each edge  $e \subset \partial T$  parametrized by  $s \in [-l_e/2, l_e/2]$ , where  $l_e$  is the length of  $e$ , we consider the following Taylor expansions

$$\frac{\partial w}{\partial \mathbf{n}_e} \Big|_e = \frac{\partial w}{\partial \mathbf{n}_e}(\mathbf{m}_e) + s \frac{\partial}{\partial \mathbf{t}_e} \left( \frac{\partial w}{\partial \mathbf{n}_e} \right)(\mathbf{m}_e) + \frac{s^2}{2} \frac{\partial^2}{\partial \mathbf{t}_e^2} \left( \frac{\partial w}{\partial \mathbf{n}_e} \right)(\mathbf{m}_e)$$

and

$$\left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle_e = \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle_e(\mathbf{m}_e) + s \frac{\partial}{\partial \mathbf{t}_e} \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle_e.$$

Since  $v|_e$  is odd about  $\mathbf{m}_e$  for  $v \in \mathbb{P}_3^0(T)$ , it follows that

$$\int_e \left( \frac{\partial w}{\partial \mathbf{n}_e} - \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle \right) v ds = \left( \frac{\partial^2 w}{\partial \mathbf{t}_e \partial \mathbf{n}_e}(\mathbf{m}_e) - \frac{\partial}{\partial \mathbf{t}_e} \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle_e \right) \int_e s v ds.$$

Consequently, by Lemma 3.2, the estimate (2) and the local inverse inequality, we obtain

$$\begin{aligned} \left| \int_e \left( \frac{\partial w}{\partial \mathbf{n}_e} - \left\langle \frac{\partial w_I}{\partial \mathbf{n}_e} \right\rangle \right) v ds \right| &\lesssim h_T^{1+\alpha} |w|_{3,\infty,\omega_T} \left( \int_{-l_e/2}^{l_e/2} s^2 ds \right)^{1/2} \|v\|_{0,e} \\ &\lesssim h_T^\alpha |w|_{3,\omega_T} \cdot h_T^{3/2} \cdot h_T^{1/2} \|\nabla v\|_{0,T}. \end{aligned}$$

This gives (12) by taking  $v = w - w_I - q_w$  in (13) (note that  $w - w_I \in \mathbb{P}_3^0(T)$  for  $w \in \mathbb{P}_3(T)$ ).

Now choose  $\phi \in \mathbb{P}_3(\omega_T)$  such that  $\|u - \phi\|_{3,\omega_T} \lesssim h_T \|u\|_{4,\omega_T}$  (see [7] for existence of such a polynomial). Then it follows from (11) and (12) that

$$\begin{aligned} \|\nabla(u - u_I - q_u)\|_{0,T} &\leq \|\nabla(u - \phi - (u - \phi)_I - q_{u-\phi})\|_{0,T} \\ &\quad + \|\nabla(\phi - \phi_I - q_\phi)\|_{0,T} \\ &\lesssim h_T^2 |u - \phi|_{3,\omega_T} + h_T^{2+\alpha} |\phi|_{3,\omega_T} \\ &\lesssim h_T^2 |u - \phi|_{3,\omega_T} + h_T^{2+\alpha} |\phi - u|_{3,\omega_T} + h_T^{2+\alpha} |u|_{3,\omega_T} \\ &\lesssim (h_T^3 + h_T^{2+\alpha}) \|u\|_{4,\omega_T}, \end{aligned}$$

which proves the desired result.  $\square$



In the following two theorems we present the main results of this paper.

**Theorem 4.2.** *Assume that the triangulation  $\mathcal{T}_h$  satisfies the condition  $(\alpha, \sigma)$ . Then we have with  $\rho = \min(\alpha, \sigma/2, 1/2)$*

$$\left( \sum_{T \in \mathcal{T}_h} \|\nabla(u - u_h - \varepsilon_T)\|_{0,T}^2 \right)^{1/2} \lesssim h^{2+\rho} (\|u\|_{4,\Omega} + |u|_{3,\infty,\Omega}).$$

*Proof.* We split  $u - u_h - \varepsilon_T$  into three terms

$$u - u_h - \varepsilon_T = (u - u_I - q_u) + (q_u - \varepsilon_T) + (u_I - u_h).$$

Let  $\tilde{\mathcal{T}}_{1,h} = \{T \in \mathcal{T}_h : \partial T \cap \partial\Omega = \emptyset \text{ and } \omega_T \subset \Omega_{1,h}\}$ . By Lemma 4.1 we obtain

$$\sum_{T \in \tilde{\mathcal{T}}_{1,h}} \|\nabla(u - u_I - q_u)\|_{0,T}^2 \lesssim h^{4+2\min(\alpha,1)} \|u\|_{4,\Omega}^2,$$

while the estimate (11) and the condition  $(\alpha, \sigma)$  lead to

$$\begin{aligned} \sum_{T \in \mathcal{T}_h \setminus \tilde{\mathcal{T}}_{1,h}} \|\nabla(u - u_I - q_u)\|_{0,T}^2 &\lesssim h^4 \sum_{T \in \mathcal{T}_h \setminus \tilde{\mathcal{T}}_{1,h}} |u|_{3,\omega_T}^2 \\ &\lesssim h^4 \left( \sum_{T \in \mathcal{T}_h \setminus \tilde{\mathcal{T}}_{1,h}} |\omega_T| \right) |u|_{3,\infty,\Omega}^2 \\ &\lesssim h^{4+\min(\sigma,1)} |u|_{3,\infty,\Omega}^2. \end{aligned}$$

Next we handle the second term for arbitrary  $T \in \mathcal{T}_h$ . By definitions (3) and (9) we have for  $v \in \mathbb{P}_3^0(T)$

$$(\nabla(q_u - \varepsilon_T), \nabla v)_T = (\Delta(u_I - u_h), v)_T - \frac{1}{2} \int_{\partial T} \left[ \frac{\partial(u_I - u_h)}{\partial \mathbf{n}_T} \right] v \, ds.$$

Then we use the estimate (2) and the local inverse inequalities to obtain

$$\begin{aligned} (\nabla(q_u - \varepsilon_T), \nabla v)_T &\lesssim \|\Delta(u_I - u_h)\|_{0,T} \|v\|_{0,T} + \left\| \left[ \frac{\partial(u_I - u_h)}{\partial \mathbf{n}_T} \right] \right\|_{0,\partial T} \|v\|_{0,\partial T} \\ &\lesssim \|\nabla(u_I - u_h)\|_{0,\omega_T} \|\nabla v\|_{0,T}, \end{aligned}$$

which gives by taking  $v = q_u - \varepsilon_T$

$$\|\nabla(q_u - \varepsilon_T)\|_{0,T} \lesssim \|\nabla(u_I - u_h)\|_{0,\omega_T}.$$

By collecting the above results, it follows that

$$\begin{aligned} \left( \sum_{T \in \mathcal{T}_h} \|\nabla(u - u_h - \varepsilon_T)\|_{0,T}^2 \right)^{1/2} &\lesssim h^{2+\min(\alpha,1)} \|u\|_{4,\Omega} + h^{2+\frac{1}{2}\min(\sigma,1)} |u|_{3,\infty,\Omega} \\ &\quad + \|\nabla(u_I - u_h)\|_{0,\Omega}. \end{aligned}$$

The proof is completed by invoking the superconvergence result (8).  $\square$

As a direct consequence of Theorem 4.2, we get the asymptotic exactness of the error estimator  $\eta$  defined by (4).

**Theorem 4.3.** Assume that the triangulation  $\mathcal{T}_h$  satisfies the condition  $(\alpha, \sigma)$  and the error  $\|\nabla(u - u_h)\|_{0,\Omega}$  is properly  $O(h^2)$ , i.e.,

$$\|\nabla(u - u_h)\|_{0,\Omega} \geq C_1(u)h^2$$

with some constant  $C_1(u) > 0$ . Then we have with  $\rho = \min(\alpha, \sigma/2, 1/2)$

$$\left| \frac{\eta}{\|\nabla(u - u_h)\|_{0,\Omega}} - 1 \right| \lesssim h^\rho (\|u\|_{4,\Omega} + |u|_{3,\infty,\Omega}) / C_1(u).$$

*Proof.* By Theorem 4.2 we have

$$\begin{aligned} \left| \|\nabla(u - u_h)\|_{0,\Omega} - \eta \right| &\leq \left( \sum_{T \in \mathcal{T}_h} \|\nabla(u - u_h) - \nabla \varepsilon_T\|_{0,T}^2 \right)^{1/2} \\ &\lesssim h^{2+\rho} (\|u\|_{4,\Omega} + |u|_{3,\infty,\Omega}), \end{aligned}$$

and thus

$$\left| \frac{\eta}{\|\nabla(u - u_h)\|_{0,\Omega}} - 1 \right| = \left| \frac{\eta - \|\nabla(u - u_h)\|_{0,\Omega}}{\|\nabla(u - u_h)\|_{0,\Omega}} \right| \lesssim \frac{h^{2+\rho} (\|u\|_{4,\Omega} + |u|_{3,\infty,\Omega})}{C_1(u)h^2}.$$

This completes the proof.  $\square$

## 5. Numerical results

In this section some numerical results are reported to illustrate the asymptotic exactness of the error estimator  $\eta$ . We solve the Poisson equation (1) with the known exact solution  $u$  and the Dirichlet boundary condition is approximated using the quadratic interpolation of  $u|_{\partial\Omega}$ .

**Example 1.** The first example is taken from [11] and is posed on the unit square  $\Omega = (0, 1)^2$  for which the exact solution is

$$u(x, y) = x^4 + y^4.$$

The initial triangulation is shown in the left of Fig. 2. Successive triangulations are created by uniform refinement which divides every triangle into four congruent subtriangles; see Fig. 2 for triangulations of refinement level 1 and 2. It is well known that this uniform refinement makes the triangulations satisfy the condition  $(\alpha, \sigma)$  with  $\alpha = 2$  and  $\sigma = 1$ .

The numerical results are displayed in Table 1 which lists the values of the  $H^1$  error  $\|\nabla(u - u_h)\|_{0,\Omega}$ , the error estimator  $\eta = \left( \sum_{T \in \mathcal{T}_h} \|\nabla \varepsilon_T\|_{0,T}^2 \right)^{1/2}$  and the effectivity index  $\theta = \eta / \|\nabla(u - u_h)\|_{0,\Omega}$ . It is evident that  $\theta$  tends to one and thus  $\eta$  is asymptotically exact as the triangulation becomes more and more refined. We remark that the effectivity indices of the error estimators presented in [11] stay near 2.7 and 1.26 for  $P3$  and  $P4$  correction spaces, respectively.

**Example 2.** The second example is concerned with the singular solution

$$u(r, \theta) = r^{\frac{2}{3}} \sin \frac{2}{3}\theta$$

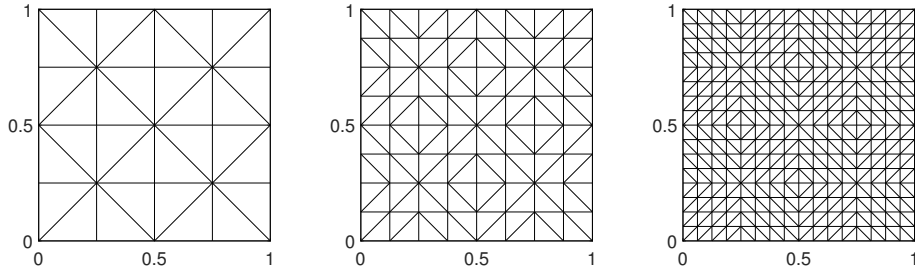


FIGURE 2. Uniform triangulations of refinement levels 0, 1, 2 for Example 1

TABLE 1.  $H^1$  errors, error estimators and effectivity indices for Example 1

level	$\ \nabla(u - u_h)\ _{0,\Omega}$	$\eta$	$\theta$
0	4.2377e-2	4.0091e-2	0.9461
1	1.0957e-2	1.0681e-2	0.9748
2	2.7942e-3	2.7606e-3	0.9880
3	7.0576e-4	7.0164e-4	0.9942
4	1.7736e-4	1.7685e-4	0.9971
5	4.4457e-5	4.4393e-5	0.9986
6	1.1129e-5	1.1121e-5	0.9993

on the L-shaped domain  $\Omega = (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ , where  $(r, \theta)$  denotes the polar coordinates. Starting with the initial triangulation shown in the left of Fig. 3, we consider two sequences of triangulations by uniform and adaptive refinement, respectively. The adaptive refinement is performed on those triangles of  $\mathcal{T}_h$  (with some neighboring triangles to avoid hanging nodes) such that

$$\eta_T > \frac{1}{2} \max_{T' \in \mathcal{T}_h} \eta_{T'}, \quad \eta_T = \|\nabla \varepsilon_T\|_{0,T}.$$

Two of the triangulations generated during adaptive refinement are shown in the middle and right of Fig. 3.

Fig. 4 plots the effectivity index  $\theta = \eta / \|\nabla(u - u_h)\|_{0,\Omega}$  with respect to the number of unknowns. For comparison purpose we also display the effectivity index of the error estimator  $\eta^*$  of [11] using the  $P_3$  correction space larger than  $\eta$ . It is observed that neither of  $\eta$  and  $\eta^*$  is asymptotically exact on uniform triangulations due to the singularity of  $u$  at the origin. But, when the singularity is resolved by adaptive refinement, our error estimator  $\eta$  seems to be asymptotically exact as the number of unknowns grows, while  $\eta^*$  is not.

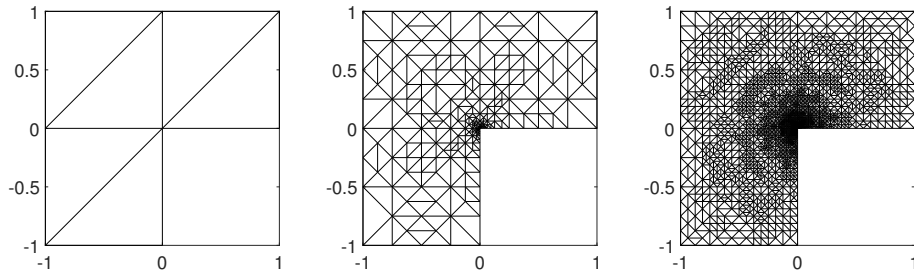


FIGURE 3. Initial and two adapted triangulations for Example 2

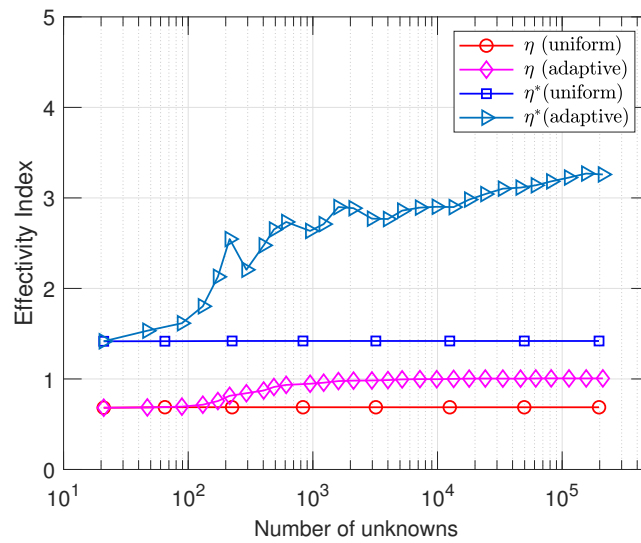


FIGURE 4. Effectivity indices on uniform and adapted triangulations for Example 2

We remark that a possible explanation for this numerical observation may be given using the theory of [18] which introduced the condition  $(\alpha, \sigma, \mu)$  to deal with adaptive triangulations near the point singularity. However, it still remains (at least theoretically) open whether adaptive refinement produces a sequence of triangulations satisfying the condition  $(\alpha, \sigma, \mu)$  and thus further investigation is necessary to explain the numerical results obtained above.

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