Bull. Korean Math. Soc. **57** (2020), No. 2, pp. 489–508
https://doi.org/10.4134/BKMS.b190357
pISSN: 1015-8634 / eISSN: 2234-3016

EVALUATIONS OF SOME QUADRATIC EULER SUMS

Xin Si and Ce Xu

Abstract. This paper develops an approach to the evaluation of quadratic Euler sums that involve harmonic numbers. The approach is based on simple integral computations of polylogarithms. By using the approach, we establish some relations between quadratic Euler sums and linear sums. Furthermore, we obtain some closed form representations of quadratic sums in terms of zeta values and linear sums. The given representations are new.

1. Introduction

Euler sums are real numbers, originally defined by Euler, that have been much studied in recent years because of their many surprising properties and the many places they appear in mathematics and mathematical physics. There are many conjectures concerning the values of Euler sums, for example see [3,6,10,13,19,20]. The subject of this paper is Euler sums, which are the infinite sums whose general term is a product of harmonic numbers (or alternating harmonic numbers) of index n and a power of n^{-1} (or $(-1)^{n-1}n^{-1}$). The nth generalized harmonic numbers and nth generalized alternating harmonic numbers are defined by

$$(1)\qquad \zeta_n(k) := \sum_{j=1}^{n} \frac{1}{j^k},\quad L_n(k) := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^k},\quad k,n \in \mathbb{N} := \{1,2,3,\ldots\},$$

where $H_n := \zeta_n(1) = \sum_{j=1}^{n} \frac{1}{j}$ is the natural harmonic number. The classical linear Euler sum is defined by

$$(2)\qquad S(p;q) := \sum_{n=1}^{\infty} \frac{1}{n^q} \sum_{k=1}^{n} \frac{1}{k^p},$$

where p, q are positive integers with $q \geq 2$ and the quantity $w := p+q$ is called the weight. The earliest results on linear sums $S(p;q)$ are due to Euler who

Received April 4, 2019; Accepted July 4, 2019.
2010 *Mathematics Subject Classification.* Primary 11M06, 11M32, 33B15.
Key words and phrases. Harmonic numbers, polylogarithm function, Euler sum, Riemann zeta function.
This work was co-supported by the National Natural Science Foundation of China (No.41801219), Natural Science Foundation of Fujian Province (No.2019J01849).

elaborated a method to reduce double sums of small weight to certain rational linear combinations of products of zeta values. In particular, he proved the simple relation in 1775 (see [3, 13])

$$S(1;k)=\sum_{n=1}^{\infty}\frac{H_n}{n^k}=\frac{1}{2}\left\{(k+2)\,\zeta(k+1)-\sum_{i=1}^{k-2}\zeta(k-i)\,\zeta(i+1)\right\}, \tag{3}$$

and determined the explicit values of zeta values function at even integers:

$$\zeta(2m)=\frac{(-1)^m B_{2m}}{2\,(2m)!}(2\pi)^{2m},$$

where $B_k\in\mathbb{Q}$ are the Bernoulli numbers defined by the generating function (see [1, 4, 5])

$$\frac{x}{e^x-1}:=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!}.$$

It is easy to verify that $B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30}$, and $B_{2m+1}=0$ for $m\geq 1$. The Riemann zeta function and alternating Riemann zeta function are defined respectively by

$$\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \Re(s)>1\quad\text{and}\quad \bar{\zeta}(s):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s},\quad \Re(s)\geq 1.$$

Obviously, for $\Re(s)>1$, $\bar{\zeta}(s)=\left(1-\frac{1}{2^{s-1}}\right)\zeta(s)$. The general multiple zeta functions are defined as

$$\zeta(s_1,s_2,\ldots,s_m):=\sum_{n_1>n_2>\cdots>n_m>0}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_m^{s_m}},$$

where $s_1+\cdots+s_m$ is called the weight and m is the multiplicity.

Euler conjectured that the linear sums $S(p;q)$ would be reducible to zeta values whenever $p+q$ is odd, and even gave what he hoped to obtain the general formula. In [6], D. Borwein, J. M. Borwein and R. Girgensohn proved conjecture and formula, and in [3], D. H. Bailey, J. M. Borwein and R. Girgensohn demonstrated that it is "very likely" that linear sums with $p+q>7$, $p+q$ even, are not reducible.

Next, we introduce the generalized Euler sums. For integers $q,p_1,\ldots,p_m$ with $q\geq 2$, we define the generalized Euler sums as

$$S(p_1,p_2,\ldots,p_m;q):=\sum_{n=1}^{\infty}\frac{X_n(p_1)\,X_n(p_2)\cdots X_n(p_m)}{n^q}, \tag{4}$$

where $X_n(p_i)=\zeta_n(p_i)$ if $p_i>0$, and $X_n(p_i)=L_n(-p_i)$ otherwise. In below, if $p<0$, we will denote it by $\overline{-p}$. For example,

$$S(2,\bar{3},5,\bar{7};q)=\sum_{n=1}^{\infty}\frac{\zeta_n(2)\,L_n(3)\,\zeta_n(5)\,L_n(7)}{n^q}.$$

Similarly, for $q \in \mathbb{N}$, we define

$$(5)\qquad S\left(p_1, p_2, \ldots, p_m; \bar{q}\right) := \sum_{n=1}^{\infty} \frac{X_n\left(p_1\right) X_n\left(p_2\right) \cdots X_n\left(p_m\right)}{n^q} (-1)^{n-1}.$$

We call $w := |p_1| + |p_2| + \cdots + |p_m| + |q|$ the weight of the Euler sums $S(p_1, p_2, \ldots, p_m; q)$. Hence, by the definition of $S(p_1, p_2, \ldots, p_m; q)$, we know that the linear sums are altogether four types:

$$S\left(p; q\right) = \sum_{n=1}^{\infty} \frac{\zeta_n\left(p\right)}{n^q}, \quad S\left(\bar{p}; q\right) = \sum_{n=1}^{\infty} \frac{L_n\left(p\right)}{n^q},$$
$$S\left(p; \bar{q}\right) = \sum_{n=1}^{\infty} \frac{\zeta_n\left(p\right)}{n^q} (-1)^{n-1}, \quad S\left(\bar{p}; \bar{q}\right) = \sum_{n=1}^{\infty} \frac{L_n\left(p\right)}{n^q} (-1)^{n-1}.$$

The study of these Euler sums was started by Euler. After that many different methods, including partial fraction expansions, Eulerian Beta integrals, summation formulas for generalized hypergeometric functions and contour integrals, have been used to evaluate these sums. The relationship between the values of the Riemann zeta function and Euler sums has been studied by many authors, for details and historical introductions, please see [2, 3, 6–20]. For example, in [13], Philippe Flajolet and Bruno Salvy gave explicit reductions to zeta values for all linear sums $S\left(p; q\right), S\left(\bar{p}; q\right), S\left(p; \bar{q}\right), S\left(\bar{p}; \bar{q}\right)$ with $w := p + q$ odd. Moreover, they proved the following conclusion: If $p_1 + p_2 + q$ is even, and $p_1 > 1, p_2 > 1, q > 1$, the quadratic sums

$$S(p_1, p_2; q) = \sum_{n=1}^{\infty} \frac{\zeta_n\left(p_1\right) \zeta_n\left(p_2\right)}{n^q}$$

are reducible to linear sums (see Theorem 4.2 in the reference [13]). It is well known that all quadratic sums $S(p_1, p_2; q)$ with $|p_1| + |p_2| + |q| \leq 4$ were reducible to zeta values and polylogarithms (explicit evaluations please see [19, 21]). In [20], we proved that all Euler sums of the form $S(1, p; q)$ for weights $p + q + 1 \in \{4, 5, 6, 7, 9\}$ with $p \geq 1$ and $q \geq 2$ are expressible polynomially in terms of zeta values. For weight 8, all such sums are the sum of a polynomial in zeta values and a rational multiple of $S(2; 6)$.

The main purpose of this paper is to evaluate some quadratic Euler sums which involve harmonic numbers and alternating harmonic numbers. In this paper, we will prove that all quadratic sums

$$S(1, p+1; p+2m) = \sum_{n=1}^{\infty} \frac{H_n \zeta_n\left(p+1\right)}{n^{p+2m}},$$
$$S(1, p+2m; p+1) = \sum_{n=1}^{\infty} \frac{H_n \zeta_n\left(p+2m\right)}{n^{p+1}}$$

are reducible to polynomials in zeta values and to linear sums, where $p, m \in \mathbb{N}$. Moreover, we also prove that, for $p \in \mathbb{N} \setminus \{1\}$, $m \in \mathbb{N} \cup \{0\}$, the quadratic combinations

$$\begin{aligned}&S(\bar{1},p+2m+1;p)+S(\bar{1},p;p+2m+1)\\&=\sum_{n=1}^{\infty}\left\{\frac{L_n\left(1\right)\zeta_n\left(p+2m+1\right)}{n^p}+\frac{L_n\left(1\right)\zeta_n\left(p\right)}{n^{p+2m+1}}\right\},\\&S(\bar{1},p+2m;p)-S(\bar{1},p;p+2m)\\&=\sum_{n=1}^{\infty}\left\{\frac{L_n\left(1\right)\zeta_n\left(p+2m\right)}{n^p}-\frac{L_n\left(1\right)\zeta_n\left(p\right)}{n^{p+2m}}\right\}\end{aligned}$$

and

$$S(1,p+2m+2;p)-S(1,p;p+2m+2)=\sum_{n=1}^{\infty}\left\{\frac{H_n\zeta_n\left(p+2m+2\right)}{n^p}-\frac{H_n\zeta_n\left(p\right)}{n^{p+2m+2}}\right\}$$

reduce to linear sums and polynomials in zeta values.

2. Main theorems and proofs

In this section, by calculating the integrals of polylogarithm functions, we will establish some explicit relationships which involve quadratic sums and linear sums. The polylogarithm function is defined as follows

$$\mathrm{Li}_p\left(x\right):=\sum_{n=1}^{\infty}\frac{x^n}{n^p},\quad \Re(p)>1,\ |x|\le 1,$$

with $\mathrm{Li}_1(x)=-\log(1-x),\ x\in[-1,1)$. First, we give the following Theorem, which will be useful in the development of the main results.

Theorem 2.1. *Let $m,p\ge 2$ be positive integers. Then the following identity holds*:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\zeta\left(m\right)\zeta_n\left(p\right)-\zeta\left(p\right)\zeta_n\left(m\right)}{n}&=\zeta\left(p\right)\sum_{n=1}^{\infty}\frac{H_n}{n^m}-\zeta\left(m\right)\sum_{n=1}^{\infty}\frac{H_n}{n^p}\\&\quad+\zeta\left(m\right)\zeta\left(p+1\right)-\zeta\left(p\right)\zeta\left(m+1\right).\end{aligned}\tag{6}$$

Proof. We construct the generating function

$$y=\sum_{n=1}^{\infty}\left\{H_n\zeta_n\left(m\right)-\zeta_n\left(m+1\right)\right\}x^{n-1},\quad x\in(-1,1).\tag{7}$$

By definition, the harmonic numbers satisfy the recurrence relation

$$\zeta_{n+1}\left(m\right)=\zeta_n\left(m\right)+\frac{1}{\left(n+1\right)^m}.$$

Then the sum on the right hand side of (7) is equal to

$$\sum_{n=1}^{\infty}\left\{H_n\zeta_n\left(m\right)-\zeta_n\left(m+1\right)\right\}x^{n-1}$$

$$= \sum_{n=1}^{\infty} \left\{ \left(H_n + \tfrac{1}{n+1} \right) \left(\zeta_n(m) + \tfrac{1}{(n+1)^m} \right) - \left(\zeta_n(m+1) + \tfrac{1}{(n+1)^{m+1}} \right) \right\} x^n$$

$$= \sum_{n=1}^{\infty} \left\{ H_n \zeta_n(m) - \zeta_n(m+1) + \tfrac{H_n}{(n+1)^m} + \tfrac{\zeta_n(m)}{n+1} \right\} x^n.$$

By simple calculation, we get

$$\sum_{n=1}^{\infty} \{ H_n \zeta_n(m) - \zeta_n(m+1) \} x^{n-1} = \sum_{n=1}^{\infty} \left\{ \frac{H_n}{(n+1)^m} + \frac{\zeta_n(m)}{n+1} \right\} \frac{x^n}{1-x}. \tag{8}$$

Multiplying (8) by $\ln^{p-1} x$ and integrating over $(0,1)$, we obtain the formula

$$\sum_{n=1}^{\infty} \frac{H_n \zeta_n(m) - \zeta_n(m+1)}{n^p} = \sum_{n=1}^{\infty} \left\{ \frac{H_n}{(n+1)^m} + \frac{\zeta_n(m)}{n+1} \right\} \{ \zeta(p) - \zeta_n(p) \}. \tag{9}$$

After some straightforward manipulations, formula (9) can be written as

$$\sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n(m)}{n^p} + \frac{H_n \zeta_n(p)}{n^m} \right\} = \zeta(p) \sum_{n=1}^{\infty} \frac{H_n}{n^m} + \sum_{n=1}^{\infty} \frac{H_n}{n^{p+m}} + \sum_{n=1}^{\infty} \frac{\zeta_n(m)}{n^{p+1}}$$
$$- \sum_{n=1}^{\infty} \frac{\zeta_n(m+1)}{n^p} + \sum_{n=1}^{\infty} \frac{\zeta_n(m)}{n} \{ \zeta(p) - \zeta_n(p) \}. \tag{10}$$

Change (m,p) to (p,m), the result is

$$\sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n(p)}{n^m} + \frac{H_n \zeta_n(m)}{n^p} \right\} = \zeta(m) \sum_{n=1}^{\infty} \frac{H_n}{n^p} + \sum_{n=1}^{\infty} \frac{H_n}{n^{m+p}} + \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^{m+1}}$$
$$- \sum_{n=1}^{\infty} \frac{\zeta_n(p+1)}{n^m} + \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n} \{ \zeta(m) - \zeta_n(m) \}. \tag{11}$$

Therefore, combining (10) and (11), we obtain the desired result. The proof of Theorem 2.1 is completed. □

Proceeding in a similar fashion to evaluation of Theorem 2.1, we consider the following function

$$y = \sum_{n=1}^{\infty} \{ \zeta_n(1, a+1) \zeta_n(p, a+1) - \zeta_n(p+1, a+1) \} x^{n+a-1}, \quad x \in (-1, 1),$$

where the partial sums $\zeta_n(p, a+1)$ for $p \geq 1$ of Hurwitz zeta function is defined as

$$\zeta_n(p, a+1) := \sum_{k=1}^{n} \frac{1}{(k+a)^p}, \quad a \notin \mathbb{N}^- := \{-1, -2, -3, \ldots\}.$$

The Hurwitz zeta function is defined by

$$\zeta(p, a+1) := \sum_{n=1}^{\infty} \frac{1}{(n+a)^p}, \quad \Re(p) > 1, \ a \notin \mathbb{N}^-.$$

By a similar argument as in the proof of Theorem 2.1, we deduce the more general identity

$$\sum_{n=1}^{\infty} \frac{\zeta(m,a+1)\zeta_n(p,a+1)-\zeta(p,a+1)\zeta_n(m,a+1)}{n+a}$$
$$=\zeta(p,a+1)\sum_{n=1}^{\infty}\frac{\zeta_n(1,a+1)}{(n+a)^m}-\zeta(m,a+1)\sum_{n=1}^{\infty}\frac{\zeta_n(1,a+1)}{(n+a)^p}$$
$$+\zeta(m,a+1)\zeta(p+1,a+1)-\zeta(m+1,a+1)\zeta(p,a+1).$$

When $a=0$, the result is formula (6).

Theorem 2.2. *Let $p\geq 2, m\geq 0$ be integers and $x\in[-1,1)$. Then the following identity holds*:

$$(-1)^{p-1}\sum_{n=1}^{\infty}\left\{\frac{\zeta_n(p+2m+1)}{n^p}+\frac{\zeta_n(p)}{n^{p+2m+1}}\right\}\left(\sum_{k=1}^{n}\frac{x^k}{k}\right) \tag{12}$$
$$=\sum_{i=1}^{p+2m}(-1)^{i-1}\mathrm{Li}_{p+2m+2-i}(x)\sum_{n=1}^{\infty}\frac{\zeta_n(p)}{n^i}x^n$$
$$-\sum_{i=1}^{p-1}(-1)^{i-1}\mathrm{Li}_{p+1-i}(x)\sum_{n=1}^{\infty}\frac{\zeta_n(p+2m+1)}{n^i}x^n$$
$$+(-1)^p\ln(1-x)\sum_{n=1}^{\infty}\left\{\frac{\zeta_n(p+2m+1)}{n^p}+\frac{\zeta_n(p)}{n^{p+2m+1}}\right\}(1-x^n).$$

Proof. By the definition of polylogarithm function and Cauchy product formula, we can verify that

$$\frac{\mathrm{Li}_m(x)}{1-x}=\sum_{n=1}^{\infty}\zeta_n(m)x^n,\ x\in(-1,1). \tag{13}$$

Now, we consider the integral

$$\int_0^x \frac{\mathrm{Li}_p(t)\mathrm{Li}_{p+2m+1}(t)}{t(1-t)}dt,\ x\in(-1,1).$$

First, by virtue of (13), we obtain

$$\int_0^x \frac{\mathrm{Li}_p(t)\mathrm{Li}_{p+2m+1}(t)}{t(1-t)}dt=\sum_{n=1}^{\infty}\zeta_n(p)\int_0^x t^{n-1}\mathrm{Li}_{p+2m+1}(t)dt$$
$$=\sum_{n=1}^{\infty}\zeta_n(p+2m+1)\int_0^x t^{n-1}\mathrm{Li}_p(t)dt. \tag{14}$$

On the other hand, using integration by parts we deduce that

$$\int_0^x t^{n-1}\mathrm{Li}_p(t)\,dt=\sum_{i=1}^{p-1}(-1)^{i-1}\frac{x^n}{n^i}\mathrm{Li}_{p+1-i}(x)+\frac{(-1)^p}{n^p}\ln(1-x)(x^n-1)$$

$$- \frac{(-1)^p}{n^p} \left(\sum_{k=1}^n \frac{x^k}{k} \right). \tag{15}$$

In fact, by using the elementary integral identity

$$\int_0^1 x^{n-1} \ln^m x \ln(1-x) dx = (-1)^{m+1} m! \left\{ \frac{H_n}{n^{m+1}} - \sum_{j=1}^m \frac{\zeta(j+1) - \zeta_n(j+1)}{n^{m+1-j}} \right\},$$

then multiplying (15) by $\frac{\ln^{m-1}(x)}{x}$ and integrating over the interval $(0,1)$, we have the following recurrence relation

$$\begin{aligned} &\int_0^1 x^{n-1} \ln^m x \mathrm{Li}_p(x) dx \\ &= m \sum_{i=1}^{p-1} \frac{(-1)^i}{n^i} \int_0^1 x^{n-1} \ln^{m-1} x \mathrm{Li}_{p+1-i}(x) dx + m!(-1)^{m+p-1} \frac{\zeta_n(m+1)}{n^p} \\ &\quad + m! \frac{(-1)^{m+p-1}}{n^p} \left\{ \frac{H_n}{n^m} - \sum_{j=1}^{m-1} \frac{\zeta(j+1) - \zeta_n(j+1)}{n^{m-j}} - \zeta(m+1) \right\}. \end{aligned} \tag{16}$$

Substituting (15) into (14), we get

$$\begin{aligned} &\int_0^x \frac{\mathrm{Li}_p(t) \mathrm{Li}_{p+2m+1}(t)}{t(1-t)} dt \\ &= \sum_{i=1}^{p+2m} (-1)^{i-1} \mathrm{Li}_{p+2m+2-i}(x) \sum_{n=1}^\infty \frac{\zeta_n(p)}{n^i} x^n \\ &\quad + (-1)^p \ln(1-x) \sum_{n=1}^\infty \frac{\zeta_n(p)}{n^{p+2m+1}} (1-x^n) \\ &\quad + (-1)^p \sum_{n=1}^\infty \frac{\zeta_n(p)}{n^{p+2m+1}} \left(\sum_{k=1}^n \frac{x^k}{k} \right) \\ &= \sum_{i=1}^{p-1} (-1)^{i-1} \mathrm{Li}_{p+1-i}(x) \sum_{n=1}^\infty \frac{\zeta_n(p+2m+1)}{n^i} x^n \\ &\quad + (-1)^{p-1} \ln(1-x) \sum_{n=1}^\infty \frac{\zeta_n(p+2m+1)}{n^p} (1-x^n) \\ &\quad + (-1)^{p-1} \sum_{n=1}^\infty \frac{\zeta_n(p+2m+1)}{n^p} \left(\sum_{k=1}^n \frac{x^k}{k} \right). \end{aligned} \tag{17}$$

By a direct calculation, we deduce the result. □

Noting that when x approach 1, by using (2.1), we obtain the result

$$\lim_{x\to 1}\left\{\mathrm{Li}_m(x)\sum_{n=1}^{\infty}\frac{\zeta_n(p)}{n}x^n-\mathrm{Li}_p(x)\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n}x^n\right\}\tag{18}$$
$$=\sum_{n=1}^{\infty}\frac{\zeta(m)\zeta_n(p)-\zeta(p)\zeta_n(m)}{n}$$
$$=\zeta(p)S(1;m)-\zeta(m)S(1;p)+\zeta(m)\zeta(p+1)-\zeta(p)\zeta(m+1),$$

where $p,m\in\mathbb{N}\setminus\{1\}$. Hence, letting $x\to 1$ and $x\to -1$ in Theorem 2.2 and combining (18), we get the following results.

Corollary 2.3. *Let $p\geq 2, m\geq 0$ be integers. Then we have*

$$(-1)^{p-1}\{S(1,p+2m+1;p)+S(1,p;p+2m+1)\}\tag{19}$$
$$=\zeta(p)S(1;p+2m+1)-\zeta(p+2m+1)S(1;p)$$
$$+\zeta(p+1)\zeta(p+2m+1)-\zeta(p)\zeta(p+2m+2)$$
$$+\sum_{i=2}^{p+2m}(-1)^{i-1}\zeta(p+2m+2-i)S(p;i)$$
$$-\sum_{i=2}^{p-1}(-1)^{i-1}\zeta(p+1-i)S(p+2m+1;i).$$

Corollary 2.4. *Let $p\geq 2, m\geq 0$ be integers. Then we have*

$$(-1)^{p}\{S(\bar{1},p+2m+1;p)+S(\bar{1},p;p+2m+1)\}\tag{20}$$
$$=\sum_{i=1}^{p+2m}(-1)^{i-1}\bar{\zeta}(p+2m+2-i)S(p;\bar{i})$$
$$-\sum_{i=1}^{p-1}(-1)^{i-1}\bar{\zeta}(p+1-i)S(p+2m+1;\bar{i})$$
$$+(-1)^{p}\ln 2\{S(p+2m+1;p)+S(p;p+2m+1)\}$$
$$+(-1)^{p}\ln 2\{S(p+2m+1;\bar{p})+S(p;\overline{p+2m+1})\}.$$

In the same way as in the proof of (12), we obtain the following Theorem.

Theorem 2.5. *For $p\in\mathbb{N}\setminus\{1\}$, $m\in\mathbb{N}\cup\{0\}$ and $x\in[-1,1)$. Then the following identity holds*:

$$(-1)^{p-1}\sum_{n=1}^{\infty}\left\{\frac{\zeta_n(p+2m)}{n^p}-\frac{\zeta_n(p)}{n^{p+2m}}\right\}\left(\sum_{k=1}^{n}\frac{x^k}{k}\right)\tag{21}$$
$$=\sum_{i=1}^{p+2m-1}(-1)^{i-1}\mathrm{Li}_{p+2m+1-i}(x)\sum_{n=1}^{\infty}\frac{\zeta_n(p)}{n^i}x^n$$

$$- \sum_{i=1}^{p-1} (-1)^{i-1} \mathrm{Li}_{p+1-i}(x) \sum_{n=1}^{\infty} \frac{\zeta_n(p+2m)}{n^i} x^n$$
$$+ (-1)^p \ln(1-x) \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(p+2m)}{n^p} - \frac{\zeta_n(p)}{n^{p+2m}} \right\} (1-x^n).$$

Proof. Similarly to the proof of Theorem 2.2, considering integral

$$\int_0^x \frac{\mathrm{Li}_p(t)\,\mathrm{Li}_{p+2m}(t)}{t(1-t)} dt,\ x \in (-1,1).$$

Then with the help of formula (2.10) we may easily deduce the result. □

Similarly, in (21), taking $x \to 1$ and $x \to -1$, by using (18), we can give the following Corollaries.

Corollary 2.6. *For integers $p \in \mathbb{N} \setminus \{1\}$ and $m \in \mathbb{N} \cup \{0\}$, we have*

(22)
$$\begin{aligned}
&(-1)^{p-1} \{S(1,p+2m;p) - S(1,p;p+2m)\} \\
&= \zeta(p)\,S(1;p+2m) - \zeta(p+2m)\,S(1;p) \\
&\quad + \zeta(p+1)\,\zeta(p+2m) - \zeta(p)\,\zeta(p+2m+1) \\
&\quad + \sum_{i=2}^{p+2m-1} (-1)^{i-1} \zeta(p+2m+1-i) S(p;i) \\
&\quad - \sum_{i=2}^{p-1} (-1)^{i-1} \zeta(p+1-i) S(p+2m;i).
\end{aligned}$$

Corollary 2.7. *For integers $p \in \mathbb{N} \setminus \{1\}$ and $m \in \mathbb{N} \cup \{0\}$, we have*

(23)
$$\begin{aligned}
&(-1)^p \{S(\bar{1},p+2m;p) - S(\bar{1},p;p+2m)\} \\
&= \sum_{i=1}^{p+2m-1} (-1)^{i-1} \bar{\zeta}(p+2m+1-i) S(p;\bar{i}) \\
&\quad - \sum_{i=1}^{p-1} (-1)^{i-1} \bar{\zeta}(p+1-i) S(p+2m;\bar{i}) \\
&\quad + (-1)^p \ln 2 \{S(p+2m;p) - S(p;p+2m)\} \\
&\quad + (-1)^p \ln 2 \left\{S(p+2m;\bar{p}) - S\left(p;\overline{p+2m}\right)\right\}.
\end{aligned}$$

Theorem 2.8. *For $l_1, l_2, m \in \mathbb{N}$ and $x, y, z \in [-1,1)$, we have the following relation*

(24)
$$\sum_{n=1}^{\infty} \frac{\zeta_n(l_1;x)\,\zeta_n(l_2;y)}{n^m} z^n + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1;x)\,\zeta_n(m;z)}{n^{l_2}} y^n$$
$$+ \sum_{n=1}^{\infty} \frac{\zeta_n(l_2;y)\,\zeta_n(m;z)}{n^{l_1}} x^n$$

$$= \sum_{n=1}^{\infty} \frac{\zeta_n(m;z)}{n^{l_1+l_2}}(xy)^n + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1;x)}{n^{m+l_2}}(yz)^n + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2;y)}{n^{l_1+m}}(xz)^n + \mathrm{Li}_m(z)\,\mathrm{Li}_{l_1}(x)\,\mathrm{Li}_{l_2}(y) - \mathrm{Li}_{l_1+l_2+m}(xyz),$$

where the partial sum $\zeta_n(l;x)$ *is defined by* $\zeta_n(l;x) := \sum_{k=1}^{n} \frac{x^k}{k^l}$.

Proof. We construct the function

$$F(x,y,z) = \sum_{n=1}^{\infty} \{\zeta_n(l_1;x)\,\zeta_n(l_2;y) - \zeta_n(l_1+l_2;xy)\}\, z^{n-1}.$$

By the definition of $\zeta_n(l;x)$, we have

$$F(x,y,z) = zF(x,y,z) + \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1;x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2;y)}{(n+1)^{l_1}} x^{n+1} \right\} z^n. \tag{25}$$

Moving $zF(x,y,z)$ from right to left and then multiplying $(1-z)^{-1}$ to the equation (25) and integrating over the interval $(0,z)$, we obtain

$$\sum_{n=1}^{\infty} \frac{\zeta_n(l_1;x)\,\zeta_n(l_2;y) - \zeta_n(l_1+l_2;xy)}{n} z^n = \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1;x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2;y)}{(n+1)^{l_1}} x^{n+1} \right\} \{\mathrm{Li}_1(z) - \zeta_n(1;z)\}. \tag{26}$$

Furthermore, using integration and the following formula

$$\sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1;x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2;y)}{(n+1)^{l_1}} x^{n+1} \right\} = \mathrm{Li}_{l_1}(x)\,\mathrm{Li}_{l_2}(y) - \mathrm{Li}_{l_1+l_2}(xy),$$

we can obtain (24). □

Putting $(x,y,z) = (-1,1,1)$, $(l_1,l_2,m) = (1,p+2m+1,p)$ and $(x,y,z) = (-1,-1,-1)$, $(l_1,l_2,m) = (1,p+2m+1,p)$ in (24), we can give the following Corollaries.

Corollary 2.9. *For integers* $p \in \mathbb{N}\backslash\{1\}$ *and* $m \in \mathbb{N}\cup\{0\}$, *the following identity holds*:

$$S(\bar{1},p+2m+1;p) + S(\bar{1},p;p+2m+1) + S(p,p+2m+1;\bar{1}) = S(p;\overline{p+2m+2}) + S(\bar{1};2p+2m+1) + S(p+2m+1;\overline{p+1}) + \ln 2\,\zeta(p+2m+1)\,\zeta(p) - \bar{\zeta}(2p+2m+2). \tag{27}$$

Corollary 2.10. *For integers* $p \in \mathbb{N}$ *and* $m \in \mathbb{N}\cup\{0\}$, *the following identity holds*:

$$S(\bar{1},\overline{p+2m+1};\bar{p}) + S(\bar{1},\bar{p};\overline{p+2m+1}) + S(\bar{p},\overline{p+2m+1};\bar{1}) = S(\bar{p};p+2m+2) + S(\bar{1};2p+2m+1) + S(\overline{p+2m+1};p+1) + \ln 2\,\bar{\zeta}(p+2m+1)\,\bar{\zeta}(p) - \bar{\zeta}(2p+2m+2). \tag{28}$$

3. Closed form of Euler sums

In this section, we give some linear relations among quadratic Euler sums by using Theorem 2.2 and Theorem 2.5. We now give the following theorems.

Theorem 3.1. *For integers $p \in \mathbb{N} \setminus \{1\}$ and $m \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}
(29)\qquad & (-1)^{p-1}\left\{S\left(2,p+2m+1;p\right)+S\left(2,p;p+2m+1\right)\right\} \\
&= \sum_{i=1}^{p+2m}\sum_{j=1}^{p+2m+1-i}(-1)^{i+j}\zeta\left(p+2m+3-i-j\right)S(p;i+j) \\
&- \sum_{i=1}^{p-1}\sum_{j=1}^{p-i}(-1)^{i+j}\zeta\left(p+2-i-j\right)S(p+2m+1;i+j) \\
&- (-1)^{p}\zeta\left(2\right)\left\{\zeta\left(p\right)\zeta\left(p+2m+1\right)+\zeta\left(2p+2m+1\right)\right\} \\
&+ (-1)^{p}\left(p+2m+1\right)S(1,p;p+2m+2) \\
&+ (-1)^{p}pS(1,p+2m+1;p+1).
\end{aligned}$$

Proof. Multiplying (12) by $\frac{1}{x}$ and integrating over (0,1), and using (15), we deduce Theorem 3.1 holds. □

Theorem 3.2. *For integers $p \in \mathbb{N} \setminus \{1\}$ and $m \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}
(30)\qquad & (-1)^{p-1}\left\{S\left(2,p+2m;p\right)-S\left(2,p;p+2m\right)\right\} \\
&= \sum_{i=1}^{p+2m-1}\sum_{j=1}^{p+2m-i}(-1)^{i+j}\zeta\left(p+2m+2-i-j\right)S(p;i+j) \\
&- \sum_{i=1}^{p-1}\sum_{j=1}^{p-i}(-1)^{i+j}\zeta\left(p+2-i-j\right)S(p+2m;i+j) \\
&- (-1)^{p-1}\zeta\left(2\right)\left\{S(p;p+2m)-S(p+2m;p)\right\} \\
&+ (-1)^{p-1}\left(p+2m\right)S(1,p;p+2m+1) \\
&- (-1)^{p-1}pS(1,p+2m;p+1).
\end{aligned}$$

Proof. By a similar argument as in the proof of Theorem 3.1, multiplying (21) by $\frac{1}{x}$ and integrating over (0,1), and combining (15), we deduce Theorem 3.2 holds. □

From [13,20], we know that for $p \in \mathbb{N} \setminus \{1\}$ and $m \in \mathbb{N} \cup \{0\}$, the quadratic sums

$$\begin{aligned}
S(1,2;2m+1) &= \frac{H_n\zeta_n\left(2\right)}{n^{2m+1}}, \\
S(2,p+2m;p) &= \frac{\zeta_n\left(2\right)\zeta_n\left(p+2m\right)}{n^{p}},
\end{aligned}$$

$$S(2,p:p+2m)=\frac{\zeta_n\left(2\right)\zeta_n\left(p\right)}{n^{p+2m}},$$

are reducible to linear sums. Hence, from (30), we have the corollary.

Corollary 3.3. *For integers* $p\in\mathbb{N}\setminus\{1\}$ *and* $m\in\mathbb{N}\cup\{0\}$*, the quadratic combination*

$$(p+2m)S(1,p;p+2m+1)-pS(1,p+2m;p+1)$$

are reducible to linear sums and to polynomials in zeta values.

On the other hand, in Corollary 2.3, we prove that for integers $p\in\mathbb{N}\setminus\{1\}$ and $m\in\mathbb{N}\cup\{0\}$, the quadratic combination

$$S(1,p+2m+1;p)+S(1,p;p+2m+1)$$

can be expressed as a rational linear combination of products of zeta values and linear sums. Replacing p by $p+1$ and m by $m-1$ in Corollary 2.3, we obtain the following corollary.

Corollary 3.4. *For* $p,m\in\mathbb{N}$*, the combination*

$$S(1,p+2m;p+1)+S(1,p+1;p+2m)$$

is a rational linear combination of products of zeta values and linear sums.

Moreover, we note that

$$\begin{aligned}&(p+2m)\,S\left(1,p;p+2m+1\right)+pS\left(1,p+1;p+2m\right)\\=&\left\{(p+2m)\,S\left(1,p;p+2m+1\right)-pS\left(1,p+2m;p+1\right)\right\}\\&+p\left\{S\left(1,p+2m;p+1\right)+S\left(1,p+1;p+2m\right)\right\}.\end{aligned}$$

Therefore, from Corollary 3.3 and Corollary 3.4, we know that the combination

$$(p+2m)\,S\left(1,p;p+2m+1\right)+pS\left(1,p+1;p+2m\right)$$

are reducible to linear sums with $p\in\mathbb{N}\setminus\{1\}$ and $m\in\mathbb{N}$. Since the quadratic sums $S(1,2;2m+1)$ reduce to linear sums and polynomials in zeta values. So, we obtain the following description of quadratic Euler sums $S(1,p+1;p+2m)$ and $S(1,p+2m;p+1)$.

Theorem 3.5. *For integers* $p\in\mathbb{N}$ *and* $m\in\mathbb{N}$*, the quadratic sums*

$$S(1,p+1;p+2m)=\sum_{n=1}^{\infty}\frac{H_n\zeta_n\left(p+1\right)}{n^{p+2m}},\ S(1,p+2m;p+1)=\sum_{n=1}^{\infty}\frac{H_n\zeta_n\left(p+2m\right)}{n^{p+1}}$$

are reducible to linear sums.

In the following examples we collect the high-order results of quadratic Euler sums. We used the following identities which can be easily derived from (19) and (30).

Example 3.1. Some illustrative examples follow.

$$
\begin{aligned}
S(1,2;3) &= -\frac{101}{48}\zeta\left(6\right)+\frac{5}{2}\zeta^2\left(3\right),\\
S(1,3;2) &= \frac{227}{48}\zeta\left(6\right)-\frac{3}{2}\zeta^2\left(3\right),\\
S(1,2;5) &= -\frac{343}{48}\zeta\left(8\right)+12\zeta\left(3\right)\zeta\left(5\right)-\frac{5}{2}\zeta\left(2\right)\zeta^2\left(3\right)-\frac{3}{4}S(2;6),\\
S(1,3;4) &= -\frac{511}{144}\zeta\left(8\right)+7\zeta\left(3\right)\zeta\left(5\right)+\zeta\left(2\right)\zeta^2\left(3\right)-\frac{25}{4}S(2;6),\\
S(1,4;3) &= \frac{443}{48}\zeta\left(8\right)-\frac{21}{2}\zeta\left(3\right)\zeta\left(5\right)-\frac{1}{2}\zeta\left(2\right)\zeta^2\left(3\right)+\frac{25}{4}S(2;6),\\
S(1,5;2) &= \frac{1063}{144}\zeta\left(8\right)-\frac{13}{2}\zeta\left(3\right)\zeta\left(5\right)+\zeta\left(2\right)\zeta^2\left(3\right)+\frac{3}{4}S(2;6),\\
S(1,2;7) &= -\frac{1331}{80}\zeta(10)+\frac{43}{4}\zeta^2(5)+\frac{41}{2}\zeta(3)\zeta(7)-7\zeta(2)\zeta(3)\zeta(5)\\
&\quad -2\zeta^2(3)\zeta(4)-\frac{5}{4}S(2;8),\\
S(1,3;6) &= -\frac{247}{40}\zeta(10)-\frac{5}{4}\zeta^2(5)-\frac{15}{2}\zeta(3)\zeta(7)+12\zeta(2)\zeta(3)\zeta(5)\\
&\quad -\frac{21}{4}S(2,8)-\zeta(2)S(2,6),\\
S(1,4;5) &= \frac{6033}{160}\zeta(10)-14\zeta^2(5)-4\zeta(3)\zeta(7)-15\zeta(2)\zeta(3)\zeta(5)\\
&\quad -\frac{1}{2}\zeta^2(3)\zeta(4)+\frac{21}{2}S(2;8)+\frac{5}{2}\zeta(2)S(2;6),\\
S(1,5;4) &= -\frac{6569}{240}\zeta(10)+16\zeta^2(5)+10\zeta(3)\zeta(7)+4\zeta(2)\zeta(3)\zeta(5)\\
&\quad +\zeta^2(3)\zeta(4)-\frac{21}{2}S(2;8),\\
S(1,6;3) &= \frac{1043}{160}\zeta(10)-\frac{17}{4}\zeta^2(5)-\frac{15}{2}\zeta(3)\zeta(7)+4\zeta(2)\zeta(3)\zeta(5)\\
&\quad -\frac{1}{2}\zeta^2(3)\zeta(4)-\frac{5}{2}\zeta(2)S(2;6)+\frac{21}{4}S(2;8),\\
S(1,7;2) &= \frac{242}{15}\zeta(10)-\frac{25}{4}\zeta^2(5)-\frac{19}{2}\zeta(3)\zeta(7)+\zeta^2(3)\zeta(4)\\
&\quad +\zeta(2)S(2;6)+\frac{5}{4}S(2;8).
\end{aligned}
$$

From (20) and (27), we have the following corollary.

Corollary 3.6. *For $p \in \mathbb{N}\setminus\{1\}$ and $m \in \mathbb{N}\cup\{0\}$, the alternating quadratic sums*

$$S(p,p+2m+1;\bar{1}) = \sum_{n=1}^{\infty}\frac{\zeta_n\left(p\right)\zeta_n\left(p+2m+1\right)}{n}(-1)^{n-1}$$

are reducible to linear sums. We have

$$
\begin{aligned}
(31)\qquad & S(p,p+2m+1;\bar{1})\\
&= S(p;\overline{p+2m+2})+S(\bar{1};2p+2m+1)+S(p+2m+1;\overline{p+1})\\
&\quad+\ln 2\zeta\left(p+2m+1\right)\zeta\left(p\right)-\bar{\zeta}\left(2p+2m+2\right)\\
&\quad+(-1)^{p-1}\sum_{i=1}^{p+2m}(-1)^{i-1}\bar{\zeta}\left(p+2m+2-i\right)S(p;\bar{i})\\
&\quad-(-1)^{p-1}\sum_{i=1}^{p-1}(-1)^{i-1}\bar{\zeta}\left(p+1-i\right)S(p+2m+1;\bar{i})\\
&\quad-\ln 2\{S(p+2m+1;p)+S(p;p+2m+1)\}\\
&\quad-\ln 2\{S(p+2m+1;\bar{p})+S(p;\overline{p+2m+1})\}.
\end{aligned}
$$

Letting $p=2,\ m=0$ in (20) and (31), we obtain

$$
\begin{aligned}
S(\bar{1},3;2)+S(\bar{1},2;3)=&\ \frac{3}{4}\zeta^2\left(3\right)+\frac{7}{4}\zeta\left(6\right)+\frac{5}{8}\zeta\left(2\right)\zeta\left(3\right)\ln 2\\
&-2\zeta\left(2\right)\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{5}{4}\zeta\left(4\right)\ln^2 2-\frac{1}{12}\zeta\left(2\right)\ln^4 2,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
S(2,3;\bar{1})=&-\frac{161}{64}\zeta\left(6\right)+\frac{31}{16}\zeta\left(5\right)\ln 2+\frac{9}{32}\zeta^2\left(3\right)+\frac{3}{8}\zeta\left(2\right)\zeta\left(3\right)\ln 2\\
&+2\zeta\left(2\right)\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{5}{4}\zeta\left(4\right)\ln^2 2+\frac{1}{12}\zeta\left(2\right)\ln^4 2+S(2;\bar{4})-S(\bar{3};3).
\end{aligned}
\tag{33}
$$

In [21], we gave the following formula

$$
\begin{aligned}
S(\bar{1},2;3)=&\ \frac{29}{8}\zeta\left(2\right)\zeta\left(3\right)\ln 2-\frac{93}{32}\zeta\left(5\right)\ln 2-\frac{1855}{128}\zeta\left(6\right)+\frac{17}{16}\zeta^2\left(3\right)\\
&-S(\bar{1};\bar{5})+S(\bar{2};4)+4S(2;\bar{4})+8S(1;\bar{5}).
\end{aligned}
\tag{34}
$$

Substituting (34) into (32), we arrive at the conclusion that

$$
\begin{aligned}
S(\bar{1},3;2)=&\ \frac{2079}{128}\zeta\left(6\right)+\frac{93}{32}\zeta\left(5\right)\ln 2-\frac{5}{16}\zeta^2\left(3\right)-3\zeta\left(2\right)\zeta\left(3\right)\ln 2\\
&-2\zeta\left(2\right)\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{5}{4}\zeta\left(4\right)\ln^2 2-\frac{1}{12}\zeta\left(2\right)\ln^4 2+S(\bar{1};\bar{5})\\
&-S(\bar{2};4)-4S(2;\bar{4})-8S(1;\bar{5}).
\end{aligned}
\tag{35}
$$

Similarly, taking $(x,y,z)=(1,1,1)\,,(l_1,l_2,m)=(2,p+2m+1,p)$ in (24), we get

$$
\begin{aligned}
(36)\qquad & S\left(2,p+2m+1;p\right)+S\left(2,p;p+2m+1\right)+S\left(p,p+2m+1;2\right)\\
&= S\left(2;2p+2m+1\right)+S\left(p;p+2m+3\right)+S\left(p+2m+1;p+2\right)\\
&\quad+\zeta\left(2\right)\zeta\left(p\right)\zeta\left(p+2m+1\right)-\zeta\left(2p+2m+3\right).
\end{aligned}
$$

From (29) and (36), we know that the quadratic sums

$$S(p,p+2m+1;2)=\sum_{n=1}^{\infty}\frac{\zeta_n\left(p\right)\zeta_n\left(p+2m+1\right)}{n^2}$$

can be expressed in terms of zeta values, linear sums and

$$S(1,p;p+2m+2)=\sum_{n=1}^{\infty}\frac{H_n\zeta_n\left(p\right)}{n^{p+2m+2}},$$
$$S(1,p+2m+1;p+1)=\sum_{n=1}^{\infty}\frac{H_n\zeta_n\left(p+2m+1\right)}{n^{p+1}}.$$

In the last of this section, we give some examples. First, in [20], we showed that all quadratic Euler sums of the form

$$S(1,m;p)=\sum_{n=1}^{\infty}\frac{H_n\zeta_n\left(m\right)}{n^p},\quad (m+p+1\le 9)$$

are reducible to $\mathbb{Q}$-linear combinations of single zeta monomial with the addition of linear sums $\{S(2;6)\}$ for weight 8 and give explicit formulas. From (29), (30) and (36), we can give the following examples

$$\begin{aligned}
2S(2,3;2)+S(2,2;3)&=\frac{13}{2}\zeta\left(3\right)\zeta\left(4\right)-3\zeta\left(7\right),\\
S(2,3;2)+S(2,2;3)&=-\frac{179}{16}\zeta\left(7\right)+8\zeta\left(3\right)\zeta\left(4\right)+\frac{5}{2}\zeta\left(2\right)\zeta\left(5\right),\\
S(2,2;4)+2S(2,4;2)&=3\zeta\left(8\right)+2S(2;6),\\
S(2,2;4)-S(2,4;2)&=\frac{1317}{36}\zeta\left(8\right)-60\zeta\left(3\right)\zeta\left(5\right)+9\zeta\left(2\right)\zeta^2\left(3\right)+\frac{31}{2}S(2;6),\\
2S(2,5;2)+S(2,2;5)&=\frac{55}{2}\zeta\left(9\right)-21\zeta\left(2\right)\zeta\left(7\right)+4\zeta\left(3\right)\zeta\left(6\right)+\frac{13}{2}\zeta\left(4\right)\zeta\left(5\right),\\
S(2,5;2)+S(2,2;5)&=-\frac{79}{72}\zeta\left(9\right)-7\zeta\left(2\right)\zeta\left(7\right)+\frac{4}{3}\zeta\left(3\right)\zeta\left(6\right)+\frac{23}{2}\zeta\left(4\right)\zeta\left(5\right)\\
&\quad+\frac{2}{3}\zeta^3\left(3\right),\\
S(2,4;3)+S(2,3;4)&=-35\zeta\left(9\right)+14\zeta\left(2\right)\zeta\left(7\right)+\frac{107}{12}\zeta\left(3\right)\zeta\left(6\right)+\frac{7}{2}\zeta\left(4\right)\zeta\left(5\right)\\
&\quad-\frac{1}{3}\zeta^3\left(3\right),\\
S(2,4;3)+S(2,3;4)+S(3,4;2)&=-\frac{77}{2}\zeta\left(9\right)+21\zeta\left(2\right)\zeta\left(7\right)+\frac{15}{4}\zeta\left(3\right)\zeta\left(6\right)\\
&\quad+3\zeta\left(4\right)\zeta\left(5\right).
\end{aligned}$$

Therefore, combining related equations, we obtain the following identities.

Example 3.2. Some results on quadratic Euler sums.

$$S(2,2;3)=\sum_{n=1}^{\infty}\frac{\zeta_n^2(2)}{n^3}=-\frac{155}{8}\zeta(7)+\frac{19}{2}\zeta(3)\zeta(4)+5\zeta(2)\zeta(5),$$

$$S(2,3;2)=\sum_{n=1}^{\infty}\frac{\zeta_n(2)\zeta_n(3)}{n^2}=\frac{131}{16}\zeta(7)-\frac{3}{2}\zeta(3)\zeta(4)-\frac{5}{2}\zeta(2)\zeta(5),$$

$$S(2,2;4)=\sum_{n=1}^{\infty}\frac{\zeta_n^2(2)}{n^4}=11S(2;6)+\frac{457}{18}\zeta(8)+6\zeta(2)\zeta^2(3)-40\zeta(3)\zeta(5),$$

$$S(2,4;2)=\sum_{n=1}^{\infty}\frac{\zeta_n(2)\zeta_n(4)}{n^2}=-\frac{9}{2}S(2;6)-\frac{403}{36}\zeta(8)-3\zeta(2)\zeta^2(3)+20\zeta(3)\zeta(5),$$

$$S(2,2;5)=\sum_{n=1}^{\infty}\frac{\zeta_n^2(2)}{n^5}=-\frac{1069}{36}\zeta(9)+\frac{4}{3}\zeta^3(3)+7\zeta(2)\zeta(7)-\frac{4}{3}\zeta(3)\zeta(6)+\frac{33}{2}\zeta(4)\zeta(5),$$

$$S(2,5;2)=\sum_{n=1}^{\infty}\frac{\zeta_n(2)\zeta_n(5)}{n^2}=\frac{2059}{72}\zeta(9)-\frac{2}{3}\zeta^3(3)-14\zeta(2)\zeta(7)+\frac{8}{3}\zeta(3)\zeta(6)-5\zeta(4)\zeta(5),$$

$$S(3,4;2)=\sum_{n=1}^{\infty}\frac{\zeta_n(3)\zeta_n(4)}{n^2}=-\frac{7}{2}\zeta(9)+7\zeta(2)\zeta(7)-\frac{31}{6}\zeta(3)\zeta(6)-\frac{1}{2}\zeta(4)\zeta(5)+\frac{1}{3}\zeta^3(3).$$

Moreover, we use Mathematica tool to check numerically each of the specific identities listed. The numerical values of nonlinear Euler sums of weights {7,8,9,10}, to 30 decimal digits, see Table 1.

In fact, by using the methods of this paper, it is possible to establish other identities of Euler sums. For example, taking $m=1$ in (16), we deduce that

$$\int_0^1 x^{n-1}\ln x\,\mathrm{Li}_p(x)dx=\sum_{i=1}^{p-1}\sum_{j=1}^{p-i}(-1)^{i+j-1}\frac{\zeta(p+2-i-j)}{n^{i+j}}+(-1)^p p\frac{H_n}{n^{p+1}}+(-1)^p\frac{\zeta_n(2)-\zeta(2)}{n^p}. \tag{37}$$

Multiplying (12) and (21) by $\frac{\ln x}{x}$, and integrating over the interval (0,1), we arrive at the conclusion that

$$(-1)^p\left[S(3,p+2m+1;p)+S(3,p;p+2m+1)\right]=(-1)^p\zeta(3)\left[S(p+2m+1;p)+S(p;p+2m+1)\right] \tag{38}$$

TABLE 1. Numerical approximation

Euler sum	Numerical values of closed form (30 decimal digits)	Numerical approximation of Euler sum (30 decimal digits)
$S(2,2;3)$	1.35125578526281388688070479101	1.35125578526281388688070478635
$S(2,3;2)$	2.04014406352629668230178759593	2.04014406352629668230178759172
$S(1,2;5)$	1.07388087034296588059339568891	1.07388087034296588059339568663
$S(1,3;4)$	1.15201859049597540982393939989	1.15201859049597540982393939372
$S(1,4;3)$	1.37755320390542981268777869872	1.37755320390542981268777869712
$S(1,5;2)$	2.45339834780017683307966649793	2.45339834780017683307966649461
$S(2,2;4)$	1.13642391274089928376327915373	1.13642391274089928376327915559
$S(2,4;2)$	1.95980117454124719492773304920	1.95980117454124719492773304287
$S(2,2;5)$	1.05972458873705638208576920975	1.05972458873705638208576920818
$S(2,5;2)$	1.92499254625584068819896689186	1.92499254625584068819896688762
$S(3,4;2)$	1.80313006078587093607835773253	1.80313006078587093607835772809
$S(1,2;7)$	1.01603499621822946463309621255	1.01603499621822946463309621221
$S(1,3;6)$	1.03017876630576928913732006061	1.03017876630576928913732005893
$S(1,4;5)$	1.06164990978502285301181351196	1.06164990978502285301181351270
$S(1,5;4)$	1.13783419529420067466663388885	1.13783419529420067466663388537
$S(1,6;3)$	1.35867450783449320806721637607	1.35867450783449320806721637012
$S(1,7;2)$	2.41561649536052525591387317796	2.41561649536052525591387317514

$$
\begin{aligned}
&+(-1)^{p+1}\frac{p\,(p+1)}{2}S\,(1,p+2m+1;p+2)\\
&+(-1)^{p+1}\frac{(p+2m+1)\,(p+2m)}{2}S\,(1,p;p+2m+3)\\
&+(-1)^{p+1}p\,[S\,(2,p+2m+1;p+1)-\zeta\,(2)\,S\,(p+2m+1;p+1)]\\
&+(-1)^{p+1}\,(p+2m+1)\,[S\,(2,p;p+2m+2)-\zeta\,(2)\,S\,(p;p+2m+2)]\\
&-\sum_{l=1}^{p-1}\sum_{i=1}^{p-l}\sum_{j=1}^{p+1-i-l}(-1)^{i+j+l}\zeta\,(p+3-i-j-l)\,S\,(p+2m+1;i+j+l)\\
&+\sum_{l=1}^{p+2m}\sum_{i=1}^{p+2m+1-l}\sum_{j=1}^{p+2m+2-i-l}(-1)^{i+j+l}\zeta\,(p+2m+4-i-j-l)\\
&S\,(p;i+j+l)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&(-1)^{p}\,[S\,(3,p+2m;p)-S\,(3,p;p+2m)] \qquad (39)\\
&=(-1)^{p}\zeta\,(3)\,[S\,(p+2m;p)-S\,(p;p+2m)]\\
&\quad+(-1)^{p+1}\frac{p\,(p+1)}{2}S\,(1,p+2m;p+2)\\
&\quad+(-1)^{p}\frac{(p+2m)\,(p+2m+1)}{2}S\,(1,p;p+2m+2)\\
&\quad+(-1)^{p+1}p\,[S\,(2,p+2m;p+1)-\zeta\,(2)\,S\,(p+2m;p+1)]\\
&\quad+(-1)^{p}\,(p+2m)\,[S\,(2,p;p+2m+1)-\zeta\,(2)\,S\,(p;p+2m+1)]
\end{aligned}
$$

$$-\sum_{l=1}^{p-1}\sum_{i=1}^{p-l}\sum_{j=1}^{p+1-i-l}(-1)^{i+j+l}\zeta\left(p+3-i-j-l\right)S\left(p+2m;i+j+l\right)$$
$$+\sum_{l=1}^{p+2m-1}\sum_{i=1}^{p+2m-l}\sum_{j=1}^{p+2m+1-i-l}(-1)^{i+j+l}\zeta\left(p+2m+3-i-j-l\right)$$
$$S\left(p;i+j+l\right).$$

Putting $m=1,p=2$ in (39), we have

$$\sum_{n=1}^{\infty}\left\{\frac{\zeta_n\left(3\right)\zeta_n\left(4\right)}{n^2}-\frac{\zeta_n\left(3\right)\zeta_n\left(2\right)}{n^4}\right\}$$
$$=-\frac{1063}{36}\zeta\left(9\right)+14\zeta\left(2\right)\zeta\left(7\right)-\frac{37}{6}\zeta\left(3\right)\zeta\left(6\right)+13\zeta\left(4\right)\zeta\left(5\right).$$

Combining related equations, we can obtain

$$\sum_{n=1}^{\infty}\frac{\zeta_n\left(2\right)\zeta_n\left(3\right)}{n^4}=\frac{937}{36}\zeta\left(9\right)+\frac{1}{3}\zeta^3\left(3\right)-7\zeta\left(2\right)\zeta\left(7\right)$$
$$+\zeta\left(3\right)\zeta\left(6\right)-\frac{27}{2}\zeta\left(4\right)\zeta\left(5\right),$$
$$\sum_{n=1}^{\infty}\frac{\zeta_n\left(2\right)\zeta_n\left(4\right)}{n^3}=-\frac{2197}{36}\zeta\left(9\right)-\frac{2}{3}\zeta^3\left(3\right)+21\zeta\left(2\right)\zeta\left(7\right)$$
$$+\frac{95}{12}\zeta\left(3\right)\zeta\left(6\right)+17\zeta\left(4\right)\zeta\left(5\right).$$

Remark 3.1. From [6,20], we have the partial fraction decomposition

$$\frac{1}{x^s(1-x)^t}=\sum_{j=1}^{s}\frac{A_j^{(s,t)}}{x^j}+\sum_{j=1}^{t}\frac{B_j^{(s,t)}}{(1-x)^j}\quad(s,t\ge 0,s+t\ge 1),$$

where

$$A_j^{(s,t)}:=\binom{s+t-j-1}{s-j}\quad\text{and}\quad B_j^{(s,t)}:=\binom{s+t-j-1}{t-j}.$$

Hence, it is easy to see that for positive integers n,m,p,

$$\int_0^1 x^{n-1}\ln^m(x)\mathrm{Li}_p(x)dx$$
$$=(-1)^m m!\sum_{k=1}^{\infty}\frac{1}{k^p(n+k)^{m+1}}$$
$$=(-1)^{m+p}m!\sum_{j=1}^{p-1}(-1)^{j+1}\binom{p+m-j-1}{m}\frac{\zeta(j+1)}{n^{p+m-j}}$$

$$+(-1)^{m+p}m!\sum_{j=1}^{m}\binom{p+m-j-1}{p-1}\frac{\zeta(j+1)-\zeta_n(j+1)}{n^{p+m-j}}$$
$$+(-1)^{m+p+1}m!\binom{p+m-1}{p-1}\frac{H_n}{n^{p+m}}.$$

Thus, multiplying (12) and (21) by $\frac{\ln^r(x)}{x}$, and integrating over the interval (0,1), we can obtain two general formulas.

Acknowledgments. The authors would like to thank the anonymous referee for his/her helpful comments, which improve the presentation of the paper. This work was co-supported by the National Natural Science Foundation of China (No.41801219), Natural Science Foundation of Fujian Province (No.2019J01849).

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, 71, Cambridge University Press, Cambridge, 1999. https://doi.org/10.1017/CBO9781107325937

[2] D. H. Bailey, J. M. Borwein, and R. E. Crandall, *Computation and theory of extended Mordell-Tornheim-Witten sums*, Math. Comp. **83** (2014), no. 288, 1795–1821. https://doi.org/10.1090/S0025-5718-2014-02768-3

[3] D. H. Bailey, J. M. Borwein, and R. Girgensohn, *Experimental evaluation of Euler sums*, Experiment. Math. **3** (1994), no. 1, 17–30. http://projecteuclid.org/euclid.em/1062621000

[4] B. C. Berndt, *Ramanujan's Notebooks. Part I*, Springer-Verlag, New York, 1985. https://doi.org/10.1007/978-1-4612-1088-7

[5] ———, *Ramanujan's Notebooks. Part II*, Springer-Verlag, New York, 1989. https://doi.org/10.1007/978-1-4612-4530-8

[6] D. Borwein, J. M. Borwein, and R. Girgensohn, *Explicit evaluation of Euler sums*, Proc. Edinburgh Math. Soc. (2) **38** (1995), no. 2, 277–294. https://doi.org/10.1017/S0013091500019088

[7] J. Borwein, P. Borwein, R. Girgensohn, and S. Parnes, *Making sense of experimental mathematics*, Math. Intelligencer **18** (1996), no. 4, 12–18.

[8] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and Petr. Lisoněk, *Special values of multiple polylogarithms*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 907–941. https://doi.org/10.1090/S0002-9947-00-02616-7

[9] J. M. Borwein and R. Girgensohn, *Evaluation of triple Euler sums*, Electron. J. Combin. **3** (1996), no. 1, Research Paper 23, approx. 27 pp.

[10] J. M. Borwein, I. J. Zucker, and J. Boersma, *The evaluation of character Euler double sums*, Ramanujan J. **15** (2008), no. 3, 377–405. https://doi.org/10.1007/s11139-007-9083-z

[11] L. Comtet, *Advanced Combinatorics*, revised and enlarged edition, D. Reidel Publishing Co., Dordrecht, 1974.

[12] M. Eie and C.-S. Wei, *Evaluations of some quadruple Euler sums of even weight*, Funct. Approx. Comment. Math. **46** (2012), part 1, 63–77. https://doi.org/10.7169/facm/2012.46.1.5

[13] P. Flajolet and B. Salvy, *Euler sums and contour integral representations*, Experiment. Math. **7** (1998), no. 1, 15–35. http://projecteuclid.org/euclid.em/1047674270

[14] P. Freitas, *Integrals of polylogarithmic functions, recurrence relations, and associated Euler sums*, Math. Comp. **74** (2005), no. 251, 1425–1440. https://doi.org/10.1090/S0025-5718-05-01747-3

[15] C. Markett, *Triple sums and the Riemann zeta function*, J. Number Theory **48** (1994), no. 2, 113–132. https://doi.org/10.1006/jnth.1994.1058

[16] I. Mező, *Nonlinear Euler sums*, Pacific J. Math. **272** (2014), no. 1, 201–226. https://doi.org/10.2140/pjm.2014.272.201

[17] A. Sofo, *Quadratic alternating harmonic number sums*, J. Number Theory **154** (2015), 144–159. https://doi.org/10.1016/j.jnt.2015.02.013

[18] P. Sun, *The sixth-order sums of the Riemann zeta function*, Acta Math. Sinica (Chin. Ser.) **50** (2007), no. 2, 373–384.

[19] C. Xu and J. Cheng, *Some results on Euler sums*, Funct. Approx. Comment. Math. **54** (2016), no. 1, 25–37. https://doi.org/10.7169/facm/2016.54.1.3

[20] C. Xu, Y. Yan, and Z. Shi, *Euler sums and integrals of polylogarithm functions*, J. Number Theory **165** (2016), 84–108. https://doi.org/10.1016/j.jnt.2016.01.025

[21] C. Xu, Y. Yang, and J. Zhang, *Explicit evaluation of quadratic Euler sums*, Int. J. Number Theory **13** (2017), no. 3, 655–672. https://doi.org/10.1142/S1793042117500336

Xin Si
School of Applied Mathematics
Xiamen University of Technology
Xiamen 361005, P. R. China
Email address: xsi@xmut.edu.cn

Ce Xu
School of Mathematical Sciences
Xiamen University
Xiamen 361005, P. R. China
Email address: 19020170155420@stu.xmu.edu.cn