

ON THE FIRST GENERALIZED HILBERT COEFFICIENT AND DEPTH OF ASSOCIATED GRADED RINGS

AMIR MAFI AND DLER NADERI

ABSTRACT. Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an ideal of R that has analytic spread $\ell(I) = d$, satisfies the G_d condition, the weak Artin-Nagata property AN_{d-2}^- and \mathfrak{m} is not an associated prime of R/I . In this paper, we show that if $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)] + 1$, then I has almost minimal j -multiplicity, $G(I)$ is Cohen-Macaulay and $r_J(I)$ is at most 2, where $J = (x_1, \dots, x_d)$ is a general minimal reduction of I and $J_i = (x_1, \dots, x_i)$. In addition, the last theorem is in the spirit of a result of Sally who has studied the depth of associated graded rings and minimal reductions for \mathfrak{m} -primary ideals.

1. Introduction

Let (R, \mathfrak{m}, k) be a Noetherian Cohen-Macaulay local ring of dimension d with maximal ideal \mathfrak{m} and infinite residue field k . For an ideal I of R , we will denote by $G(I) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ the associated graded algebra of I , and by $\mathcal{F}(I) = \bigoplus_{n=0}^{\infty} I^n/\mathfrak{m}I^n$ the fiber cone of I , that reflect various algebraic and geometric properties of the ideal I . The dimension of $G(I)$ is always d and the dimension of $\mathcal{F}(I)$ is called the analytic spread of I and is denoted by $\ell(I)$. The associated graded rings are indeed interesting since they encode many useful properties. Provided that both the ideal I and the ring R satisfy some good properties, the associated graded ring of I reveals itself as a key tool in the study of many invariants such as the depth or the regularity mainly by means of the Hilbert polynomial of I , and its Hilbert coefficients. For classes of ideals that are not necessarily \mathfrak{m} -primary, the concept of the j -multiplicity was introduced as a generalization of the Hilbert multiplicity in 1993 by Achilles and Manaresi [1] and further this topic was studied in [2], [3] and [4]. Polini and Xie in [14] investigate the depth of the associated graded ring of an arbitrary ideal using the j -multiplicity. On the other hand, the numerical information on the Hilbert coefficient has been used to obtain information on the depth

Received March 15, 2019; Revised August 23, 2019; Accepted September 19, 2019.

2010 *Mathematics Subject Classification.* 13A30, 13D40, 13H15, 13C14.

Key words and phrases. Generalized Hilbert coefficient, minimal reduction, associated graded ring.

of associated graded ring. For instance, Northcott in [11] showed that the bound $e_1(I) \geq e_0(I) - \lambda(R/I) = \lambda(R/J) - \lambda(R/I) = \lambda(I/J)$, always holds for any minimal reduction J of I . When equality holds, the ideal I enjoys nice properties. Indeed, it was shown that $e_1(I) = \lambda(I/J)$ if and only if the reduction number of I is at most 1, and when this is the case, the associated graded ring of I is Cohen-Macaulay (see [8] and [13]). Xie in [19] generalized Northcott's inequality to R -ideal as follows:

$$j_1(I) \geq \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$$

and she proved that $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$ if and only if $r(I) \leq 1$ and when this is the case, the associated graded ring of I is Cohen-Macaulay. Sally in [16] studied \mathfrak{m} -primary ideal I satisfying the condition $e_1(I) = \lambda(I/J) + 1$, with $e_2(I) > 0$ if $d \geq 2$ and she proved that if $e_1(I) = \lambda(I/J) + 1$, then $\text{depth}(G(I)) \geq d - 1$ and $r_J(I) \leq 2$ for any minimal reduction J of I . This paper generalizes the above results to ideals of maximal analytic spread. In Section 1, we recall the definition of generalized Hilbert-samuel function of I and residual intersections. In Section 2, we prove that if (R, \mathfrak{m}) is a Cohen-Macaulay local ring, I is a non \mathfrak{m} -primary R -ideal which satisfies $\ell(I) = d$, the G_d condition, the AN_{d-2}^- and $\text{depth}(R/I) \geq \min\{1, \dim R/I\}$, then for a general minimal reduction $J = (x_1, \dots, x_d)$ of I and $J_i = (x_1, \dots, x_i)$, one has that $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$. When $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)] + 1$ then $\text{depth}(G(I)) \geq d - 1$ and $r_J(I) \leq 2$ for any general minimal reduction $J = (x_1, \dots, x_d)$ of I .

2. Preliminary

In this paper, we always assume that (R, \mathfrak{m}, k) is a Noetherian local ring of dimension d with maximal ideal \mathfrak{m} and infinite residue field k . Northcott and Rees [12] defined a minimal reduction of I to be a d -generated ideal $J \subseteq I$ of R such that $JI^n = I^{n+1}$ for some nonnegative integers n . We denote $r_J(I) = \min\{n \in \mathbb{Z} : JI^n = I^{n+1}\}$. The reduction number $r(I)$ defined as $r(I) = \min\{r_J(I) : J \text{ is minimal reduction of } I\}$. The reduction number $r(I)$ is said to be independent if $r(I) = r_J(I)$ for all minimal reduction J of I .

Let $I = (a_1, \dots, a_t)$ and write $x_i = \sum_{j=1}^t \lambda_{ij} a_j$ for $1 \leq i \leq s$ and $\lambda_{ij} \in R^{st}$. The elements x_1, \dots, x_s form a sequence of general elements in I (equivalently x_1, \dots, x_s are general in I) if there exists a Zariski dense open subset U of k^{st} such that the image $(\bar{\lambda}_{ij}) \in U$. When $s = 1$, $x = x_1$ is said to be general in I . Furthermore, general $\ell(I)$ elements in I form a minimal reduction J whose $r_J(I)$ coincides with the reduction number $r(I)$ (see [9] or [17]). One says that J is a general minimal reduction of I if it is generated by $\ell(I)$ general elements in I .

Let $G(I) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ be the associated graded ring of I . Since the homogeneous components of $G(I)$ may not have finite length, so we consider

the ideal of $G(I)$ of elements supported on \mathfrak{m} and defined as follow:

$$W = \{\xi \in G(I) \mid \exists s > 0 \text{ such that } \xi \cdot \mathfrak{m}^s = 0\} = H_{\mathfrak{m}}^0(G) = \bigoplus_{n=0}^{\infty} H_{\mathfrak{m}}^0(I^n/I^{n+1}).$$

Since W is annihilated by a large power of \mathfrak{m} , it is a finite graded module over $G(I) \otimes_R R/\mathfrak{m}^t$ for some $t \geq 0$, hence its Hilbert-Samuel function

$$H_I(n) = H_W(n) = \sum_{j=0}^n \lambda(H_{\mathfrak{m}}^0(I^j/I^{j+1}))$$

and Hilbert-Samuel polynomial

$$P_I(n) = \sum_{i=0}^d (-1)^i j_i(I) \binom{n+d-i}{d-i}$$

are well defined. Polini and Xie defined $P_I(n)$ to be the generalized Hilbert-Samuel polynomial of I and $j_i(I)$, $0 \leq i \leq d$, the generalized Hilbert coefficients of I . The normalized leading coefficient $j_0(I)$ is called the j -multiplicity of I (see [1] or [15]). The next normalized coefficient $j_1(I)$ is called the generalized first Hilbert coefficient. Xie provided a formula relating the length $\lambda(I^{n+1}/JI^n)$ to the difference $P_I(n) - H_I(n)$, where I is an R -ideal (generalization of fundamental Lemma of Huneke [8, Lemma 2.4] and extension of this Lemma by Huckaba [6, Theorem 2.4]) and found a formula for the first generalized Hilbert coefficient (see [19, Theorem 3.2, Corollary 3.3]). If I is \mathfrak{m} -primary, each homogeneous component of $G(I)$ has finite length, thus $W = G(I)$ and the generalized Hilbert-Samuel function coincides with the usual Hilbert-Samuel function; in particular, the generalized Hilbert coefficients $j_i(I)$, $0 \leq i \leq d$, coincide with the usual Hilbert coefficients $e_i(I)$. We now recall some definitions and facts from the theory of residual intersections which will be used frequently in the rest of the paper.

- (i) The ideal I is said to satisfy the G_s condition if for every $\mathfrak{p} \in V(I)$ with $\text{height } \mathfrak{p} = i < s$, the ideal $I_{\mathfrak{p}}$ is generated by i elements, i.e., $I_{\mathfrak{p}} = (x_1, \dots, x_i)_{\mathfrak{p}}$ for some x_1, \dots, x_i in I .
- (ii) Let $J_s = (x_1, \dots, x_s)$, where x_1, \dots, x_s are elements in I . Then $J_s : I$ is called an s -residual intersection of I if $I_{\mathfrak{p}} = (x_1, \dots, x_s)_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq s - 1$.
- (iii) An s -residual intersection $J_s : I$ is called geometric if $I_{\mathfrak{p}} = (x_1, \dots, x_s)_{\mathfrak{p}}$ for every $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}}$ at most s .
- (iv) Let R be Cohen-Macaulay, the ideal I has the Artin-Nagata property AN_s^- if, for every $0 \leq i \leq s$ and every geometric i -residual intersection $J_i : I$ of I , one has that $R/J_i : I$ is Cohen-Macaulay.

It was shown that if I satisfies the G_s condition, then for general elements x_1, \dots, x_s in I and each $0 \leq i < s$, the ideal $J_i : I$ is a geometric i -residual intersection of I , and $J_s : I$ is an s -residual intersection of I (see [18] and [14]). It is well-known that the G_d condition and the Artin-Nagata property AN_{d-2}^- are automatically satisfied by any \mathfrak{m} -primary ideal in a Cohen-Macaulay local

ring. From now on, we will assume I has maximal analytic spread $\ell(I) = d$, satisfies the G_d condition and $J = (x_1, \dots, x_d)$ is a general minimal reduction of I , where x_1, \dots, x_d are general elements in I . We write $R^i = R/(J_i : I^\infty)$, where $J_i : I^\infty = \{r \in R \mid \exists n > 0 \text{ such that } rI^n \subseteq J_i\}$ and $J_i = (x_1, \dots, x_i)$ (with the convention $J_i = (0)$ if $i \leq 0$) for $i = 0, 1, \dots, d-1$. By the above hypothesis, R^{d-1} is a 1-dimensional Cohen-Macaulay local ring and IR^{d-1} is primary to the maximal ideal $\mathfrak{m}R^{d-1}$. Therefore, one has the Hilbert-Samuel function of I on R^{d-1} :

$$H_{IR^{d-1}}(n) = \lambda(I^n R^{d-1}/I^{n+1} R^{d-1}) \text{ for } n \geq 0.$$

Hence the generalized Hilbert-Samuel function $H_{IR^{d-1}}(n)$ and the generalized Hilbert-Samuel polynomial $P_{IR^{d-1}}(n)$ are respectively the usual Hilbert-Samuel function and the usual Hilbert-Samuel polynomial of IR^{d-1} . Let I be an R -ideal with analytic spread $\ell(I) = d$. We recall that I has minimal j -multiplicity if $j(I) = \lambda(IR^{d-1}/I^2 R^{d-1})$, I has almost minimal j -multiplicity if $j(I) = \lambda(IR^{d-1}/I^2 R^{d-1}) + 1$ and the index of nilpotency of an R -ideal I with respect to a reduction J is defined to be the integer $s_J(I) = \min\{n \mid I^{n+1} \subseteq J\}$ [14, Definition 2.4 and Definition 4.1].

3. Main result

In this section we find a lower bound for the first generalized Hilbert coefficient $j_1(I)$, when I is not an \mathfrak{m} -primary ideal with $\ell(I) = d$, and satisfies the G_d condition and the AN_{d-2}^- , J is a general minimal reduction of I (see Theorem 3.6). Also, we prove that if $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)] + 1$, then $G(I)$ is almost Cohen-Macaulay (see Theorem 3.7).

Remark 3.1. Assume R is Cohen-Macaulay. Let I be an R -ideal which satisfies $\ell(I) = d$, the G_d condition and AN_{d-2}^- . Let $J = (x_1, \dots, x_d)$ be a general minimal reduction of I and $J_i = (x_1, \dots, x_i)$. Since I satisfies $\ell(I) = d$ and the G_d condition, then by [14, Lemma 3.1], $J_i : I$ is a geometric i -residual intersection of I for $0 \leq i \leq d-1$. By the weak Artin-Nagata property AN_{d-2}^- , one has that $J_i : I^\infty = J_i : I = J_i : x_{i+1}$ and $(J_i : I^\infty) \cap I = (J_i : I) \cap I = J_i$ for $0 \leq i \leq d-1$ [14, Lemma 3.2] and [18]. Therefore

$$\begin{aligned} I^{n+1} \cap (J_i : I^\infty) &= I^{n+1} \cap (J_i : I^\infty) \cap I \\ &= I^{n+1} \cap J_i \end{aligned}$$

and by similar argument we have $J I^n \cap (J_i : I^\infty) = J I^n \cap J_i$ for $n \geq 0$ and $0 \leq i \leq d-1$.

Lemma 3.2. *Assume R is Cohen-Macaulay. Let I be a non \mathfrak{m} -primary R -ideal which satisfies $\ell(I) = d$, the G_d condition, the AN_{d-2}^- , $J = (x_1, \dots, x_d)$ general minimal reduction of I and \mathfrak{m} is not an associated prime of R/I . Then $I^2 \cap J_i = J_i I$ for any $0 \leq i \leq d-1$.*

Proof. Since $\text{depth}(R/I) \geq 1$, by [14, Lemma 3.2(f)], we have $J_{d-1} :_{I^2} I^\infty = J_{d-1}I$. Then by Remark 3.1, $I^2 \cap J_{d-1} = J_{d-1}I$. Now we show that $I^2 \cap J_{d-2} = J_{d-2}I$. Indeed,

$$I^2 \cap J_{d-2} \subseteq I^2 \cap J_{d-1} = J_{d-1}I = J_{d-2}I + x_{d-1}I.$$

Intersecting both sides with J_{d-2} , we have

$$\begin{aligned} I^2 \cap J_{d-2} &= (J_{d-2}I + x_{d-1}I) \cap J_{d-2} \\ &= J_{d-2}I + (J_{d-2} \cap x_{d-1}I) \\ &= J_{d-2}I + (J_{d-2} \cap x_{d-1}R \cap x_{d-1}I) \\ &= J_{d-2}I + x_{d-1}(J_{d-2} : x_{d-1} \cap I) \\ &= J_{d-2}I + x_{d-1}J_{d-2} \\ &= J_{d-2}I. \end{aligned}$$

By the similar argument we have $I^2 \cap J_i = J_iI$ for any $0 \leq i \leq d-3$. \square

Lemma 3.3. *Assume R is Cohen-Macaulay. Let I be an R -ideal which satisfies $\ell(I) = d \geq 3$, the G_d condition, the AN_{d-2}^- , $J = (x_1, \dots, x_d)$ a general minimal reduction of I and $\text{depth}(R/I) \geq \min\{1, \dim R/I\}$. Then $\text{depth}(R^1/IR^1) \geq \min\{1, \dim(R^1/IR^1)\}$, where $R^1 = R/J_1 : I$ and IR^1 is the image of I in the quotient ring R^1 .*

Proof. If I is an \mathfrak{m} -primary ideal the assertion is clear, so we assume that $\dim R/I \geq 1$. Set $R^0 = R/0 : I$. Then R^0 is Cohen-Macaulay since I satisfies AN_{d-2}^- . Note that $\dim R^0 = \dim R = d$, $\text{grade}(IR^0) \geq 1$, IR^0 still satisfies the G_d condition, AN_{d-2}^- , $\ell(IR^0) = \ell(I) = d$ (see for instance [18]). Hence $\text{depth}(R^0/x_1R^0) = d-1$ or $\text{depth}(R/J_1 + 0 : I) = d-1$. By the exact sequence

$$0 \longrightarrow (J_1 : I)/J_1 + (0 : I) \longrightarrow R/J_1 + (0 : I) \longrightarrow R/(J_1 : I) \longrightarrow 0$$

one has $\text{depth}((J_1 : I)/J_1 + (0 : I)) \geq d-1$. Since

$$\begin{aligned} (J_1 : I) + I/(0 : I) + I &= (J_1 : I) + I + (0 : I)/(0 : I) + I \\ &\cong (J_1 : I)/((0 : I) + I) \cap (J_1 : I) \\ &= (J_1 : I)/(0 : I) + J_1 \end{aligned}$$

we have $\text{depth}((J_1 : I) + I/(0 : I) + I) \geq d-1$. Furthermore by the exact sequence

$$0 \longrightarrow 0 : I \longrightarrow R/I \longrightarrow R/((0 : I) + I) \longrightarrow 0,$$

and since $\text{depth}(0 : I) = d$ (see [18]), we have $\text{depth}(R/((0 : I) + I)) \geq 1$. Now by the following exact sequence

$$0 \longrightarrow (J_1 : I) + I/(0 : I) + I \longrightarrow R/(0 : I) + I \longrightarrow R/(J_1 : I) + I \longrightarrow 0$$

we have $\text{depth}(R/(J_1 : I) + I) \geq 1$. Therefore

$$\text{depth}(R^1/IR^1) \geq \min\{1, \dim(R^1/IR^1)\}. \quad \square$$

Remark 3.4. Assume R is Cohen-Macaulay of dimension d . Let I be an \mathfrak{m} -primary ideal of R and $I^2 \cap J = JI$ for a minimal reduction J of I . By formula of Huckaba and Marley [7, Theorem 4.7], we have $e_1(I) \geq \lambda(I/J) + \lambda(I^2/JI)$, and if $e_1(I) = \lambda(I/J) + 1$, then $\text{depth}(G(I)) \geq d - 1$.

Theorem 3.5. *Assume R is Cohen-Macaulay of dimension 2. Let I be a non \mathfrak{m} -primary R -ideal which satisfies $\ell(I) = 2$, the G_2 condition, the AN_0^- and \mathfrak{m} is not an associated prime of R/I . Then for a general minimal reduction $J = (x_1, x_2)$ of I , one has that $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_1 :_R I + (0 :_R I + I) :_R \mathfrak{m}^\infty)]$.*

Proof. By [19, Corollary 3.3] and Remark 3.1, we have

$$\begin{aligned} j_1(I) &= \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) - \sum_{n=0}^{\infty} \lambda(I^{n+1} \cap J_1/JI^n \cap J_1) \\ &\quad + \lambda(R/(I + J_1 : I)) - \lambda(H_{\mathfrak{m}}^0(R/I + 0 : I)) \\ &\quad + \sum_{n=1}^{\infty} [\lambda(\widetilde{L}_n^0) - \lambda(L_n^0) + \lambda(N_n^0)], \end{aligned}$$

where

$$\begin{aligned} \widetilde{L}_n^0 &= I^n \cap J_1/[I^{n+1} \cap J_1 + x_1 I^{n-1}], \\ L_n^0 &= I^{n+1} :_{I^n \cap J_1} \mathfrak{m}^\infty/[I^{n+1} \cap J_1 + x_1(I^n :_{I^{n-1}} \mathfrak{m}^\infty)], \\ N_n^0 &= (I^n \cap J_1 + I^{n+1}) :_{I^n} \mathfrak{m}^\infty/[I^n \cap J_1 + (I^{n+1} :_{I^n} \mathfrak{m}^\infty)]. \end{aligned}$$

Recall $R^1 = R/(J_1 : I^\infty)$ and $IR^1 = (I + J_1 : I^\infty)/(J_1 : I^\infty)$. By the fact that $\lambda(I^{n+1}/JI^n) < \infty$ and $\lambda(I^{n+1} \cap J_1/JI^n \cap J_1) = \lambda(I^{n+1} \cap (JI^n + J_1 : I^\infty)/JI^n) < \infty$ (see [14, Lemma 4.6]), one has

$$\begin{aligned} \lambda(I^{n+1}R^1/JI^nR^1) &= \lambda((I^{n+1} + J_1 : I^\infty)/(JI^n + J_1 : I^\infty)) \\ &= \lambda(I^{n+1}/JI^n + (I^{n+1} \cap J_1 : I^\infty)) \\ &= \lambda(I^{n+1}/JI^n) - \lambda(I^{n+1} \cap J_1/JI^n \cap J_1). \end{aligned}$$

Since $\dim(R/I) \geq 1$ (I is a non \mathfrak{m} -primary ideal) by hypothesis $\text{depth}(R/I) \geq 1$, then by [14, Lemma 3.2(f)], $I^2 \cap J_1 : I^\infty = J_1I$. Now by summing these equations over all $n \geq 1$ and using the fact that $\lambda(IR^1/JR^1) = \lambda(I/J)$, $\lambda(I^2R^1/JIR^1) = \lambda(I^2/(JI + (I^2 \cap J_1 : I^\infty))) = \lambda(I^2/JI)$, we see that

$$\begin{aligned} &\sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) - \sum_{n=0}^{\infty} \lambda(I^{n+1} \cap J_1/JI^n \cap J_1) \\ &= \sum_{n=0}^{\infty} \lambda(I^{n+1}R^1/JI^nR^1) \\ &= \lambda(I/J) + \lambda(I^2/JI) + \sum_{n=2}^{\infty} \lambda(I^{n+1}/(JI^n + (I^{n+1} \cap J_1 : I^\infty))). \end{aligned}$$

Claim 1.

$$\lambda(R/(I+J_1 : I)) - \lambda(H_{\mathfrak{m}}^0(R/I+0 : I)) \geq \lambda[R/(J_1 :_R I + (0 :_R I + I) :_R \mathfrak{m}^\infty)].$$

Proof of Claim 1. We can write

$$\begin{aligned} & \lambda(R/(I+J_1 : I)) - \lambda(H_{\mathfrak{m}}^0(R/I+0 : I)) \\ &= \lambda(R/(I+J_1 : I)) - \lambda(H_{\mathfrak{m}}^0(R/I)) + \lambda(H_{\mathfrak{m}}^0(R/I)) - \lambda(H_{\mathfrak{m}}^0(R/I+0 : I)). \end{aligned}$$

By applying the long exact sequence of local cohomology on the following exact sequence

$$0 \longrightarrow 0 : I \longrightarrow R/I \longrightarrow R/(0 : I + I) \longrightarrow 0,$$

and since $\text{depth}(0 : I) = 2$ (see [18]), we have

$$\lambda(H_{\mathfrak{m}}^0(R/I)) - \lambda(H_{\mathfrak{m}}^0(R/I+0 : I)) \geq 0.$$

To complete the proof of Claim 1, we need to show that

$$\lambda(R/(I+J_1 : I)) - \lambda(H_{\mathfrak{m}}^0(R/I)) \geq \lambda[R/(J_1 :_R I + (0 :_R I + I) :_R \mathfrak{m}^\infty)].$$

One has

$$\begin{aligned} & \lambda(R/(I+J_1 : I)) - \lambda(H_{\mathfrak{m}}^0(R/I)) \\ &= [\lambda(R/(J+J_1 : I)) - \lambda(I/J)] - [\lambda(J : \mathfrak{m}^\infty/J) - \lambda(I/J)] \\ &= \lambda(R/(J+J_1 : I)) - \lambda(J : \mathfrak{m}^\infty/J) \\ &= \lambda(R/(J_1 : I + J : \mathfrak{m}^\infty)) + \lambda(J_1 : I + J : \mathfrak{m}^\infty/J + J_1 : I) - \lambda(J : \mathfrak{m}^\infty/J) \\ &= \lambda(R/(J_1 : I + J : \mathfrak{m}^\infty)) + \lambda(J : \mathfrak{m}^\infty/(J+J_1 : I) \cap J : \mathfrak{m}^\infty) - \lambda(J : \mathfrak{m}^\infty/J) \\ &= \lambda(R/(J_1 : I + J : \mathfrak{m}^\infty)) - \lambda((J+J_1 : I) \cap J : \mathfrak{m}^\infty/J) \\ &= \lambda(R/(J_1 : I + J : \mathfrak{m}^\infty)) - \lambda(J/J) \\ &= \lambda(R/(J_1 : I + J : \mathfrak{m}^\infty)) \\ &\geq \lambda(R/(J_1 : I + 0 : I + I : \mathfrak{m}^\infty)), \end{aligned}$$

where the first equality holds because

$$\begin{aligned} \lambda(R/(I+J_1 : I)) &= \lambda(R/(J+J_1 : I)) - \lambda(I+J_1 : I/J+J_1 : I) \\ &= \lambda(R/(J+J_1 : I)) - \lambda(I/J + (J_1 : I \cap I)) \\ &= \lambda(R/(J+J_1 : I)) - \lambda(I/J) \end{aligned}$$

and because $\lambda(I/J) < \infty$, we also have

$$\lambda(H_{\mathfrak{m}}^0(R/I)) = \lambda(I : \mathfrak{m}^\infty/I) = \lambda(J : \mathfrak{m}^\infty/I) = \lambda(J : \mathfrak{m}^\infty/J) - \lambda(I/J)$$

the sixth equality follows by [19, (9)] that $(J_1 : I) \cap (J : \mathfrak{m}^\infty) = J_1$. \square

Claim 2.

$$\lambda(\widetilde{L}_n^0) - \lambda(L_n^0) \geq 0$$

Proof of Claim 2. There is a map

$$I^{n+1} :_{I^n \cap J_1} m^\infty \longrightarrow I^n \cap J_1 / I^{n+1} \cap J_1 + x_1 I^{n-1}$$

with kernel

$$\begin{aligned} & [I^{n+1} :_{I^n \cap J_1} m^\infty] \cap [I^{n+1} \cap J_1 + J_1 I^{n-1}] \\ &= I^{n+1} \cap J_1 + [I^{n+1} :_{I^n \cap J_1} m^\infty] \cap J_1 I^{n-1} \\ &= I^{n+1} \cap J_1 + [J_1 I^n :_{I^n \cap J_1} m^\infty] \cap J_1 I^{n-1} \\ &= I^{n+1} \cap J_1 + x_1 (I^n :_{I^{n-1}} m^\infty). \end{aligned}$$

We see that $\lambda(\widetilde{L}_n^0) - \lambda(L_n^0) \geq 0$. \square

Then by using the formula for $j_1(I)$, Claim 1 and Claim 2 we have $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_1 :_R I + (0 :_R I + I) :_R m^\infty)]$, as required. \square

Theorem 3.6. *Assume R is Cohen-Macaulay of dimension $d \geq 2$. Let I be a non \mathfrak{m} -primary R -ideal which satisfies $\ell(I) = d$, the G_d condition, the AN_{d-2}^- and \mathfrak{m} is not an associated prime of R/I . Then for a general minimal reduction $J = (x_1, \dots, x_d)$ of I , one has that $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R m^\infty)]$.*

Proof. We prove the theorem by induction on d . The case $d = 2$ has been proved in Theorem 3.5. Let $d \geq 3$ and assume that the theorem holds for $d-1$. Let $R^1 = R/J_1 : I$, where $J_1 = (x_1)$. Observe that R^1 is a Cohen-Macaulay ring of dimension $d-1$ and $\ell(IR^1) = d-1$. Also, $IR^1 = (I + J_1 : I)/J_1 : I \cong I/(x_1)$ satisfies G_{d-1} and AN_{d-3}^- (see [14, Lemma 3.2] and [18]). By Lemma 3.3, $\text{depth}(R^1/IR^1) \geq 1$ and by [15, Proposition 2.5], $j_1(I) = j_1(IR^1)$ and so we have

$$\begin{aligned} j_1(I) &= j_1(IR^1) \\ &\geq \lambda(IR^1/JR^1) + \lambda(I^2R^1/JR^1IR^1) \\ &\quad + \lambda[R^1/(x_2, \dots, x_{d-1})R^1 :_{R^1} IR^1 \\ &\quad + (x_2, \dots, x_{d-2})R^1 :_{R^1} IR^1 + IR^1 :_{R^1} m^\infty] \\ &= \lambda(I/J) + \lambda(I^2/(JI + (I^2 \cap J_1 : I))) \\ &\quad + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R m^\infty)] \\ &= \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R m^\infty)], \end{aligned}$$

where the last equality follows by Lemma 3.2. \square

Theorem 3.7. *Assume R is Cohen-Macaulay. Let I be a non \mathfrak{m} -primary R -ideal which satisfies $\ell(I) = d$, the G_d condition, the AN_{d-2}^- and \mathfrak{m} is not an associated prime of R/I . Then for a general minimal reduction $J = (x_1, \dots, x_d)$ of I , with $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R m^\infty)] + 1$, I has almost minimal j -multiplicity, the reduction number equals the index of nilpotency and is at most 2, and $G(I)$ is Cohen-Macaulay.*

Proof. By Theorem 3.6 we have $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$. From the assumption one obtains that $\lambda(I^2/JI)$ is at most 1, so the length of I^2R^{d-1}/JIR^{d-1} is at most one, and thus the length of I^2R^{d-1}/x_dIR^{d-1} is at most 1. The ideal I then has almost minimal j -multiplicity, based on [14, Definition 4.1]. Thus [14, Corollary 4.9] implies $\text{depth}(G(I))$ is at least $d-1$. If $\lambda(I^2/JI) = 0$, then I has minimal j -multiplicity and the reduction number is 1; in particular $G(I)$ is Cohen-Macaulay by [14, Corollary 3.5]. If $\lambda(I^2/JI) = 1$, then the reduction number of I is at least 2, and the index of nilpotency of I is $K = \lambda(I^2/JI) + 1 = 2$. If $j_1(I) = \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$, then by the formula of $j_1(I)$, we have $\sum_{n=2}^{\infty} \lambda(I^{n+1}/(JI^n + (I^{n+1} \cap J_1 : I^\infty))) = 0$. If $\text{depth}(G(I)) \geq d-1$, then $I^{n+1} \cap J_i = J_i I^n$ for every $n \geq 0$ and $0 \leq i \leq d-1$. Therefore we have $\sum_{n=2}^{\infty} \lambda(I^{n+1}/JI^n) = 0$ and so the reduction number is at most 2, thus it equals K . Then $G(I)$ is Cohen-Macaulay by [10, Theorem 4.1]. \square

Example 3.8. Let $R = k[[x, y, z]]$ be a polynomial ring where k is a field and $\mathfrak{q} = (x^4, xz, yz)$. Set $S = R/\mathfrak{q}$ and let $I = (x, y)$ be an ideal of S . Then S is a 1-dimensional Cohen-Macaulay local ring and I is a Cohen-Macaulay prime ideal that has $\ell(I) = 1$, G_1 condition and AN_{-1}^- . By using Macaulay 2 [5], we have $\text{depth}(S/I) = 1$ and the generalized Hilbert-Samuel polynomial is $P_I(n) = 4(n+1) - 7$. Hence $j_0(I) = 4$ and $j_1(I) = 7$. $J = (y)$ is a minimal reduction of I (see [10, Example 2.2]) and again by Macaulay 2 we have $\lambda(I/J) = 3$, $\lambda(I^2/JI) = 2$ and $\lambda[R/(0 :_R I + I :_R \mathfrak{m}^\infty)] = 1$. Therefore

$$j_1(I) > \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(0 :_R I + I :_R \mathfrak{m}^\infty)].$$

Example 3.9. Let $R = k[[x, y, z]]$ be polynomial ring where k is a field and let $\mathfrak{q} = (x^3, xz, yz)$. Set $S = R/\mathfrak{q}$ and let $I = (x, y)$ be an ideal of S . Then S is a 1-dimensional Cohen-Macaulay local ring and I is a Cohen-Macaulay prime ideal that has $\ell(I) = 1$, G_1 condition and AN_{d-2}^- (automatically satisfied since $d = 1$). By using Macaulay 2 [5], we have $\text{depth}(S/I) = 1$ and the generalized Hilbert-Samuel polynomial is $P_I(n) = 3(n+1) - 4$. Hence $j_0(I) = 3$ and $j_1(I) = 4$. $J = (y)$ is a minimal reduction of I (see [10, Example 2.2]) and again by Macaulay 2 we have $\lambda(I/J) = 2$, $\lambda(I^2/JI) = 1$ and $\lambda[R/(0 :_R I + I :_R \mathfrak{m}^\infty)] = 1$ then we have,

$$j_1(I) = \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(0 :_R I + I :_R \mathfrak{m}^\infty)].$$

References

- [1] R. Achilles and M. Manaresi, *Multiplicity for ideals of maximal analytic spread and intersection theory*, J. Math. Kyoto Univ. **33** (1993), no. 4, 1029–1046. <https://doi.org/10.1215/kjm/1250519127>
- [2] ———, *Multiplicities of a bigraded ring and intersection theory*, Math. Ann. **309** (1997), no. 4, 573–591. <https://doi.org/10.1007/s002080050128>

- [3] C. Ciupercă, *A numerical characterization of the S_2 -ification of a Rees algebra*, J. Pure Appl. Algebra **178** (2003), no. 1, 25–48. [https://doi.org/10.1016/S0022-4049\(02\)00157-3](https://doi.org/10.1016/S0022-4049(02)00157-3)
- [4] H. Flenner, L. O’Carroll, and W. Vogel, *Joins and intersections*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1999. <https://doi.org/10.1007/978-3-662-03817-8>
- [5] D. R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2>.
- [6] S. Huckaba, *A d -dimensional extension of a lemma of Huneke’s and formulas for the Hilbert coefficients*, Proc. Amer. Math. Soc. **124** (1996), no. 5, 1393–1401. <https://doi.org/10.1090/S0002-9939-96-03182-6>
- [7] S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. (2) **56** (1997), no. 1, 64–76. <https://doi.org/10.1112/S0024610797005206>
- [8] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), no. 2, 293–318. <https://doi.org/10.1307/mmj/1029003560>
- [9] C. Huneke and I. Swanson, *Integral Closure of Ideals, Rings, and Modules*, London Mathematical Society Lecture Note Series, **336**, Cambridge University Press, Cambridge, 2006.
- [10] P. Mantero and Y. Xie, *Generalized stretched ideals and Sally’s conjecture*, J. Pure Appl. Algebra **220** (2016), no. 3, 1157–1177. <https://doi.org/10.1016/j.jpaa.2015.08.013>
- [11] D. G. Northcott, *A note on the coefficients of the abstract Hilbert function*, J. London Math. Soc. **35** (1960), 209–214. <https://doi.org/10.1112/jlms/s1-35.2.209>
- [12] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158. <https://doi.org/10.1017/s0305004100029194>
- [13] A. Ooishi, *Δ -genera and sectional genera of commutative rings*, Hiroshima Math. J. **17** (1987), no. 2, 361–372. <http://projecteuclid.org/euclid.hmj/1206130073>
- [14] C. Polini and Y. Xie, *j -multiplicity and depth of associated graded modules*, J. Algebra **379** (2013), 31–49. <https://doi.org/10.1016/j.jalgebra.2013.01.001>
- [15] ———, *Generalized Hilbert functions*, Comm. Algebra **42** (2014), no. 6, 2411–2427. <https://doi.org/10.1080/00927872.2012.756884>
- [16] J. D. Sally, *Hilbert coefficients and reduction number 2*, J. Algebraic Geom. **1** (1992), no. 2, 325–333.
- [17] N. V. Trung, *Constructive characterization of the reduction numbers*, Compositio Math. **137** (2003), no. 1, 99–113.
- [18] B. Ulrich, *Artin-Nagata properties and reductions of ideals*, in Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992), 373–400, Contemp. Math., 159, Amer. Math. Soc., Providence, RI, 1994. <https://doi.org/10.1090/conm/159/01519>
- [19] Y. Xie, *Generalized Hilbert coefficients and Northcott’s inequality*, J. Algebra **461** (2016), 177–200. <https://doi.org/10.1016/j.jalgebra.2016.05.009>

AMIR MAFI
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF KURDISTAN
 P.O. BOX: 416, SANANDAJ, IRAN
 Email address: A.Mafi@ipm.ir

DLER NADERI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KURDISTAN
P.O. BOX: 416, SANANDAJ, IRAN
Email address: dler.naderi65@gmail.com