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NOTES ON FINITELY GENERATED FLAT MODULES

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Abstract. In this paper, the projectivity of finitely generated flat modules of a commutative ring are studied from a topological point of view. Then various interesting results are obtained. For instance, it is shown that if a ring has either finitely many minimal primes or finitely many maximal ideals then every finitely generated flat module over it is projective. It is also shown that if a particular subset of the prime spectrum of a ring satisfies some certain ascending or descending chain conditions, then every finitely generated flat module over this ring is projective. These results generalize some major results in the literature on the projectivity of finitely generated flat modules.

1. Introduction

Studying the projectivity of finitely generated flat modules has been the main topic of many articles over the years and it is still of current interest, see e.g. [1], [2], [4], [7, §4E], [10], [12], [15] and [17]. The main motivation behind in the investigating the projectivity of f.g. flat modules stems from the fact that "every f.g. flat module over a local ring is free". We use f.g. in place of "finitely generated". Note that in general there are f.g. flat modules which are not necessarily projective, see [12, Example 2.9] see also [5, Tag 00NY].

In this article we have applied the spectral (Zariski and flat) and compact spectral (patch) topologies of the prime spectrum Spec(R) in order to investigate the projectivity of f.g. flat R-modules. The obtained results from this method generalize some major results in the literature on the projectivity of f.g. flat modules. In fact, Theorem 3.11 vastly generalizes [7, Theorem 4.38], [4, Corollary 1.5], [10, Fact 7.5] and [11, Corollary 3.57] in the commutative case. Also Theorem 3.13 generalizes [10, Proposition 7.6]. In summary, Theorems 3.1, 3.2, 3.11 and 3.13 and Corollaries 3.4, 3.5, 3.6, 3.7 and 3.9 are the main results of this paper. In this paper, all rings are commutative.

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2. Preliminaries

Let R be a commutative ring. Then there is a (unique) topology over $\operatorname{Spec}(R)$ such that the collection of subsets $V(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \in \mathfrak{p}\}$ with $f \in R$ forms a sub-basis for the opens of this topology. It is called the flat (or, inverse) topology. Therefore, the collection of subsets $V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) : I \subseteq \mathfrak{p}\}$ where I runs through the set of finitely generated ideals of R forms a basis for the flat opens. It is proved that the flat closed subsets of $\operatorname{Spec}(R)$ are precisely of the form $\operatorname{Im} \varphi^*$ where $\varphi : R \to A$ is a "flat" ring map. Recall that if $\varphi : R \to A$ is a ring map, then the induced map $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ given by $\mathfrak{p} \rightsquigarrow \varphi^{-1}(\mathfrak{p})$ is denoted by φ^* or by $\operatorname{Spec}(\varphi)$. Also, by a flat ring map $\varphi : R \to A$ we mean φ is a ring map and that A as an R-module, induced via φ, is a flat R-module. There is a (unique) topology over $\operatorname{Spec}(R)$ such that the collection of subsets $D(f) \cap V(g)$ with $f, g \in R$ forms a sub-basis for the opens of this topology where $D(f) = \operatorname{Spec}(R) \setminus V(f)$. It is called the patch (or, constructible) topology. Therefore the collection of subsets $D(f) \cap V(I)$ with $f \in R$ and I runs through the set of finitely generated ideals of R is a basis for the patch opens. It is also proved that the patch closed subsets of $\operatorname{Spec}(R)$ are precisely of the form $\operatorname{Im} \varphi^*$ where $\varphi : R \to A$ is a ring map. Clearly the patch topology is finer than the flat and Zariski topologies. The flat topology behaves as the dual of the Zariski topology. For instance, if $\mathfrak{p}$ is a prime ideal of R, then its closure with respect to the flat topology comes from the canonical ring map $R \to R_{\mathfrak{p}}$. In fact, $\Lambda(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R : \mathfrak{q} \subseteq \mathfrak{p}\}$. Here $\Lambda(\mathfrak{p})$ denotes the closure of $\{\mathfrak{p}\}$ in $\operatorname{Spec} R$ with respect to the flat topology. Recall that a subset E of $\operatorname{Spec}(R)$ is said to be stable under the generalization (resp. specialization) if for any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R with $\mathfrak{p} \subset \mathfrak{q}$ (resp. $\mathfrak{q} \subset \mathfrak{p}$) if $\mathfrak{q} \in E$, then $\mathfrak{p} \in E$. Therefore, a subset of $\operatorname{Spec}(R)$ is flat closed if and only if it is patch closed and stable under the generalization. Dually, a subset of $\operatorname{Spec}(R)$ is Zariski closed if and only if it is patch closed and stable under the specialization. A subset of $\operatorname{Spec}(R)$ is said to be a double-closed if it is closed with respect to the both flat and Zariski topologies. For more details see [3] or [13]. We shall freely use the above facts in this paper.

3. Main results

Let R be a ring. If E is a subset of $\operatorname{Spec}(R)$, then we define $\mathcal{F}(E) = \bigcup_{\mathfrak{p} \in E} \Lambda(\mathfrak{p})$ and $\mathcal{Z}(E) = \bigcup_{\mathfrak{p} \in E} V(\mathfrak{p})$. We have then the following interesting result.

Theorem 3.1. (i) *If E is Zariski closed, then $\mathcal{F}(E)$ is flat closed.*

(ii) *If E is patch closed, then $\mathcal{Z}(E)$ is Zariski closed.*

(iii) *If E is stable under the generalization and $E = V(I)$ for some ideal I of R, then R/J is R-flat and $E = V(J)$ where J is the kernel of the canonical map $R \to S^{-1}R$ with $S = 1 + I$.*

Proof. (i) Suppose $E = V(I)$ for some ideal I of R. We claim that $\mathcal{F}(E) = \operatorname{Im}\pi^*$ where $\pi : R \to S^{-1}R$ is the canonical map with $S = 1+I$. The inclusion $\mathcal{F}(E) \subseteq \operatorname{Im}\pi^*$ is obvious. To prove the reverse inclusion, let $\mathfrak{q}$ be a prime ideal of R such that $\mathfrak{q} \cap S = \emptyset$. There exists a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $S^{-1}\mathfrak{p}$ is a maximal ideal of $S^{-1}R$. We have $I \subseteq \mathfrak{p}$. If not, then choose some element $f \in I \setminus \mathfrak{p}$. Clearly $(\mathfrak{p} + Rf) \cap S \neq \emptyset$. Thus there are elements $r \in R$ and $g \in I$ such that $1 + rf + g \in \mathfrak{p}$. But this is a contradiction since $\mathfrak{p} \cap S = \emptyset$. Therefore $\mathfrak{q} \in \mathcal{F}(E)$.

(ii) We have $E = \operatorname{Im}\varphi^*$ for some ring morphism $\varphi : R \to A$. It follows that $\mathcal{Z}(E) = V(I)$ where $I = \operatorname{Ker}\varphi$. Because the inclusion $\mathcal{Z}(E) \subseteq V(I)$ is obvious. To prove the reverse inclusion, pick $\mathfrak{p} \in V(I)$. Let $\mathfrak{q}$ be a minimal prime of I such that $\mathfrak{q} \subseteq \mathfrak{p}$. Thus there exists a prime ideal of A which lying over $\mathfrak{q}/I$, because it is well known that in an extension of rings, for every minimal prime of the subring then there exists a (minimal) prime of the extended ring which lying over it, see e.g. [13, Lemma 3.9]. It follows that $\mathfrak{q} \in E$.

(iii) Clearly $E \subseteq V(J)$ by the fact that S is not contained in any element in E. By the proofs of (i) and (ii), we have $\mathcal{Z}\big(\mathcal{F}(E)\big) = V(J)$. It follows that $V(J) \subseteq E$ because E is stable under the generalization. Thus $E = V(J)$. Using this, then by [12, Theorem 2.5], in order to show that R/J is R-flat, it suffices to prove that $\operatorname{Ann}(f) + J = R$ for any $f \in J$. If not, then there exists some $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{Ann}(f) + J \subseteq \mathfrak{p}$. Thus $\mathfrak{p} \in V(J) = V(I)$, which implies that $I \subseteq \mathfrak{p}$. Since $f \in J$, it follows that there exists some $a \in I$ such that $(1 + a).f = 0$. Thus $1 + a \in \operatorname{Ann}(f)$. So $1 + a \in \mathfrak{p}$. Hence $1 \in \mathfrak{p}$, a contradiction. Therefore, $\operatorname{Ann}(f) + J = R$. □

An ideal I of a ring R is called a pure ideal if the canonical ring map $R \to R/I$ is a flat ring map.

As a first application of Theorem 3.1 we obtain the following result.

Theorem 3.2. *Let R be a ring. Then the assignment $I \rightsquigarrow V(I)$ is a bijective map from the set of pure ideals of R onto the set of Zariski closed subsets of* $\operatorname{Spec}(R)$ *which are stable under the generalization.*

Proof. First we show that this map is well-defined. That is, if R/I is R-flat, then we have to show that $V(I)$ is stable under the generalization. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals of R such that $I \subseteq \mathfrak{p}$ and $\mathfrak{q} \subseteq \mathfrak{p}$. Suppose there is some $f \in I$ such that $f \notin \mathfrak{q}$. It follows that $\operatorname{Ann}(f) \subseteq \mathfrak{q}$. So $\operatorname{Ann}(f)+I \subseteq \mathfrak{p}$. But this is a contradiction since $\operatorname{Ann}(f) + I = R$ by [12, Theorem 2.5]. Thus $\mathfrak{q} \in V(I)$. Then we show that this map is injective. Let I and J be two ideals of R such that R/I and R/J are R-flat and $V(I) = V(J)$. Take $f \in I$. If $f \notin J$, then by [12, Theorem 2.5], $\operatorname{Ann}(f) + J \neq R$. Thus there exists a prime ideal $\mathfrak{p}$ of R such that $\operatorname{Ann}(f) + J \subseteq \mathfrak{p}$. It follows that $\operatorname{Ann}(f) + I \subseteq \mathfrak{p}$. But this is a contradiction since $\operatorname{Ann}(f) + I = R$. Therefore $I = J$. The surjectivity of this map implies from Theorem 3.1(iii). □

Remark 3.3. In regarding with Theorem 3.2, note that a subset of $\mathrm{Spec}(R)$ is Zariski closed and stable under the generalization if and only if it is flat closed and stable under the specialization, see [13, Theorem 3.11].

Corollary 3.4. *Let I be an ideal of a ring R such that $\sqrt{I}$ is a pure ideal. Then $I = \sqrt{I}$.*

Proof. If $f \in I$, then by [12, Theorem 2.5], $\mathrm{Ann}(f) + \sqrt{I} = R$. It follows that $\sqrt{\mathrm{Ann}(f)} + \sqrt{I} = R$ and so $\mathrm{Ann}(f) + I = R$. Thus again by [12, Theorem 2.5], R/I is R-flat. Then, by Theorem 3.2, $I = \sqrt{I}$. □

Corollary 3.5. *If I is a pure ideal of a reduced ring R, then $I = \sqrt{I}$.*

Proof. By [12, Theorem 2.5], $\mathrm{Supp}(I) = \mathrm{Spec}(R) \setminus V(I) \subseteq \mathrm{Supp}(\sqrt{I})$ where $\mathrm{Supp}(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) : I_{\mathfrak{p}} \neq 0\}$. Conversely, if $\mathfrak{p} \in \mathrm{Supp}(\sqrt{I})$, then there exists some $f \in \sqrt{I}$ such that $f/1 \neq 0$. If $I_{\mathfrak{p}} = 0$, then there exist $s \in R \setminus \mathfrak{p}$ and a natural number $n \geq 1$ such that $sf^n = 0$. It follows that $sf = 0$ since R is reduced. But this is a contradiction. Therefore $\mathfrak{p} \in \mathrm{Supp}(I)$. Hence, $R/\sqrt{I}$ is R-flat. Thus by Theorem 3.2, $I = \sqrt{I}$. □

Corollary 3.6. *Let I and J be two ideals of a reduced ring R such that I is a pure ideal and $V(I) = V(J)$. Then $I = J$.*

Proof. We have $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$. By Corollary 3.5, $I = \sqrt{I}$. Hence $\sqrt{J} = I$ is a pure ideal. This yields that $J = \sqrt{J}$, see Corollary 3.4. □

Corollary 3.7. *Let I be an ideal of a ring R. If I is a pure ideal of R, then $V(I)$ is a flat closed subset of* $\mathrm{Spec}(R)$*. If moreover, R is a reduced ring, then the converse holds.*

Proof. If I is a pure ideal, then the canonical ring map $\pi : R \to R/I$ is a flat ring map. Thus by the definition of the flat topology, $V(I) = \mathrm{Im}\,\pi^*$ is a flat closed subset of $\mathrm{Spec}(R)$. Conversely, if $V(I)$ is a flat closed subset of $\mathrm{Spec}(R)$, then it is stable under the generalization. Therefore by Theorem 3.1(iii), there exists a pure ideal J of R such that $V(I) = V(J)$. Then by Corollary 3.6, $I = J$. □

Lemma 3.8. *If I is a pure ideal of a ring R, then for each finite subset $\{f_1, \ldots, f_n\}$ of I there exists some $g \in I$ such that $f_i = f_i g$ for all i.*

Proof. By [12, Theorem 2.5], R/I is R-flat if and only if $\mathrm{Ann}_R(f) + I = R$ for all $f \in I$. Thus for each pair (f, f') of elements of I then there exist $h, h' \in I$ such that $f = fh$ and $f' = f'h'$. Clearly $g := h + h' - hh' \in I$, $f = fg$ and $f' = f'g$. □

Corollary 3.9. *For an ideal I of a ring R then the following are equivalent.*

(i) *I is a pure ideal.*

(ii) *For each R-module M then the canonical morphism $I\otimes_R M\to M$ given by $a\otimes m\rightsquigarrow am$ is injective.*

(iii) *$IJ=I\cap J$ for all ideals J of R.*

Proof. (i)⇒(ii) Suppose $\sum_i a_im_i=0$ where $a_i\in I$ and $m_i\in M$ for all i. By Lemma 3.8, there exists some $b\in I$ such that $a_i=a_ib$ for all i. We have then $\sum_i a_i\otimes m_i=b\otimes(\sum_i a_im_i)=0$.

(ii)⇒(iii) If $f\in I\cap J$, then the pure tensor $f\otimes(1+J)$ of $I\otimes_R R/J$ is zero by (ii). Then using the canonical isomorphism $I\otimes_R R/J\to I/IJ$, we get that $f\in IJ$.

(iii)⇒(i) By [9, Theorem 7.7], it suffices to show that for each ideal J of R then the canonical morphism $J\otimes_R R/I\to J(R/I)$ given by $a\otimes(r+I)\rightsquigarrow ra+I$ is an isomorphism. But this morphism, using the hypothesis, is the composition of the following two canonical isomorphisms:

$$J\otimes_R R/I\xrightarrow{\simeq}J/IJ=J/I\cap J\xrightarrow{\simeq}(I+J)/I=J(R/I).\qquad\square$$

A ring R is called an S-ring ("S" refers to Sakhajev) if every f.g. flat R-module is R-projective.

We have improved the following well known result by adding (iv)-(vii) as new equivalents. The equivalency of the classical criteria are also proved by new methods.

Theorem 3.10. *For a ring R the following conditions are equivalent.*

(i) *The ring R is an S-ring.*

(ii) *Every cyclic flat R-module is R-projective.*

(iii) *R/I is R-projective whenever it is R-flat where I is an ideal of R.*

(iv) *Every Zariski closed subset of* $\mathrm{Spec}(R)$ *which is stable under the generalization is Zariski open.*

(v) *Every patch closed subset of* $\mathrm{Spec}(R)$ *which is stable under the generalization and specialization is patch open.*

(vi) *Every flat closed subset of* $\mathrm{Spec}(R)$ *which is stable under the specialization is flat open.*

(vii) *Each double-closed subset of* $\mathrm{Spec}(R)$ *is of the form $V(e)$ where $e\in R$ is an idempotent.*

(viii) *For every sequence $(f_n)_{n\geq1}$ of elements of R if $f_n=f_nf_{n+1}$ for all n, then there exists some k such that f_k is an idempotent and $f_n=f_k$ for all $n\geq k$.*

(ix) *For every sequence $(g_n)_{n\geq1}$ of elements of R if $g_{n+1}=g_ng_{n+1}$ for all n, then there exists some k such that g_k is an idempotent and $g_n=g_k$ for all $n\geq k$.*

Proof. The implications (i)⇒(ii)⇒(iii) are obvious.

(iii)⇒(iv) Suppose $E\subseteq\mathrm{Spec}(R)$ is stable under the generalization and $E=V(I)$ for some ideal I of R. By Theorem 3.1, there is an ideal J such that R/J is R-flat and $E=V(J)$. Thus, by the hypothesis, R/J is R-projective.

It follows that E is Zariski open because it is well known that the support of a projective module is Zariski open, see [8, Lemma 6.2] or [14, Corollary 3.5] and also see [16, Lemma 1.1].

(iv)⇔(v) It follows from the fact that a subset of $\mathrm{Spec}(R)$ is Zariski closed if and only if it is patch closed and is stable under the specialization.

(v)⇔(vi) It follows from the fact that a subset of $\mathrm{Spec}(R)$ is flat closed if and only if it is patch closed and is stable under the generalization.

(vi)⇒(vii) Let E be a double-closed subset of $\mathrm{Spec}(R)$. It follows that E is Zariski closed and is stable under the generalization. Thus, using the equivalency (vi)⇔(iv), we get that E is also Zariski open. Therefore E is a clopen (both open and closed) subset of $\mathrm{Spec}(R)$. But it is well known that the map $f \rightsquigarrow D(f)$ is a bijective map from the set of idempotents of R onto the set of clopens of $\mathrm{Spec}(R)$, see [5, Tag 00EE]. Thus there exists an idempotent $e' \in R$ such that $E = D(e') = V(e)$ where $e := 1 - e'$.

(vii)⇒(iv) There is nothing to prove.

(iv)⇒(i) Let M be a f.g. flat R-module. To prove the assertion, by [12, Theorem 2.8], it suffices to show that for each natural number n, $\psi^{-1}(\{n\})$ is Zariski open where ψ is the rank map of M, see [12, Remark 2.4]. We have $\psi^{-1}(\{n\}) = \mathrm{Supp}\, N \cap \big(\mathrm{Spec}(R) \setminus \mathrm{Supp}\, N'\big)$ where $N = \Lambda^n(M)$ and $N' = \Lambda^{n+1}(M)$. But $\mathrm{Supp}\, N$ and $\mathrm{Supp}\, N'$ are Zariski closed since N and N' are f.g. R-modules (recall the fact that if M is a f.g. R-module, then $\mathrm{Supp}(M) = V(I)$ where $I = \mathrm{Ann}_R(M)$). But N is a flat R-module. By applying [12, Corollary 2.6] then we observe that the support of a f.g. flat module is stable under the generalization. Thus by the hypothesis, $\mathrm{Supp}\, N$ is Zariski open. Therefore $\psi^{-1}(\{n\})$ is Zariski open.

(iii)⇒(viii) Let $I = (f_n : n \geq 1)$. Clearly $\mathrm{Ann}_R(f) + I = R$ for all $f \in I$. It follows that R/I is R-flat and so, by the hypothesis, it is R-projective. Therefore by [12, Lemma 2.7], there exists $g \in I$ such that $I = Rg$. It follows that there is some $d \geq 1$ such that $Rg = Rf_d$ since $I = \bigcup_{n \geq 1} Rf_n$. Let $k = d + 1$. There exists some $r \in R$ such that $f_k = rf_d = rf_d f_k = f_k^2$. We also have $f_{k+1} = r' f_k$ for some $r' \in R$. It follows that $f_{k+1} = f_{k+1} f_k = f_k$ and by the induction we obtain that $f_n = f_k$ for all $n \geq k$.

(viii)⇒(iii) Let I be an ideal of R such that R/I is R-flat. We shall prove that I is generated by an idempotent element. To do this we act as follows. Let $\mathcal{I}$ be the set of ideals of the form Re where $e \in I$ is an idempotent element. Let $\{Re_n : n \geq 1\}$ be an ascending chain of elements of $\mathcal{I}$. For each n there is some $r_n \in R$ such that $e_n = r_n e_{n+1}$. It follows that $e_n = e_n e_{n+1}$. Thus, by the hypothesis, the chain $Re_1 \subseteq Re_2 \subseteq \cdots$ is stationary. Therefore, by the axiom of choice, $\mathcal{I}$ has at least a maximal element Re. We also claim that if $J = (f_n : n \geq 1)$ is a countably generated ideal of R with $J \subseteq I$, then there exists an idempotent $e' \in I$ such that $J \subseteq Re'$. Because, by Lemma 3.8, there is an $g_1 \in I$ such that $f_1 = f_1 g_1$. Then for the pair (g_1, f_2), again by Lemma 3.8, we may find an $g_2 \in I$ such that $g_1 = g_1 g_2$ and $f_2 = f_2 g_2$. Therefore, in this way, we obtain a sequence (g_n) of elements of I such that $J \subseteq L = (g_n : n \geq 1)$ and $g_n = g_n g_{n+1}$ for all $n \geq 1$. But, by the hypothesis, there exists some $k \geq 1$ such that g_k is an idempotent and $g_n = g_k$ for all $n \geq k$. It follows that $L = Rg_k$. This establishes the claim. Now pick $f \in I$. Then, by what we have proved above, there is an idempotent $e' \in I$ such that $Re \subseteq (e, f) \subseteq Re'$. By the maximality of Re, we obtain that $e = e'$. Thus $I = Re$ and so R/I as an R-module is isomorphic to $R(1-e)$. Therefore R/I is R-projective.

(viii)⇔(ix) Let (f_n) be a sequence of elements of R. Put $g_n := 1 - f_n$ for all n. Then $f_n = f_n f_{n+1}$ if and only if $g_{n+1} = g_n g_{n+1}$. □

As a consequence of Theorem 3.10, we obtain the following result which in turn vastly generalizes some previous results in the literature specially including [7, Theorem 4.38], [4, Corollary 1.5], [10, Fact 7.5] and [11, Corollary 3.57] in the commutative case.

Theorem 3.11. *Let R be a ring which has either finitely many minimal primes or finitely many maximal ideals. Then R is an S-ring.*

Proof. Let F be a patch closed subset of $\operatorname{Spec}(R)$ which is stable under the generalization and specialization. By Theorem 3.10, it suffices to show that it is a patch open. First assume that $\operatorname{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. There exists some s with $1 \leq s \leq n$ such that $\mathfrak{p}_s, \mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_n \notin F$ but $\mathfrak{p}_i \in F$ for all $i < s$. It follows that $\operatorname{Spec}(R) \setminus F = \bigcup_{i=s}^{n} V(\mathfrak{p}_i)$. Therefore F is Zarsiki open in this case and so it is patch open. Now assume that $\operatorname{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_d\}$. Similarly, there exists some k with $1 \leq k \leq d$ such that $\mathfrak{m}_k, \mathfrak{m}_{k+1}, \ldots, \mathfrak{m}_d \notin F$ but $\mathfrak{m}_i \in F$ for all $i < k$. We have $\operatorname{Spec}(R) \setminus F = \bigcup_{i=k}^{d} \Lambda(\mathfrak{m}_i)$. Therefore F is a flat open in this case and so it is patch open. □

Remark 3.12. In relation with Theorem 3.11, note that though every projective module over a local ring is free (see [6] or [5, Tag 0593]), but in general this is not necessarily true even for a semi-local ring (a ring with finitely many maximal ideals) which is not local. As a specific example, let $n > 1$ be a natural number which has at least two distinct prime factors and let $n = p_1^{s_1} \cdots p_k^{s_k}$ be its prime factorization where p_i are distinct prime numbers and $s_i \geq 1$ for all i. Each A_i can be considered as an R-module through the canonical ring map $R \to A_i$ where $R = \mathbb{Z}/n\mathbb{Z}$ and $A_i = \mathbb{Z}/p_i^{s_i}\mathbb{Z}$. By the Chinese remainder theorem, R as a module over itself is isomorphic to the direct sum $\bigoplus_{i=1}^{k} A_i$. Thus each A_i is R-projective since it is a direct summand of the free R-module R. But none of them is R-free since every non-zero free R-module has at least n elements while $p_i^{s_i} < n$ for all i. Note that R is a semi-local ring with the maximal ideals $p_i\mathbb{Z}/n\mathbb{Z}$.

Theorem 3.13. *Let X be a subset of $\operatorname{Spec}(R)$ with the property that for each maximal ideal $\mathfrak{m}$ of R there exists some $\mathfrak{p} \in X$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. If the collection of subsets $X \cap V(f)$ with $f \in R$ satisfies either the ascending chain condition or the descending chain condition, then R is an S-ring.*

Proof. By [12, Corollary 3.4], it suffices to show that R/J is an S-ring where $J = \bigcap_{\mathfrak{p}\in X} \mathfrak{p}$. Let (x_n) be a sequence of elements of R/J such that $x_n = x_n x_{n+1}$ for all n. Suppose $x_n = a_n + J$ for all n. Let $E_n = X \cap V(a_n)$ and let $F_n = X \cap V(1 - a_n)$. Clearly $E_n \supseteq E_{n+1}$, $F_n \subseteq F_{n+1}$ and $X = E_n \cup F_{n+1}$. First assume the descending chain condition. Then there exists some $d \geq 1$ such that $E_n = E_d$ for all $n \geq d$. Therefore $X = E_n \cup F_n$ for all $n > d$. Thus $a_n(1 - a_n) \in \mathfrak{p}$ for all $\mathfrak{p} \in X$ and all $n > d$. It follows that $x_n = x_n^2$ for all $n > d$. We claim that the ascending chain $V(1 - x_{d+1}) \subseteq V(1 - x_{d+2}) \subseteq \cdots$ eventually stabilizes. If not, then we may find some $k > d$ such that $V(1 - x_k)$ is a proper subset of $V(1 - x_{k+1})$. Thus there exists a prime ideal $\mathfrak{q}$ of R such that $J \subseteq \mathfrak{q}$ and $1 - a_{k+1} \in \mathfrak{q}$ but $a_k \in \mathfrak{q}$. There is a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{q} \subseteq \mathfrak{m}$. By the hypothesis, there is a $\mathfrak{p} \in X$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. Clearly $1 - a_{k+1}, a_k \in \mathfrak{p}$. This means that E_{k+1} is a proper subset of E_k. But this is a contradiction. This establishes the claim. Therefore there exists some ℓ with $\ell \geq d + 1$ such that $V(1 - x_n) = V(1 - x_\ell)$ for all $n \geq \ell$. But we have $D(x_n) = V(1 - x_n) = V(1 - x_\ell) = D(x_\ell)$ and the x_n are idempotent for all $n \geq \ell$. This yields that $x_n = x_\ell$ for all $n \geq \ell$. Thus by Theorem 3.10(viii), R/J is an S-ring in the case of the descending chain condition. Apply a similar argument as above for the chain $F_1 \subseteq F_2 \subseteq \cdots$ in the case of the ascending chain condition. □

Corollary 3.14. *If the collection of subsets* $\mathrm{Min}(R) \cap V(f)$ *with* $f \in R$ *satisfies either the ascending chain condition or the descending chain condition, then* R *is an S-ring.*

Proof. It implies from Theorem 3.13 by taking $X = \mathrm{Min}(R)$. □

Corollary 3.15 ([10, Proposition 7.6])**.** *If the collection of subsets* $\mathrm{Max}(R) \cap V(f)$ *with* $f \in R$ *satisfies either the ascending chain condition or the descending chain condition, then* R *is an S-ring.*

Proof. In Theorem 3.13, put $X = \mathrm{Max}(R)$. □

Proposition 3.16. *The direct product of a family of rings* $(R_i)_{i\in I}$ *is an S-ring if and only if* I *is a finite set and each* R_i *is an S-ring.*

Proof. Let $R = \prod_{i\in I} R_i$ be an S-ring. We may assume that all of the rings R_i are non-zero. Suppose I is an infinite set. Consider a well-ordered relation $<$ on I. Let i_1 be the least element of I and for each natural number $n \geq 1$, by induction, let i_{n+1} be the least element of $I \setminus \{i_1, \ldots, i_n\}$. Now we define $x_n = (r_{n,i})_{i\in I}$ as an element of R by $r_{n,i} = 1$ for all $i \in \{i_1, \ldots, i_n\}$ and $r_{n,i} = 0$ for all $i \in I \setminus \{i_1, \ldots, i_n\}$. Clearly the sequence (x_n) satisfies the condition $x_n = x_n x_{n+1}$. Thus, by Theorem 3.10, there is some k such that $x_n = x_k$ for all $n \geq k$. But this is a contradiction. Thus I should be a finite set. The remaining assertions, by applying Theorem 3.10(viii), are straightforward. □

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