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MULTIPLIERS OF DIRICHLET-TYPE SUBSPACES OF BLOCH SPACE

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Abstract. Let $M(X, Y)$ denote the space of multipliers from X to Y, where X and Y are analytic function spaces. As we known, for Dirichlet-type spaces $\mathcal{D}^p_\alpha$, $M(\mathcal{D}^p_{p-1}, \mathcal{D}^q_{q-1}) = \{0\}$, if $p \neq q$, $0 < p, q < \infty$. If $0 < p, q < \infty$, $p \neq q$, $0 < s < 1$ such that $p + s, q + s > 1$, then $M(\mathcal{D}^p_{p-2+s}, \mathcal{D}^q_{q-2+s}) = \{0\}$. However, $X \cap \mathcal{D}^p_{p-1} \subseteq X \cap \mathcal{D}^q_{q-1}$ and $X \cap \mathcal{D}^p_{p-2+s} \subseteq X \cap \mathcal{D}^q_{q-2+s}$ whenever X is a subspace of the Bloch space $\mathcal{B}$ and $0 < p \leq q < \infty$. This says that the set of multipliers $M(X \cap \mathcal{D}^p_{p-2+s}, X \cap \mathcal{D}^q_{q-2+s})$ is nontrivial. In this paper, we study the multipliers $M(X \cap \mathcal{D}^p_{p-2+s}, X \cap \mathcal{D}^q_{q-2+s})$ for distinct classical subspaces X of the Bloch space $\mathcal{B}$, where $X = \mathcal{B}$, $BMOA$ or $\mathcal{H}^\infty$.

1. Introduction

Let $\mathbb{D}$ denote the unit disk of the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$, the unit circle. Denote by $\mathcal{H}(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. The Bloch space $\mathcal{B}$, consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

Let $f \in \mathcal{H}(\mathbb{D})$. For $0 < p < \infty$, $0 < r < 1$, set

$$M^p_p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

and

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

The Hardy space $\mathcal{H}^p (0 < p \leq \infty)$ is defined as the space of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{H}^p} = \sup_{0<r<1} M_p(r, f) < \infty.$$

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For the theory about the Hardy space $\mathcal{H}^p$, we refer the readers to [6]. The $BMOA$ space is the set of those $f \in \mathcal{H}^1$ whose boundary values have bounded mean oscillation on the unit circle $\partial\mathbb{D}$ [10]. It is well known that $BMOA$ is contained in the Bloch space $\mathcal{B}$ continuously.

The weighted Dirichlet-type space $\mathcal{D}^p_\alpha(0 < p < \infty, \alpha > -1)$ is the class of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{\mathcal{D}^p_\alpha} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dA_\alpha(z) < \infty,$$

here $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ and $dA(z) = \frac{1}{\pi}dxdy$ is the normalized Lebesgue area measure. It is well known that when $p < \alpha+1$, $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$, the Bergman space [7]. If $p > \alpha+2$, then $\mathcal{D}^p_\alpha \subseteq \mathcal{H}^\infty$. Therefore, when $\alpha+1 \le p \le \alpha+2$, $\mathcal{D}^p_\alpha$ is a proper Dirichlet-type space. The spaces $\mathcal{D}^p_{p-1}$ are closely related with Hardy spaces. In fact, $\mathcal{D}^2_1 = \mathcal{H}^2$. Notice that when $0 < p \le 2$, $\mathcal{D}^p_{p-1} \subseteq \mathcal{H}^p$ [7]. When $2 \le p < \infty$, $\mathcal{H}^p \subseteq \mathcal{D}^p_{p-1}$ [14].

For $g \in \mathcal{H}(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g f(z) = g(z)f(z), \ z \in \mathbb{D}, \ f \in \mathcal{H}(\mathbb{D}).$$

Let X, Y be the norm spaces of analytic functions in $\mathbb{D}$. We denote by $M(X,Y)$ the space of multipliers from X to Y, in other words,

$$M(X,Y) = \{g \in \mathcal{H}(\mathbb{D}) : fg \in Y, \ \forall f \in X\}.$$

For convenience, we write $M(X) := M(X,X)$. Denote the norm of the multiplication operator M_g by $\|M_g\|$. From [2,3], we see that

$$M(\mathcal{B}) = \mathcal{H}^\infty \cap \mathcal{B}_{\log}. \tag{1}$$

Here $\mathcal{B}_{\log}$ is the logarithmic Bloch space, consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_{\log}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| \left(\log \frac{2}{1-|z|^2}\right) < \infty.$$

In [15], we have that

$$M(BMOA) = BMOA_{\log} \cap \mathcal{H}^\infty, \tag{2}$$

where $BMOA_{\log}$ is the space of those functions $f \in \mathcal{H}^1$ such that the positive Borel measure $(1-|z|^2)|f'(z)|^2dA(z)$ is a 2-logarithmic Carleson measure. In other words, $f \in BMOA_{\log}$ if and only if $f \in \mathcal{H}^1$ such that

$$\sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|}\right)^2 \int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)dA(z) < \infty,$$

where φ_a is the disk automorphism which interchange the origin and a, that is

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \ z \in \mathbb{D}. \tag{3}$$

The multipliers of Dirichlet-type space $\mathcal{D}^p_\alpha$ have been studied in [8,9,11,12]. In [8], the authors proved that for $1 < p \le q < \infty$, a function $g \in \mathcal{H}(\mathbb{D})$ belongs to $M(\mathcal{D}^p_{p-2}, \mathcal{D}^q_{q-2})$ if and only if $g \in \mathcal{H}^\infty$ and the positive Borel measure μ

defined by $d\mu(z) = |g'(z)|^q(1-|z|^2)^{q-2}dA(z)$ is a q-Carleson measure for $\mathcal{D}^q_{q-2}$. If $1 < q < p < \infty$, then $M(\mathcal{D}^p_{p-2}, \mathcal{D}^q_{q-2}) = \{0\}$.

It is standard that if $0 < p, q < \infty$ and $p \neq q$, then we have

$$M(\mathcal{D}^p_{p-1}, \mathcal{D}^q_{q-1}) = \{0\}.$$

Let X be a non-zero subspace of the Bloch space $\mathcal{B}$. The space $X \cap \mathcal{D}^p_\alpha$ is equipped with the norm

$$\|f\|_{X\cap\mathcal{D}^p_\alpha} = \|f\|_X + \|f\|_{\mathcal{D}^p_\alpha}.$$

Lemma 1 in [5] says that if $0 < p \leq q < \infty$, then $X \cap \mathcal{D}^p_{p-1} \subseteq X \cap \mathcal{D}^q_{q-1}$. It follows that the set of multipliers $M(X \cap \mathcal{D}^p_{p-1}, X \cap \mathcal{D}^q_{q-1})$ is nontrivial.

By Corollary 1 in [12] and Theorem 2 in [9], for all $p \neq q$ and $0 < s < 1$,

$$M(\mathcal{D}^p_{p-2+s}, \mathcal{D}^q_{q-2+s}) = \{0\}.$$

But when $0 < p \leq q < \infty$, if $f \in X \cap \mathcal{D}^p_{p-2+s}$, then

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^q(1-|z|^2)^{q-2+s}dA(z) &\leq \|f\|^{q-p}_{\mathcal{B}} \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z)\\ &\leq \|f\|^{q-p}_{\mathcal{B}}\|f\|^p_{\mathcal{D}^p_{p-2+s}}\\ &\leq C\|f\|^{q-p}_X\|f\|^p_{\mathcal{D}^p_{p-2+s}}\\ &\leq C\|f\|^q_{X\cap\mathcal{D}^p_{p-2+s}}.\end{aligned}$$

Hence $f \in X \cap \mathcal{D}^q_{q-2+s}$ and $\|f\|_{X\cap\mathcal{D}^q_{q-2+s}} \leq C\|f\|_{X\cap\mathcal{D}^p_{p-2+s}}$. In other words, $X\cap\mathcal{D}^p_{p-2+s} \subseteq X\cap\mathcal{D}^q_{q-2+s}$. So the set of multipliers $M(X\cap\mathcal{D}^p_{p-2+s}, X\cap\mathcal{D}^q_{q-2+s})$ is also nontrivial.

From [5], we see that if $q > 1$ and $0 < p \leq q < \infty$, then

$$M(\mathcal{B}\cap\mathcal{D}^p_{p-1}, \mathcal{B}\cap\mathcal{D}^q_{q-1}) = M(\mathcal{B})$$

and

$$M(BMOA\cap\mathcal{D}^p_{p-1}, BMOA\cap\mathcal{D}^q_{q-1}) = M(BMOA).$$

If $0 < p \leq q < \infty$, then

$$M(\mathcal{H}^\infty\cap\mathcal{D}^p_{p-1}, \mathcal{H}^\infty\cap\mathcal{D}^q_{q-1}) = \mathcal{H}^\infty\cap\mathcal{D}^q_{q-1}.$$

Motivated by [8] and [5], it is natural to ask what is the set of multipliers $M(X\cap\mathcal{D}^p_{p-2+s}, X\cap\mathcal{D}^q_{q-2+s})$ when $0 < s < 1$. In this paper, we characterize the multipliers $M(X\cap\mathcal{D}^p_{p-2+s}, X\cap\mathcal{D}^q_{q-2+s})$ when $0 < s < 1$, $X = \mathcal{B}$, $X = BMOA$ or $X = \mathcal{H}^\infty$, respectively. Our main results are stated as follows.

Theorem 1.1. *Suppose that $g \in \mathcal{H}(\mathbb{D})$, $0 < p \leq q < \infty$, $0 < s < 1$ satisfying $p + s > 1$. Define the positive Borel measure μ by $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$, then*

(i) *$g \in M(\mathcal{B}\cap\mathcal{D}^p_{p-2+s}, \mathcal{B}\cap\mathcal{D}^q_{q-2+s})$ if and only if $g \in M(\mathcal{B})$ and μ is a q-Carleson measure for $\mathcal{B}\cap\mathcal{D}^p_{p-2+s}$.*

(ii) $g \in M(BMOA \cap \mathcal{D}^p_{p-2+s}, BMOA \cap \mathcal{D}^q_{q-2+s})$ *if and only if* $g \in M(BMOA)$ *and* μ *is a* q*-Carleson measure for* $BMOA \cap \mathcal{D}^p_{p-2+s}$.

(iii) $M(\mathcal{H}^\infty \cap \mathcal{D}^p_{p-2+s}, \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s}) = \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s}$.

Theorem 1.2. *Suppose* $0 < q < p < \infty$, $0 < s < 1$ *with* $q + s > 1$. *Then*

(i) $M(\mathcal{B} \cap \mathcal{D}^p_{p-2+s}, \mathcal{B} \cap \mathcal{D}^q_{q-2+s}) = \{0\}$.

(ii) $M(BMOA \cap \mathcal{D}^p_{p-2+s}, BMOA \cap \mathcal{D}^q_{q-2+s}) = \{0\}$.

(iii) $M(\mathcal{H}^\infty \cap \mathcal{D}^p_{p-2+s}, \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s}) = \{0\}$.

Throughout this paper, C denotes a positive constant depending only on indexes $p, q, s, \ldots$, it is not necessary to be the same from one line to another. Let f and g be two positive functions. For convenience, we write $f \preceq g$, if $f \le Cg$ holds, where C is a positive constant independent of f and g. If $f \preceq g$ and $g \preceq f$, then we say $f \asymp g$.

2. Preliminary

In this section, we state some definitions and lemmas which will be used in the paper. Let I be an arc of $\partial\mathbb{D}$. Denote the normalized Lebesgue measure of I by $|I|$, that is, $|I| = \frac{1}{2\pi}\int_I |d\xi|$. For an arc $I \subseteq \partial\mathbb{D}$, the Carleson square based on I is defined by

$$S(I) := \left\{ z \in \mathbb{D} : 1 - |I| \le |z| < 1, \frac{z}{|z|} \in I \right\}.$$

If $I = \partial\mathbb{D}$, then we set $S(I) = \mathbb{D}$. Let μ be a positive Borel measure on $\mathbb{D}$. For $0 \le \alpha < \infty$, $0 < s < \infty$, we say that μ is an α-logarithmic s-Carleson measure if there exists a constant $C > 0$ such that for all arcs $I \subseteq \partial\mathbb{D}$,

$$\mu(S(I)) \le C \frac{|I|^s}{(\log \frac{2}{|I|})^\alpha}.$$

If $\alpha = 0$, then μ is called an s-Carleson measure. If $\alpha = 0$ and $s = 1$, then μ is said to be a Carleson measure. Recall that an $f \in \mathcal{H}^1$ belongs to the space $BMOA$ if and only if the positive Borel measure $|f'(z)|^2(1 - |z|^2)dA(z)$ is a Carleson measure.

Let $(X, \|\cdot\|_X)$ be a normed space of analytic functions. Then a positive Borel measure μ on $\mathbb{D}$ is said to be an s-Carleson measure for X, if there exists a constant $C > 0$ such that for all $f \in X$,

$$\int_{\mathbb{D}} |f(z)|^s d\mu(z) \le C\|f\|_X^s.$$

The following lemma can be found in Theorem 2 of [17], which plays an important role in the proofs of theorems.

Lemma 2.1. *Suppose that* $0 \le \alpha < \infty$ *and* $0 < s < \infty$. *Then a positive Borel measure* μ *on* $\mathbb{D}$ *is an* α*-logarithmic* s*-Carleson measure if and only if*

$$\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|}\right)^\alpha \int_{\mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2}\right)^s d\mu(z) < \infty.$$

We will make use of the lacunary power series (also called power series with Hadamard gaps) of a function $f \in \mathcal{H}(\mathbb{D})$, that is, f is of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \; z \in \mathbb{D},$$

with $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all k. Several known results on lacunary power series will be used in this paper. We put them together in the following statement, see [1, 4, 5, 13, 19].

Lemma 2.2. *Suppose that $0 < p < \infty$, $\alpha > -1$. $f \in \mathcal{H}(\mathbb{D})$ which is given by a lacunary power series, $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$, $z \in \mathbb{D}$. Then*

(i) *$f \in \mathcal{D}^p_\alpha$ if and only if $\sum_{k=0}^{\infty} n_k^{p-\alpha-1} |a_k|^p < \infty$, and*

$$\|f - f(0)\|^p_{\mathcal{D}^p_\alpha} \asymp \sum_{k=0}^{\infty} n_k^{p-\alpha-1} |a_k|^p.$$

(ii) *$f \in \mathcal{H}^\infty$ if and only if $\sum_{k=0}^{\infty} |a_k| < \infty$, and*

$$\|f\|_{\mathcal{H}^\infty} \asymp \sum_{k=0}^{\infty} |a_k|.$$

(iii) *$f \in \mathcal{B}$ if and only if $\sup_k |a_k| < \infty$, and*

$$\|f\|_{\mathcal{B}} \asymp \sup_k |a_k|.$$

The following estimate can be found in [13].

Lemma 2.3. *Suppose that $\beta > -1$, $s > 0$ and $f \in \mathcal{H}(\mathbb{D})$ with $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$, $z \in \mathbb{D}$. Then*

$$\sum_{k=1}^{\infty} n_k^{-(\beta+1)} |a_k|^s \asymp \int_0^1 (1-r)^\beta |f(re^{i\theta})|^s dr$$

for all $\theta \in \mathbb{R}$.

The following lemma is useful in theory of analytic function spaces and operator theory, see [18].

Lemma 2.4. *Suppose that $z \in \mathbb{D}$, c is real, $t > -1$, and*

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1 - \bar{w}z|^{2+t+c}} dA(w).$$

(i) *If $c < 0$, then as a function of z, $I_{c,t}$ is bounded on $\mathbb{D}$.*

(ii) *If $c = 0$, then*

$$I_{c,t}(z) \asymp \log \frac{1}{1-|z|^2} \quad \textit{as} \quad |z| \to 1^-.$$

(iii) *If $c > 0$, then*

$$I_{c,t}(z) \asymp \frac{1}{(1-|z|^2)^c} \quad \textit{as} \quad |z| \to 1^-.$$

We will use the following estimate to prove our results, which can be found in [16].

Lemma 2.5. *For $s > -1$, $r, t > 0$ with $0 < r + t - s - 2 < r$, there exists a constant $C > 0$ such that for any $a, b \in \mathbb{D}$,*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{a}z|^r |1-\bar{b}z|^t} dA(z) \le \frac{C}{(1-|a|^2)^{r+t-s-2}}.$$

3. Proof of main results

Proof of Theorem 1.1. (i) First suppose that $g \in M(\mathcal{B} \cap \mathcal{D}^p_{p-2+s}, \mathcal{B} \cap \mathcal{D}^q_{q-2+s})$. For any $a \in \mathbb{D}$, let φ_a be defined by (3) and f_a be defined by

$$f_a(z) = \log \frac{1}{1-\bar{a}z}, \ z \in \mathbb{D}.$$

A simple computation shows that $\sup_{a\in\mathbb{D}} \|\varphi_a\|_{\mathcal{B}} < \infty$ and $\sup_{a\in\mathbb{D}} \|\varphi_a\|_{\mathcal{D}^p_{p-2+s}} < \infty$. This implies that $\varphi_a \in \mathcal{B} \cap \mathcal{D}^p_{p-2+s}$ and $\sup_{a\in\mathbb{D}} \|\varphi_a\|_{\mathcal{D}^p_{p-2+s}\cap\mathcal{B}} < \infty$. We have $g\varphi_a \in \mathcal{B} \cap \mathcal{D}^q_{q-2+s}$ and

$$\begin{aligned}
(1-|z|^2)|(g\varphi_a)'(z)| &\le \|g\varphi_a\|_{\mathcal{B}} \\
&\le \|g\varphi_a\|_{\mathcal{B}\cap\mathcal{D}^q_{q-2+s}} \\
&\le \|M_g\| \ \|\varphi_a\|_{\mathcal{B}\cap\mathcal{D}^p_{p-2+s}} \le C\|M_g\|,
\end{aligned}$$

that is,

$$(1-|z|^2)|g'(z)\varphi_a(z) + g(z)\varphi_a'(z)| \le C\|M_g\|.$$

Taking $z = a$, using the fact that $\varphi_a(a) = 0$ and $|\varphi_a'(a)| = \frac{1}{1-|a|^2}$ we get

$$|g(a)| \le C\|M_g\|,$$

which implies that $g \in \mathcal{H}^\infty$.

It is obvious that $f_a'(z) = \frac{\bar{a}}{1-\bar{a}z}$ and $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathcal{B}} < \infty$. By Lemma 2.4, there is a constant $C > 0$ independent of a such that

$$\begin{aligned}
\int_{\mathbb{D}} |f_a'(z)|^p (1-|z|^2)^{p-2+s} dA(z) &\le \int_{\mathbb{D}} \frac{(1-|z|^2)^{p-2+s}}{|1-\bar{a}z|^p} dA(z) \\
&= \int_{\mathbb{D}} \frac{(1-|z|^2)^{p-2+s}}{|1-\bar{a}z|^{2+p-2+s-s}} dA(z) \\
&\le C.
\end{aligned}$$

This implies that $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathcal{D}^p_{p-2+s}} < \infty$. Hence, we have $f_a \in \mathcal{B} \cap \mathcal{D}^p_{p-2+s}$ and $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathcal{B}\cap\mathcal{D}^p_{p-2+s}} < \infty$. So $gf_a \in \mathcal{B} \cap \mathcal{D}^q_{q-2+s}$ and

$$(1-|z|^2)|(gf_a)'(z)| \le \|gf_a\|_{\mathcal{B}\cap\mathcal{D}^q_{q-2+s}} \le \|M_g\| \ \|f_a\|_{\mathcal{B}\cap\mathcal{D}^p_{p-2+s}} \le C\|M_g\|. \tag{4}$$

On the other hand, since $g \in \mathcal{H}^\infty$,

$$(1-|z|^2)|g(z)f_a'(z)| \le \|g\|_{\mathcal{H}^\infty}\|f_a\|_{\mathcal{B}} \le C\|g\|_{\mathcal{H}^\infty}. \tag{5}$$

Combining (4) and (5) we deduce that

$$(1-|z|^2)|g'(z)f_a(z)| \le C(\|M_g\| + \|g\|_{\mathcal{H}^\infty}).$$

Taking $z=a$ we obtain

$$(1-|a|^2)|g'(a)|\log\frac{1}{1-|a|^2} \le C,$$

which shows that $g \in \mathcal{B}_{\log}$. From (1) we see that $g \in M(\mathcal{B})$.

We next show that $d\mu(z) = |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)$ is a q-Carleson measure for $\mathcal{B}\cap\mathcal{D}^p_{p-2+s}$. Let $f \in \mathcal{B}\cap\mathcal{D}^p_{p-2+s}$. Since $g \in \mathcal{H}^\infty$, we have

$$\begin{aligned}\int_{\mathbb{D}} |g(z)|^q|f'(z)|^q(1-|z|^2)^{q-2+s}dA(z) &\le \|g\|^q_{\mathcal{H}^\infty}\|f\|^{q-p}_{\mathcal{B}}\|f\|^p_{\mathcal{D}^p_{p-2+s}}\\ &\le \|g\|^q_{\mathcal{H}^\infty}\|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-2+s}}.\end{aligned} \tag{6}$$

Note that $gf \in \mathcal{B}\cap\mathcal{D}^q_{q-2+s}$,

$$\begin{aligned}\int_{\mathbb{D}} |(gf)'(z)|^q(1-|z|^2)^{q-2+s}dA(z) &\le \|gf\|^q_{\mathcal{B}\cap\mathcal{D}^q_{q-2+s}}\\ &\le \|M_g\|^q\|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-2+s}}.\end{aligned} \tag{7}$$

Combining (6) and (7) implies

$$\int_{\mathbb{D}} |f(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+s}dA(z) \le C(\|g\|^q_{\mathcal{H}^\infty} + \|M_g\|^q)\|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-2+s}}.$$

That is, $d\mu(z) = |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)$ is a q-Carleson measure for $\mathcal{B}\cap\mathcal{D}^p_{p-2+s}$.

Suppose that $g \in M(\mathcal{B})$ and $d\mu(z) = |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)$ is a q-Carleson measure for $\mathcal{B}\cap\mathcal{D}^p_{p-2+s}$, we prove that $g \in M(\mathcal{B}\cap\mathcal{D}^p_{p-2+s}, \mathcal{B}\cap\mathcal{D}^q_{q-2+s})$. For any $f \in \mathcal{B}\cap\mathcal{D}^p_{p-2+s}$, we have $gf \in \mathcal{B}$. It remains to prove that $gf \in \mathcal{D}^q_{q-2+s}$. Since $d\mu(z) = |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)$ is a q-Carleson measure for $\mathcal{B}\cap\mathcal{D}^p_{p-2+s}$, there is a constant $C>0$ independent of f such that

$$\int_{\mathbb{D}} |f(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+s}dA(z) \le C\|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-2+s}}. \tag{8}$$

Combining (6) and (8) we see that

$$\int_{\mathbb{D}} |(gf)'(z)|^q(1-|z|^2)^{q-2+s}dA(z) \le C\|f\|^q_{\mathcal{B}\cap\mathcal{D}^p_{p-2+s}},$$

which implies that $gf \in \mathcal{D}^q_{q-2+s}$.

The idea of proofs of (ii) and (iii) is similar to that of (i). For the completeness of the paper, we give their proofs briefly below.

(ii) Assume that $g \in M(BMOA\cap\mathcal{D}^p_{p-2+s}, BMOA\cap\mathcal{D}^q_{q-2+s})$. For any $a \in \mathbb{D}$, let φ_a and f_a be defined as in the proof of (i). An easy computation shows that $\sup_{a\in\mathbb{D}}\|f_a\|_{\mathcal{D}^p_{p-2+s}} < \infty$. Since $\frac{1}{2\pi}\int_0^{2\pi}|\log\frac{1}{1-\bar{a}e^{i\theta}}|d\theta < \infty$, we have $f_a \in \mathcal{H}^1$. Since $f_a'(z) = \frac{\bar{a}}{1-\bar{a}z}$, by Lemma 2.5, there exists a constant $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{D}}|f_a'(z)|^2(1-|\varphi_b(z)|^2)dA(z) &= \int_{\mathbb{D}}\frac{|a|^2}{|1-\bar{a}z|^2}\frac{(1-|b|^2)(1-|z|^2)}{|1-\bar{b}z|^2}dA(z)\\ &\le (1-|b|^2)\int_{\mathbb{D}}\frac{1-|z|^2}{|1-\bar{a}z|^2|1-\bar{b}z|^2}dA(z)\\ &\le C.\end{aligned}$$

Hence, the Borel measure $|f_a'(z)|^2(1-|z|^2)dA(z)$ is a Carleson measure by Lemma 2.1, so $f_a \in BMOA$. Since C is independent of a, we deduce that $\sup_{a\in\mathbb{D}}\|f_a\|_{BMOA} < \infty$. Hence, $f_a \in BMOA\cap\mathcal{D}^p_{p-2+s}$ and $\sup_{a\in\mathbb{D}}\|f_a\|_{BMOA\cap\mathcal{D}^p_{p-2+s}} < \infty$. In addition, a similar argument implies $g \in \mathcal{H}^\infty$. So $gf_a \in BMOA\cap\mathcal{D}^q_{q-2+s}$. Hence, there exists a constant $C > 0$ such that for any arc I,

$$\int_{S(I)}|(gf_a)'(z)|^2(1-|z|^2)dA(z) \le C|I| \tag{9}$$

and

$$\int_{S(I)}|f_a'(z)|^2(1-|z|^2)dA(z) \le C|I|. \tag{10}$$

Then by $g \in \mathcal{H}^\infty$, (9) and (10) we obtain

$$\int_{S(I)}|g'(z)|^2|f_a(z)|^2(1-|z|^2)dA(z) \le C|I|. \tag{11}$$

Take $a = (1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the center of I, then for any $z \in S(I)$,

$$|1-\bar{a}z| \asymp 1-|a| = |I|,\ \ |f_a(z)| \asymp \log\frac{1}{|I|}.$$

Thus (11) implies that

$$\left(\log\frac{1}{|I|}\right)^2\int_{S(I)}|g'(z)|^2(1-|z|^2)dA(z) \le C|I|,$$

in other words, $g \in BMOA_{\log}$. Therefore $g \in M(BMOA)$ from (2).

We turn to show that $|g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)$ is a q-Carleson measure for $BMOA\cap\mathcal{D}^p_{p-2+s}$. For every $f \in BMOA\cap\mathcal{D}^p_{p-2+s}$, we have $gf \in BMOA\cap\mathcal{D}^q_{q-2+s}$ and

$$\begin{aligned}\int_{\mathbb{D}}|(gf)'(z)|^q(1-|z|^2)^{q-2+s}dA(z) &\le \|gf\|^q_{\mathcal{D}^q_{q-2+s}}\\ &\le \|gf\|^q_{BMOA\cap\mathcal{D}^q_{q-2+s}}\\ &\le \|M_g\|^q\|f\|^q_{BMOA\cap\mathcal{D}^p_{p-2+s}}.\end{aligned} \tag{12}$$

A similar argument as in the proof of (i) shows that

$$\int_{\mathbb{D}} |g(z)|^q |f'(z)|^q (1-|z|^2)^{q-2+s} dA(z) \le \|g\|^q_{\mathcal{H}^\infty} \|f\|^q_{BMOA\cap\mathcal{D}^p_{p-2+s}}. \tag{13}$$

Combining (12) and (13) yields

$$\int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|^2)^{q-2+s} dA(z) \le C(\|g\|^q_{\mathcal{H}^\infty} + \|M_g\|^q) \|f\|^q_{BMOA\cap\mathcal{D}^p_{p-2+s}}.$$

We conclude that $d\mu(z) = |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)$ is a q-Carleson measure for $BMOA \cap \mathcal{D}^p_{p-2+s}$.

Conversely, for any $f \in BMOA \cap \mathcal{D}^p_{p-2+s}$, we have $gf \in BMOA$. We only need to prove $gf \in \mathcal{D}^q_{q-2+s}$. By hypothesis, there exists a constant $C > 0$ independent of f such that

$$\int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|^2)^{q-2+s} dA(z) \le C\|f\|^q_{BMOA\cap\mathcal{D}^p_{p-2+s}}. \tag{14}$$

By (13) and (14) we obtain

$$\int_{\mathbb{D}} |(gf)'(z)|^q (1-|z|^2)^{q-2+s} dA(z) \le C\|f\|^q_{BMOA\cap\mathcal{D}^p_{p-2+s}}.$$

That is, $gf \in \mathcal{D}^q_{q-2+s}$.

(iii) We only need to show

$$M(\mathcal{H}^\infty \cap \mathcal{D}^p_{p-2+s}, \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s}) \supseteq \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s},$$

since the converse is obvious.

Let $g \in \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s}$. For any $f \in \mathcal{H}^\infty \cap \mathcal{D}^p_{p-2+s}$, we have $gf \in \mathcal{H}^\infty$. It remains to prove that $gf \in \mathcal{D}^q_{q-2+s}$. These hypothesis imply

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|^2)^{q-2+s} dA(z) &\le \|f\|^q_{\mathcal{H}^\infty} \|g\|^q_{\mathcal{D}^q_{q-2+s}} \\ &\le \|f\|^q_{\mathcal{H}^\infty\cap\mathcal{D}^p_{p-2+s}} \|g\|^q_{\mathcal{D}^q_{q-2+s}}\end{aligned}$$

and

$$\int_{\mathbb{D}} |g(z)|^q |f'(z)|^q (1-|z|^2)^{q-2+s} dA(z) \le \|g\|^q_{\mathcal{H}^\infty} \|f\|^q_{\mathcal{H}^\infty\cap\mathcal{D}^p_{p-2+s}}.$$

Hence

$$\int_{\mathbb{D}} |(gf)'(z)|^q (1-|z|^2)^{q-2+s} dA(z) \le C(\|g\|^q_{\mathcal{D}^q_{q-2+s}} + \|g\|^q_{\mathcal{H}^\infty}) \|f\|^q_{\mathcal{H}^\infty\cap\mathcal{D}^p_{p-2+s}}.$$

The proof is complete. □

Proof of Theorem 1.2. (i) Suppose that $g \in M(\mathcal{B} \cap \mathcal{D}^p_{p-2+s}, \mathcal{B} \cap \mathcal{D}^q_{q-2+s})$ and $g \neq 0$, then $g \in \mathcal{B} \cap \mathcal{D}^q_{q-2+s}$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k},\ a_k = n_k^{\frac{s-1}{q}},\ z \in \mathbb{D},$$

with $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all k. Since $\sum_{k=1}^{\infty} |a_k| < \infty$, by Lemma 2.2 we have $f \in \mathcal{H}^{\infty} \subseteq \mathcal{B}$. It is not difficult to see that $\sum_{k=0}^{\infty} n_k^{1-s}|a_k|^p < \infty$, Lemma 2.2 yields $f \in \mathcal{D}^p_{p-2+s}$. Hence $f \in \mathcal{B} \cap \mathcal{D}^p_{p-2+s}$ and $fg \in \mathcal{B} \cap \mathcal{D}^q_{q-2+s}$. We have

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|(gf)'(z)|^q dA(z) \leq \|gf\|^q_{\mathcal{D}^q_{q-2+s}} < \infty$$

and

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|g'(z)f(z)|^q dA(z) \leq \|f\|^q_{\mathcal{H}^{\infty}} \|g\|^q_{\mathcal{D}^q_{q-2+s}} < \infty.$$

These imply

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|g(z)f'(z)|^q dA(z) < \infty. \tag{15}$$

On the other hand, $f'(z) = \sum_{k=0}^{\infty} a_k n_k z^{n_k-1}$, by Lemma 2.3 we see that

$$\int_0^1 (1-r)^{q-2+s}|f'(re^{i\theta})|^q dr \asymp \sum_{k=0}^{\infty} n_k^{-(q+s-1)}|a_k n_k|^q = \infty.$$

Since $g \in \mathcal{D}^q_{q-2+s} \subseteq \mathcal{H}^q$ (see [9], p. 1877), g has a finite and nonzero radial limit almost everywhere on the boundary of $\mathbb{D}$. Thus

$$\int_0^1 (1-r)^{q-2+s}|f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty$$

for almost all $\theta \in \mathbb{R}$ (see [9], p. 1878). This is in contradiction to (15).

(ii) Assume that $g \in M(BMOA \cap \mathcal{D}^p_{p-2+s}, BMOA \cap \mathcal{D}^q_{q-2+s})$ and $g \neq 0$, then $g \in BMOA \cap \mathcal{D}^q_{q-2+s}$. Let $a_k = (2^k)^{\frac{s-1}{q}}$, $k = 1, 2, \ldots$, and

$$f(z) = \sum_{k=0}^{\infty} a_k z^{2^k}, \ z \in \mathbb{D}.$$

Then $f \in \mathcal{H}^{\infty} \cap \mathcal{D}^p_{p-2+s}$ by Lemma 2.2. Hence $f \in BMOA \cap \mathcal{D}^p_{p-2+s}$ and $fg \in BMOA \cap \mathcal{D}^q_{q-2+s}$. So

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|(gf)'(z)|^q dA(z) \leq \|gf\|^q_{\mathcal{D}^q_{q-2+s}} < \infty$$

and

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|g'(z)f(z)|^q dA(z) \leq \|f\|^q_{\mathcal{H}^{\infty}} \|g\|^q_{\mathcal{D}^q_{q-2+s}} < \infty.$$

We get

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|g(z)f'(z)|^q dA(z) < \infty.$$

Since $f'(z) = \sum_{k=0}^{\infty} 2^k a_k z^{2^k-1}$, from Lemma 2.3,

$$\int_0^1 (1-r)^{q-2+s}|f'(re^{i\theta})|^q dr \asymp \sum_{k=0}^{\infty}(2^k)^{-(q+s-1)}|a_k 2^k|^q = \infty.$$

Therefore, for almost all $\theta \in \mathbb{R}$,

$$\int_0^1 (1-r)^{q-2+s}|f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty.$$

This is a contradiction.

(iii) Assume $g \in M(\mathcal{H}^\infty \cap \mathcal{D}^p_{p-2+s}, \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s})$ and $g \neq 0$, then $g \in \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s}$. Let $f \in \mathcal{H}(\mathbb{D})$ be defined as in the proof of (i). The same argument as in the proof of (i) shows that $f \in \mathcal{H}^\infty \cap \mathcal{D}^p_{p-2+s}$. So $fg \in \mathcal{H}^\infty \cap \mathcal{D}^q_{q-2+s}$, i.e.,

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|(gf)'(z)|^q dA(z) \leq \|gf\|^q_{\mathcal{D}^q_{q-2+s}}.$$

In addition,

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|g'(z)f(z)|^q dA(z) \leq \|f\|^q_{\mathcal{H}^\infty}\|g\|^q_{\mathcal{D}^q_{q-2+s}}.$$

We have

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s}|g(z)f'(z)|^q dA(z) < \infty.$$

On the other hand, by Lemma 2.3 we deduce that

$$\int_0^1 (1-r)^{q-2+s}|f'(re^{i\theta})|^q dr = \infty.$$

This together with $g \in \mathcal{D}^q_{q-2+s} \subseteq \mathcal{H}^q$ yields

$$\int_0^1 (1-r)^{q-2+s}|f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty$$

for almost all $\theta \in \mathbb{R}$ ([9], p. 1878). We obtain a contradiction. This finishes the proof. □

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