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FREDHOLM TOEPLITZ OPERATORS ON THE DIRICHLET SPACES OF THE POLYDISK

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Abstract. We study the Toeplitz operators on the holomorphic and pluriharmonic Dirichlet spaces of the polydisk in terms of when Toeplitz operator is Fredholm operator there. Consequently, we describe the essential spectrum of Toeplitz operators.

1. Introduction

Let D be the unit disk in the complex plane $\mathbf{C}$. For a fixed integer n, the unit polydisk D^n of $\mathbf{C}^n$ is the cartesian product of n copies of D and $V = V_n$ is the Lebesgue volume measure on D^n normalized so that $V(D^n) = 1$.

The Sobolev space $\mathcal{S}$ is the completion of the space $C^1(D^n)$ for which

$$\|f\| = \left\{ \left| \int_{D^n} f\, dV \right|^2 + \int_{D^n} \{|\mathcal{R}f(z)|^2 + |\widetilde{\mathcal{R}}f(z)|^2\}\, dV(z) \right\}^{1/2} < \infty,$$

where

$$\mathcal{R}f(z) = \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z), \quad \widetilde{\mathcal{R}}f(z) = \sum_{i=1}^{n} \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(z)$$

for $z = (z_1, \ldots, z_n) \in D^n$. Then $\mathcal{S}$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{D^n} f\, dV \overline{\int_{D^n} g\, dV} + \int_{D^n} \{\mathcal{R}f\overline{\mathcal{R}g} + \widetilde{\mathcal{R}}f\overline{\widetilde{\mathcal{R}}g}\}\, dV. \tag{1}$$

The Dirichlet space $\mathcal{D}$ is the subspace of $\mathcal{S}$ consisting of all holomorhpic functions. And the pluriharmonic Dirichlet space $\mathcal{D}_{ph}$ is the space of all pluriharmonic functions f in $\mathcal{S}$. Note that $f \in C^2(D^n)$ is a pluriharmonic if and only if the function $\varphi_{a,b} : \mathbf{C} \to D^n$ defined by $\varphi_{a,b}(\lambda) = f(a + \lambda b)$ is harmonic for each $a \in D^n$ and $b \in \mathbf{C}^n$. Thus $\mathcal{D}_{ph}$ is also a closed subspace of $\mathcal{S}$.

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We put

$$\mathcal{L}^{1,\infty} = \left\{\varphi \in \mathcal{S} \ : \varphi,\, \frac{\partial \varphi}{\partial z_j}, \frac{\partial \varphi}{\partial \bar{z}_j} \in L^\infty,\ j = 1, \ldots, n\right\},$$

where the derivatives are taken in the sense of distributions. Sobolev's embedding theorem ([1], Theorem 5.4) shows that each function in $\mathcal{L}^{1,\infty}$ can be extended to a continuous function on the closed polydisk $\overline{D^n}$. Hence we will use the same notation between a function in $\mathcal{L}^{1,\infty}$ and its continuous extension to $\overline{D^n}$. Note that $\mathcal{R}\varphi, \widetilde{\mathcal{R}}\varphi \in L^\infty$.

Let P and Q be the Hilbert space orthogonal projections from $\mathcal{S}$ onto $\mathcal{D}$ and $\mathcal{D}_{ph}$, respectively. Given a function $u \in \mathcal{L}^{1,\infty}$, the *Toeplitz operators* T_u on $\mathcal{D}$ and T_u^{ph} on $\mathcal{D}_{ph}$ with symbol u are defined by

$$T_u f = P(uf), \quad T_u^{ph}\varphi = Q(u\varphi)$$

for $f \in \mathcal{D}$ and $\varphi \in \mathcal{D}_{ph}$, respectively. Then T_u on $\mathcal{D}$ and T_u^{ph} on $\mathcal{D}_{ph}$ are bounded linear operators.

On the Bergman space of the ball, McDonald ([8]) studied the Fredholm properties of a Toeplitz operators and Cao ([2]) considered the same problem on the holomorphic Dirichlet space. Also Lee ([5] and [6]) characterized the Fredholm Toeplitz operators on the holomorphic and pluriharmonic Dirichlet spaces of the ball. In this paper, we deal with the same problem of when a Toeplitz operator is to be Fredholm operator on the holomorphic and pluriharmonic Dirichlet spaces of the polydisk. Now we introduce our main theorems.

Theorem 1.1. *Let $u \in \mathcal{L}^{1,\infty}$. Then T_u is Fredholm on $\mathcal{D}$ if and only if u has no zero on ∂D^n.*

Theorem 1.2. *Let $u \in \mathcal{L}^{1,\infty}$. Then T_u^{ph} is Fredholm on $\mathcal{D}_{ph}$ if and only if u has no zero on ∂D^n.*

2. Preliminaries

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each α_k is a nonnegative integer, we will write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. We will also write

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

for $z = (z_1, \ldots, z_n) \in D^n$.

Let A^2 be the well known Bergman space consisting of all holomorphic functions in L^2 where $L^2 = L^2(D^n, V)$ denotes the usual Lebesgue space on D^n. Note that $\mathcal{D} \subset A^2$ and moreover

$$||f||_2^2 \leq \|\mathcal{R}f\|_2^2 \leq \|f\|^2 \tag{2}$$

holds for all $f \in \mathcal{D}$. Throughout the paper, we use the notations

$$||\varphi||_2 = \left(\int_{D^n} |\varphi|^2\, dV\right)^{\frac{1}{2}} \quad \text{and} \quad \langle \varphi, \psi\rangle_2 = \int_{D^n} \varphi\overline{\psi}\, dV$$

for $\varphi, \psi \in L^2$.

Note that each point evaluation is a bounded linear functional on A^2: see Chapter 2 of [10] for details and related facts. Thus each point evaluation is a bounded linear functional on $\mathcal{D}$ and $\mathcal{D}_{ph}$ either. For each $z \in D^n$, it follows that there exists a unique kernel functions $K_z \in \mathcal{D}$ and $R_z \in \mathcal{D}_{ph}$ which have the reproducing property:

$$f(z) = \langle f, K_z \rangle \quad \text{and} \quad \varphi(z) = \langle \varphi, R_z \rangle$$

for $f \in \mathcal{D}$ and $\varphi \in \mathcal{D}_{ph}$, respectively.

As is well known, a real valued function on D^n is pluriharmonic if and only if it is the real part of a holomorphic function on D^n. Hence we can express $\mathcal{D}_{ph} = \mathcal{D} + \overline{\mathcal{D}}$ and

$$R_z = K_z + \overline{K_z} - 1;$$

see Chapter 4 of [9]. From this, we obtain the relation between P and Q as follows:

$$Q(\varphi) = P(\varphi) + \overline{P(\overline{\varphi})} - P(\varphi)(0) \tag{3}$$

for $\varphi \in \mathcal{S}$.

Let B be the well known Bergman projection which is the orthogonal projection from L^2 onto A^2 and its explicit formula can be written as

$$B\psi(z) = \int_{D^n} \psi(w)\overline{B_z(w)}\, dV(w), \qquad z \in D^n$$

for $\psi \in L^2$. Here B_z is the Bergman kernel given by

$$B_z(w) = \prod_{i=1}^{n} \frac{1}{(1-\overline{z_i}w_i)^2}, \qquad w \in D^n.$$

Since

$$\prod_{i=1}^{n} \frac{1}{(1-\overline{z_i}w_i)^2} = \prod_{i=1}^{n} \sum_{\alpha_i=0}^{\infty} (1+\alpha_i)(\overline{z_i}w_i)^{\alpha_i} = \sum_{|\alpha|\geq 0} \prod_{i=1}^{n} (1+\alpha_i)\overline{z}^{\alpha} w^{\alpha}$$

for $z, w \in D^n$, we have

$$B\psi(z) = \sum_{|\alpha|\geq 0} \prod_{i=1}^{n} (1+\alpha_i) z^{\alpha} \int_{D^n} \overline{w}^{\alpha} \psi(w)\, dV(w) \tag{4}$$

for $z \in D^n$. On the other hand, since

$$\int_D |\lambda^{\beta}|^2\, dV_1(\lambda) = \frac{1}{\beta+1}$$

for every integer $\beta \geq 0$, one can see

$$||z^{\alpha}||^2 = |\alpha|^2 \prod_{i=1}^{n} \frac{1}{\alpha_i + 1}$$

for each multi-index α. Note that the set $\{z^\alpha : |\alpha| \geq 0\}$ spans a dense subset of $\mathcal{D}$. Thus it can be easily seen that the reproducing kernel K_z on $\mathcal{D}$ has the following explicit formula

$$K_z(w) = 1 + \sum_{|\alpha|>0} \frac{\prod_{i=1}^n (1+\alpha_i)}{|\alpha|^2} \overline{z}^\alpha w^\alpha \tag{5}$$

for $z, w \in D^n$. Since $K_z(0) = 1$ for all $z \in D^n$, it follows from (5) that

$$P\psi(z) = \int_{D^n} \psi \, dV + \sum_{|\alpha|>0} \frac{\prod_{i=1}^n (1+\alpha_i)}{|\alpha|} z^\alpha \int_{D^n} \overline{w}^\alpha \mathcal{R}\psi(w) \, dV(w) \tag{6}$$

for $z \in D^n$. Thus, for $\psi \in \mathcal{S}$, we have by (4),

$$\begin{aligned} \mathcal{R}(P\psi)(z) &= \sum_{|\alpha|>0} \prod_{i=1}^n (1+\alpha_i) z^\alpha \int_{D^n} \overline{w}^\alpha \mathcal{R}\psi(w) \, dV(w) \\ &= B(\mathcal{R}\psi)(z) - B(\mathcal{R}\psi)(0), \qquad z \in D^n. \end{aligned} \tag{7}$$

Note that the following mean value property holds for holomorphic functions $f \in L^1$:

$$f(z) = \int_{D^n} f |b_z|^2 \, dV, \qquad z \in D^n, \tag{8}$$

where b_a denotes the normalized Bergman kernel of A^2 defined by

$$b_a(z) = \frac{B_a(z)}{\|B_a\|_2} = \frac{(1-|a_1|^2)\cdots(1-|a_n|^2)}{(1-\overline{a_1}z_1)^2 \cdots (1-\overline{a_n}z_n)^2}.$$

Since f^2 is holomorphic, we have by (8)

$$f(z)^2 = \int_{D^n} f^2 |b_z|^2 \, dV, \qquad z \in D^n.$$

Taking the modulus on both sides, we obtain

$$|f(z)|^2 \leq \int_{D^n} |f|^2 |b_z|^2 \, dV,$$

so that

$$|f(0)|^2 \leq \int_{D^n} |f|^2 \, dV = \|f\|_2^2$$

for all holomorphic $f \in L^1$; see [3] for details. Combining this with (2), we have the useful estimation as follows:

$$|f(0)| \leq \|f\|_2 \leq \|\mathcal{R}f\|_2 \leq \|f\|. \tag{9}$$

3. Fredholm Toeplitz operators

For each $a \in D^n$, we let $E_a = \mathcal{R}K_a$. Then the explicit formula of E_a is

$$E_a(z) = \sum_{|\alpha|>0} \frac{\prod_{i=1}^n (1+\alpha_i)}{|\alpha|} \overline{a}^\alpha z^\alpha.$$

Note that $\|\mathcal{R}e_a\|_2 = \|e_a\| = 1$ for all $a \in D^n$ where

$$e_a(z) = \frac{E_a(z)}{\|E_a\|}, \qquad a, z \in D^n.$$

Since $\|B_a\|_2 = \prod_{i=1}^n (1-|a_i|^2)^{-1}$ and $\|E_a\|^2 = B_a(a) - 1$ for all $a \in D^n$, we have

$$\lim_{a\to\partial D^n} \frac{\|B_a\|_2}{\|E_a\|} = \lim_{a\to\partial D^n} \frac{1}{\sqrt{1-(1-|a_1|^2)^2\cdots(1-|a_n|^2)^2}} = 1. \tag{10}$$

The following results are taken from [7].

Lemma 3.1. *e_a converges weakly to 0 in $\mathcal{D}$ as $a \to \partial D^n$.*

Lemma 3.2. *The identity operator from $\mathcal{D}$ into A^2 is compact. In particular, if a sequence f_k converging weakly to 0 in $\mathcal{D}$, then $\|f_k\|_2 \to 0$ as $k \to \infty$.*

Let $b^2 = A^2 + \overline{A^2}$ be the pluriharmonic Bergman space consisting of all pluriharmonic functions in L^2. Let $\varphi = f + \overline{g} \in \mathcal{D}_{ph}$ for some $f, g \in \mathcal{D}$ with $f(0) = 0$. Since $\|\varphi\|^2 = \|f\|^2 + \|g\|^2$, we have by (2)

$$\|\varphi\|_2 \le \|f\|_2 + \|g\|_2 \le \|f\| + \|g\| \le 2\|\varphi\|.$$

Using this with Lemma 3.2, we can see that the identity operator from $\mathcal{D}_{ph}$ into b^2 is bounded.

Recall that $\mathcal{D}_{ph} = \mathcal{D}_0 + \overline{\mathcal{D}}$ where $\mathcal{D}_0 = \{f \in \mathcal{D} : f(0) = 0\}$.

Proposition 3.3. *Let $\varphi_j = f_j + \overline{g_j} \in \mathcal{D}_0 + \overline{\mathcal{D}}$ be a sequence. If φ_j converges to 0 weakly in $\mathcal{D}_{ph}$, then f_j and g_j converge to 0 weakly in $\mathcal{D}$.*

Proof. Let $h \in \mathcal{D}$. Since $f_j(0) = 0$, we have

$$\begin{aligned}\langle f_j, h\rangle &= \langle \varphi_j - \overline{g_j}, h\rangle = \langle \varphi_j, h\rangle - \overline{h(0) g_j(0)} = \langle \varphi_j, h\rangle - \overline{h(0)}\varphi_j(0) \\ &= \langle \varphi_j, h\rangle - \overline{h(0)}\langle \varphi_j, 1\rangle\end{aligned}$$

for each j. If $\varphi_j \to 0$ weakly in $\mathcal{D}_{ph}$, then $\langle \varphi_j, h\rangle \to 0$ and $\langle \varphi_j, 1\rangle \to 0$ as $j \to \infty$. Hence $f_j \to 0$ weakly in $\mathcal{D}$. Also we have

$$\langle g_j, h\rangle = \langle \overline{\varphi_j} - \overline{f_j}, h\rangle = \langle \overline{\varphi_j}, h\rangle - \overline{h(0) f_j(0)} = \overline{\langle \varphi_j, \overline{h}\rangle} \to 0$$

as $j \to \infty$, which implies $g_j \to 0$ weakly in $\mathcal{D}$. Thus we have the desired result. □

Proposition 3.4. *If h_j converges to 0 weakly in $\mathcal{D}$, then h_j and $\overline{h_j}$ converge to 0 weakly in $\mathcal{D}_{ph}$.*

Proof. For $\varphi = f + \overline{g} \in \mathcal{D}_0 + \overline{\mathcal{D}}$, we have

$$\langle h_j, \varphi\rangle = \langle h_j, f + \overline{g}\rangle = \langle h_j, f\rangle + \langle h_j, \overline{g}\rangle = \langle h_j, f\rangle + g(0)\langle h_j, 1\rangle$$

for each j. Combining $h_j \to 0$ weakly in $\mathcal{D}$ with $f, 1 \in \mathcal{D}$, we have $\langle h_j, f+\overline{g}\rangle \to 0$ as $j \to \infty$. Thus $h_j \to 0$ weakly in $\mathcal{D}_{ph}$, so that $\overline{h_j} \to 0$ weakly in $\mathcal{D}_{ph}$. Thus we have the desired result. □

Finally the identity operator from $\mathcal{D}_{ph}$ into b^2 is compact as follows.

Lemma 3.5. *The identity operator from $\mathcal{D}_{ph}$ into b^2 is compact.*

Proof. Let $\varphi_j = f_j + \overline{g_j} \in \mathcal{D}_0 + \overline{\mathcal{D}}$ and $\varphi_j \to 0$ weakly in $\mathcal{D}_{ph}$. By Proposition 3.3, f_j and g_j converge to 0 weakly in $\mathcal{D}$, so that we have $\|f_j\|_2 \to 0$ and $\|g_j\|_2 \to 0$ as $j \to \infty$. From this we conclude that

$$\|\varphi_j\|_2 \le \|f_j\|_2 + \|g_j\|_2 \to 0$$

as $j \to \infty$. Thus we have the desired result. □

For $u \in \mathcal{L}^{1,\infty}$, we let S_u denote the Bergman space Toeplitz operator on A^2 defined by

$$S_u f = B(uf)$$

for all $f \in A^2$. Clearly S_u is a bounded linear operator on A^2. Then we have the Berezin transform $\widehat{S_u S_v}$ is continuous up to $\overline{D^n}$ and

$$\widehat{S_u S_v} = uv \quad \text{on} \quad \partial D^n \tag{11}$$

for given two bounded symbols u, v which are continuous on $\overline{D^n}$. Here $\widehat{L}$ of L is the Berezin transform on D^n defined by

$$\widehat{L}(a) = \langle L b_a, b_a\rangle_2, \qquad a \in D^n;$$

see [7] for details.

We let $\mathcal{B}$ denote the C^*-algebra consisting of all bounded operators on $\mathcal{D}$ (resp. $\mathcal{D}_{ph}$). Also, let $\mathcal{K}$ be the algebra of all compact operators on $\mathcal{D}$ (resp. $\mathcal{D}_{ph}$). An operator $L \in \mathcal{B}$ is said to be Fredholm if $L + \mathcal{K}$ is invertible in the quotient algebra $\mathcal{B}/\mathcal{K}$. Recall that $L \in \mathcal{B}$ is Fredholm if and only if there exist $L_1, L_2 \in \mathcal{B}$ such that $L_1 L - I, L L_2 - I \in \mathcal{K}$. Also, if there exists a sequence $\{f_j\}$ of unit vectors in $\mathcal{D}$ (resp. $\mathcal{D}_{ph}$) for which $f_j \to 0$ weakly and $\|L f_j\| \to 0$ as $j \to \infty$, then L can't be Fredholm; see Chapter 6 of [4] for example. Throughout the paper, L^* denotes the adjoint operator of a bounded operator L.

Theorem 3.6. *Let $u \in \mathcal{L}^{1,\infty}$. Then T_u is Fredhlom on $\mathcal{D}$ if and only if u has no zero on ∂D^n.*

Proof. Suppose T_u is Fredhlom on $\mathcal{D}$ and $u(\zeta) = 0$ for some $\zeta \in \partial D^n$. Note that

$$\|T_u e_a\|^2 \le \|u e_a\|^2 = \left|\int_{D^n} u e_a \, dV\right|^2 + \|\mathcal{R}(u e_a)\|_2^2 + \|\widetilde{\mathcal{R}}(u e_a)\|_2^2$$

for all $a \in D^n$. Using Lemmas 3.1 and 3.2, we obtain

$$\left| \int_{D^n} u e_a \, dV \right|^2 \leq \|u\|_\infty^2 \|e_a\|_2^2 \to 0$$

and similarly

$$\|\widetilde{\mathcal{R}}(ue_a)\|_2^2 \leq \|e_a \widetilde{\mathcal{R}} u\|_2^2 \leq \|\widetilde{\mathcal{R}} u\|_\infty^2 \|e_a\|_2^2 \to 0$$

as $a \to \zeta$. It remains to estimate $\|\mathcal{R}(ue_a)\|_2^2$. Note that

$$\|e_a \mathcal{R} u\|_2^2 \leq \|\mathcal{R} u\|_\infty^2 \|e_a\|_2^2 \to 0$$

and

$$\begin{aligned} |\langle e_a \mathcal{R} u, u \mathcal{R} e_a \rangle_2| &\leq \frac{1}{\|E_a\|} \int_{D^n} |e_a (\mathcal{R} u) \overline{u(B_a - 1)}| \, dV \\ &\leq \|\mathcal{R} u\|_\infty \|u\|_\infty \frac{\|B_a\|_2 + 1}{\|E_a\|} \|e_a\|_2 \to 0 \end{aligned}$$

as $a \to \zeta$. Also, we have

$$\begin{aligned} \|u \mathcal{R} e_a\|_2^2 &= \frac{1}{\|E_a\|^2} \langle u(B_a - 1), u(B_a - 1) \rangle_2 \\ &\leq \frac{1}{\|E_a\|^2} (\|u B_a\|_2^2 + 2 |\langle u B_a, u \rangle_2| + \|u\|_2^2) \\ &\leq \left(\frac{\|B_a\|_2}{\|E_a\|} \right)^2 \langle S_{|u|^2} b_a, b_a \rangle_2 + \frac{\|u\|_\infty^2 (2\|B_a\|_2 + 1)}{\|E_a\|^2} \\ &\leq \left(\frac{\|B_a\|_2}{\|E_a\|} \right)^2 \widehat{S_{|u|^2}}(a) + \frac{3\|u\|_\infty^2 \|B_a\|_2}{\|E_a\|^2} \end{aligned}$$

for all $a \in D^n$. Recall that $|u|^2$ is continuous on $\overline{D^n}$. Combining these observations with (10) and (11), we obtain

$$\begin{aligned} \lim_{a \to \zeta} \|\mathcal{R}(ue_a)\|_2^2 &= \lim_{a \to \zeta} \left(\|e_a \mathcal{R} u\|_2^2 + \langle e_a \mathcal{R} u, u \mathcal{R} e_a \rangle_2 + \langle u \mathcal{R} e_a, e_a \mathcal{R} u \rangle_2 + \|u \mathcal{R} e_a\|_2^2 \right) \\ &= \lim_{a \to \zeta} \|u \mathcal{R} e_a\|_2^2 \\ &\leq \lim_{a \to \zeta} \widehat{S_{|u|^2}}(a) = |u(\zeta)|^2. \end{aligned}$$

Thus the assumption $u(\zeta) = 0$ yields

$$\lim_{a \to \zeta} \|T_u e_a\|^2 \leq \|u e_a\|^2 \leq |u(\zeta)|^2 = 0.$$

Since the sequence $\{e_a\}$ of unit vectors converges weakly to 0 in $\mathcal{D}$, T_u can't be Fredholm on $\mathcal{D}$. Hence u has no zero on ∂D^n.

To prove the converse, assume u has no zero on ∂D^n. Since u has no zero on ∂D^n, we can choose a bounded continuous function v on $\overline{D^n}$ with $uv = 1$ on ∂D^n. According to (11), we have

$$S_u \widehat{S_v - I} = S_u \widehat{S_v - S_1} = uv - 1 = 0$$

on ∂D^n, and so $S_uS_v - I$ is compact. Also $S_vS_u - I$ is compact by the similar argument. Thus S_u is Fredholm on A^2.

Now suppose T_u is not Fredholm on $\mathcal{D}$. Then, there is a sequence $\{k_j\}$ of unit vectors in $\mathcal{D}$ converging weakly to 0 such that

$$\|T_uk_j\| \to 0 \quad \text{or} \quad \|T_u^*k_j\| \to 0$$

as $j \to \infty$; see Chapter 6 of [4] for example.

First consider the case $\|T_uk_j\| \to 0$ as $j \to \infty$. To get a contradiction, we consider $\langle L\mathcal{R}(T_uk_j), \mathcal{R}k_j\rangle_2$.

$$\begin{aligned}\langle L\mathcal{R}(T_uk_j), \mathcal{R}k_j\rangle_2 &= \langle L\mathcal{R}[P(uk_j)], \mathcal{R}k_j\rangle_2 \\ &= \langle L(B[\mathcal{R}(uk_j)] - B[\mathcal{R}(uk_j)](0)), \mathcal{R}k_j\rangle_2 \\ &= \langle LB[\mathcal{R}(uk_j)], \mathcal{R}k_j\rangle_2 - B[\mathcal{R}(uk_j)](0)\langle L1, \mathcal{R}k_j\rangle_2\end{aligned}$$

and

$$\begin{aligned}\langle LB[\mathcal{R}(uk_j)], \mathcal{R}k_j\rangle_2 &= \langle LB[(\mathcal{R}u)k_j], \mathcal{R}k_j\rangle_2 + \langle LB[u(\mathcal{R}k_j)], \mathcal{R}k_j\rangle_2 \\ &= \langle LB[(\mathcal{R}u)k_j], \mathcal{R}k_j\rangle_2 + \langle LS_u(\mathcal{R}k_j), \mathcal{R}k_j\rangle_2.\end{aligned}$$

Since S_u is Fredholm on A^2, there exists a bounded operator L on A^2 such that $LS_u - I$ is compact on A^2. From $\mathcal{R}k_j \to 0$ weakly in A^2, we have

$$\langle (LS_u - I)\mathcal{R}k_j, \mathcal{R}k_j\rangle_2 \to 0$$

as $j \to \infty$. Since $|k_j(0)| \leq \|k_j\|_2 \to 0$, we see that $\langle \mathcal{R}k_j, \mathcal{R}k_j\rangle_2 \to 1$ as $j \to \infty$. From this, we have

$$\lim_{j\to\infty} \langle (LS_u - I)\mathcal{R}k_j, \mathcal{R}k_j\rangle_2 = \lim_{j\to\infty} \langle LS_u(\mathcal{R}k_j), \mathcal{R}k_j\rangle_2 - 1,$$

which gives

$$\lim_{j\to\infty} \langle LS_u(\mathcal{R}k_j), \mathcal{R}k_j\rangle_2 = 1.$$

These facts with Lemma 3.5 implies

$$|\langle LB[(\mathcal{R}u)k_j], \mathcal{R}k_j\rangle_2| \leq \|L\|\|\mathcal{R}u\|_\infty \|k_j\|_2 \|\mathcal{R}k_j\|_2 \to 0.$$

By the above facts and using Lemma 5 of [7] with $\langle L1, \mathcal{R}k_j\rangle_2 \to 0$, we have

$$\langle L\mathcal{R}(T_uk_j), \mathcal{R}k_j\rangle_2 \to 1 \tag{12}$$

as $j \to \infty$. On the other hand, since $\|T_uk_j\| \to 0$ and $\|\mathcal{R}k_j\|_2 \to 1$, we see

$$|\langle L\mathcal{R}(T_uk_j), \mathcal{R}k_j\rangle_2| \leq \|L\mathcal{R}(T_uk_j)\|_2\|\mathcal{R}k_j\|_2 \leq \|L\|\|T_uk_j\|\|\mathcal{R}k_j\|_2 \to 0$$

as $j \to \infty$, which is a contradiction to (12).

Now applying this argument to the other case, we can see that the fact $\|T_u^*k_j\| \to 0$ yields a contradiction. Hence T_u is Fredholm on $\mathcal{D}$, which completes the proof. □

Given $u \in \mathcal{L}^{1,\infty}$, the (*little*) *Hankel operator* $H_u : \mathcal{D} \to \overline{\mathcal{D}}$ with symbol u is defined by

$$H_u f = \overline{P(u\overline{f})}$$

for $f \in \mathcal{D}$.

Proposition 3.7. *For $u \in \mathcal{L}^{1,\infty}$, the Hankel operator H_u is compact on $\mathcal{D}$.*

Proof. Let $f_j \to 0$ weakly on $\mathcal{D}$ as $j \to \infty$. From (7) and the L^2-boundedness of B, we have

$$\begin{aligned} \|H_u f_j\|^2 = \|P(u\overline{f_j})\|^2 &= |P(u\overline{f_j})(0)|^2 + \|\mathcal{R}[P(u\overline{f_j})]\|^2 \\ &\le \|u\|_\infty^2 \|f_j\|^2 + \|B[(\mathcal{R}u)\overline{f_j}] - B[(\mathcal{R}u)\overline{f_j}](0)\|_2^2 \\ &\le \|u\|_\infty^2 \|f_j\|^2 + 4\|B[(\mathcal{R}u)\overline{f_j}]\|_2^2 \\ &\le \|u\|_\infty^2 \|f_j\|^2 + 4\|(\mathcal{R}u)\overline{f_j}\|_2^2 \\ &\le \|u\|_\infty^2 \|f_j\|^2 + 4\|\mathcal{R}u\|_\infty^2 \|\overline{f_j}\|_2^2 \\ &\le \left(\|u\|_\infty^2 + 4\|\mathcal{R}u\|_\infty^2\right) \|f_j\|_2^2 \end{aligned}$$

for each j. Recall that the compactness of the identity operator from $\mathcal{D}$ in A^2 implies $\lim_{j\to\infty} \|f_j\|_2^2 = 0$. From this, we have $\|H_u f_j\| \to 0$ as $j \to \infty$. Thus H_u is compact on $\mathcal{D}$ as we desired. The proof is complete. □

Lemma 3.8. *For $u \in \mathcal{L}^{1,\infty}$ and $f \in \mathcal{D}$, we have the followings.*

(a) $\|T_u^{ph} f\|^2 = \|T_u f\|^2 - |\langle f, T_u^* 1\rangle|^2 + \|H_{\overline{u}} f\|^2$.
(b) $\|T_u^{ph} \overline{f}\|^2 = \|H_u f\|^2 - |\langle f, T_{\overline{u}}^* 1\rangle|^2 + \|T_{\overline{u}} f\|^2$.
(c) $\|(T_u^{ph})^* f\|^2 = \|T_u^* f\|^2 - |\langle f, T_u 1\rangle|^2 + \|H_u^* \overline{f}\|^2$.

Proof. Let $f \in \mathcal{D}$. Then we have by (3)

$$T_u^{ph} f = P(uf) + \overline{P(\overline{uf})} - P(uf)(0) = T_u f + H_{\overline{u}} f - T_u f(0),$$

so that

$$\|T_u^{ph} f\|^2 = \|T_u f + H_{\overline{u}} f\|^2 - |T_u f(0)|^2 = \|T_u f\|^2 + \|H_{\overline{u}} f\|^2 - |T_u f(0)|^2.$$

Since $T_u f(0) = \langle T_u f, 1\rangle = \langle f, T_u^* 1\rangle$, we have (a). Similarly one can prove (b). Now we prove (c).

$$\begin{aligned} (T_u^{ph})^* f(z) = \langle (T_u^{ph})^* f, R_z\rangle &= \langle (T_u^{ph})^* f, K_z + \overline{K_z} - 1\rangle \\ &= \langle (T_u)^* f, K_z - 1\rangle + \langle K_z, H_u^* \overline{f}\rangle \\ &= (T_u)^* f(z) + \overline{H_u^* \overline{f}(z)} \end{aligned}$$

for every $z \in D^n$. Thus we have (c) following the similar method in the proof of (a). The proof is complete. □

Now we introduce the new notations as follows: for given $u \in \mathcal{L}^{1,\infty}$, we define bounded linear operators A_u, B_u, C_u from $\mathcal{D}_0 + \overline{\mathcal{D}}$ to $\mathcal{D}_{ph}$ by

$$\begin{aligned} A_u(f+\overline{g}) &= T_u f + \overline{T_{\overline{u}} g}, \\ B_u(f+\overline{g}) &= H_{\overline{u}} f + \overline{H_u g}, \\ C_u(f+\overline{g}) &= -\langle f, T_u^* 1\rangle - \overline{\langle g, T_{\overline{u}}^* 1\rangle}, \end{aligned} \tag{13}$$

respectively. Then we can decompose T_u^{ph} into the sums of the operators A_u, B_u and C_u.

Lemma 3.9. *For $u \in \mathcal{L}^{1,\infty}$, we have $T_u^{ph} = A_u + B_u + C_u$.*

Proof. Let $\varphi = f + \overline{g} \in \mathcal{D}_0 + \overline{\mathcal{D}}$. From (3), we have

$$\begin{aligned} T_u^{ph}\varphi &= P(uf) + \overline{P(\overline{uf})} - P(uf)(0) + P(u\overline{g}) + \overline{P(\overline{u}g)} - P(u\overline{g})(0) \\ &= T_u f + H_{\overline{u}} f - P(uf)(0) + \overline{T_{\overline{u}} g} + \overline{H_u g} - P(u\overline{g})(0). \end{aligned}$$

Here we obtain by the reproducing property

$$P(uf)(0) = \langle T_u f, K_0\rangle = \langle T_u f, 1\rangle = \langle f, T_u^* 1\rangle$$

and

$$P(u\overline{g})(0) = \overline{P(\overline{u}g)(0)} = \overline{\langle g, T_{\overline{u}}^* 1\rangle}.$$

Using (13) with the above, we get

$$\begin{aligned} T_u^{ph}\varphi &= T_u f + H_{\overline{u}} f - \langle f, T_u^* 1\rangle + \overline{T_{\overline{u}} g} + \overline{H_u g} - \overline{\langle g, T_{\overline{u}}^* 1\rangle} \\ &= A_u\varphi + B_u\varphi + C_u\varphi \end{aligned}$$

for $\varphi = f + \overline{g} \in \mathcal{D}_0 + \overline{\mathcal{D}}$. Thus we have the desired results. □

The following result shows that the relation between T_u and A_u for the Fredholm operator.

Lemma 3.10. *Let $u \in \mathcal{L}^{1,\infty}$. Then T_u^{ph} is Fredhlom on $\mathcal{D}_{ph}$ if and only if A_u has no zero on ∂D^n.*

Proof. Let $\varphi_j = f_j + \overline{g_j}$ be a sequence in $\mathcal{D}_0 + \overline{\mathcal{D}}$ and $\varphi_j \to 0$ weakly in $\mathcal{D}_{ph}$. Then Proposition 3.3 shows f_j and g_j converge weakly to 0 in $\mathcal{D}$. Compactness of H_u and $H_{\overline{u}}$ by Proposition 3.7 implies that $\|H_{\overline{u}} f_j\| \to 0$ and $\|H_u g_j\| \to 0$ as $j \to \infty$. Thus B_u is compact. Also C_u is compact by the definition. Using Lemma 3.9, we have A_u is compact as desired result. The proof is complete. □

Theorem 3.11. *Let $u \in \mathcal{L}^{1,\infty}$. Then T_u^{ph} is Fredhlom on $\mathcal{D}_{ph}$ if and only if u has no zero on ∂D^n.*

Proof. We first assume T_u^{ph} is Fredhlom on $\mathcal{D}_{ph}$ and u has a zero on ∂D^n. Then T_u is not Fredholm on $\mathcal{D}$. If T_u is not left Fredholm on $\mathcal{D}$. Then there exists a sequence $\{f_j\}$ of unit vectors in $\mathcal{D}$ which is weakly convergent to zero and $\|T_u f_j\| \to 0$. Using Lemma 3.8 and Proposition 3.7, we have

$$\lim_{j\to\infty} \|T_u^{ph} f_j\|^2 = \lim_{j\to\infty} \left(\|T_u f_j\|^2 - |\langle f_j, T_u^* 1\rangle|^2 + \|H_{\overline{u}} f_j\|^2\right) = 0.$$

Also $\{f_j\}$ converges weakly to 0 in $\mathcal{D}_{ph}$ by Proposition 3.4, so that T_u^{ph} is not left Fredholm on $\mathcal{D}_{ph}$. Thus it is contradiction. Now we consider the case T_u is not right Fredholm on $\mathcal{D}$. By the similar way, there exists a sequence $\{g_j\}$ of unit vectors in $\mathcal{D}$ such that $g_j \to 0$ weakly in $\mathcal{D}$ and $\|T_u^* g_j\| \to 0$ as $j \to \infty$. Using Lemma 3.8 and Proposition 3.7 again, we have

$$\lim_{j\to\infty} \left(\|(T_u^{ph})^* g_j\|^2 = \|T_u^* g_j\|^2 - |\langle g_j, T_u 1\rangle|^2 + \|H_u^* \overline{g_j}\|^2\right) = 0$$

since $\{\overline{g_j}\}$ converges weakly to 0 in $\overline{\mathcal{D}}$. Applying Proposition 3.4 again, we see that $\{f_j\}$ converges weakly to 0 in $\mathcal{D}_{ph}$, so that T_u^{ph} is not right Fredholm on $\mathcal{D}_{ph}$. Thus it is contradiction. It means that u has no zero on ∂D^n.

To prove the converse, we suppose u has no zero on ∂D^n. Then $\overline{u}$ has no zero on ∂D^n, which implies T_u and $T_{\overline{u}}$ are Fredholm on $\mathcal{D}$. Since T_u and $T_{\overline{u}}$ are left Fredholm on $\mathcal{D}$, there exist bounded linear operators L and M on $\mathcal{D}$ such that $LT_u - I$ and $MT_{\overline{u}} - I$ are compact on $\mathcal{D}$. Now we define T from $\mathcal{D}_0 + \overline{\mathcal{D}}$ to $\mathcal{D}_{ph}$ by

$$T(f + \overline{g}) = Lf + \overline{Mg}$$

for $f + \overline{g} \in \mathcal{D}_0 + \overline{\mathcal{D}}$. Then one can see that T is well defined and linear. Also it is bounded because L and M are bounded on $\mathcal{D}$. We just show that $TA_u - I$ is compact in $\mathcal{D}_{ph}$. Note that $T_u f(0) \neq 0$ in general. Thus we have with a simple computation

$$\begin{aligned} &(TA_u - I)(F + \overline{G}) \\ &= T\left(T_u F - T_u F(0) + \overline{\overline{T_u F(0)} + T_{\overline{u}} G}\right) - (F + \overline{G}) \\ &= LT_u F - T_u F(0) L1 + T_u F(0)\overline{M1} + \overline{MT_{\overline{u}} G} - (F + \overline{G}) \end{aligned} \tag{14}$$

for $F + \overline{G} \in \mathcal{D}_0 + \overline{\mathcal{D}}$. Let $\varphi_j = f_j + \overline{g_j}$ in $\mathcal{D}_0 + \overline{\mathcal{D}}$ converges weakly to 0 in $\mathcal{D}_{ph}$. We obtain by (14),

$$(TA_u - I)(\varphi_j) = [LT_u - I](f_j) + \overline{[MT_{\overline{u}} - I](g_j)} + T_u f_j(0)[\overline{M1} - L1]$$

for each j. By Proposition 3.3, f_j and g_j converge weakly to 0 in $\mathcal{D}$. Using the compactness of $LT_u - I$ and $MT_{\overline{u}} - I$ on $\mathcal{D}$, we obtain $[LT_u - I](f_j) \to 0$ and $\overline{[MT_{\overline{u}} - I](g_j)} \to 0$ in $\mathcal{D}$ as $j \to \infty$. Also note that $T_u f_j(0) = \langle f_j, T_u^* 1\rangle$ for each j. From this we have $T_u f_j(0) \to 0$ in $\mathcal{D}$. Thus A_u is left Fredholm on $\mathcal{D}_{ph}$. It is easy to show that A_u is right Fredholm on $\mathcal{D}_{ph}$, since L is linear and $Lf(0) = 0$. Following the same argument, we have for

$$(A_u T - I)(\varphi_j) = [T_u L - I](f_j) + \overline{[T_{\overline{u}} M - I](g_j)},$$

where L and M are bounded linear on $\mathcal{D}$ such that $T_u L - I$ and $T_u M - I$ are compact. The rest of the proof runs as before. Thus we conclude A_u is right Fredholm on $\mathcal{D}_{ph}$. Finally A_u is Fredholm on $\mathcal{D}_{ph}$. Hence Lemma 3.10 gives T^{ph} is Fredholm on $\mathcal{D}_{ph}$. The proof is complete. □

Recall that the essential spectrum $\sigma_e(L)$ of $L \in \mathcal{B}$ is defined to be the spectrum of $L + \mathcal{K}$ in $\mathcal{B}/\mathcal{K}$. Thus the following is a simple consequence of Theorem 3.6 and 3.11.

Corollary 3.12. *For $u \in \mathcal{L}^{1,\infty}$, we have $\sigma_e(T_u) = u(\partial D^n)$ and $\sigma_e(T_u^{ph}) = u(\partial D^n)$.*

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