Bull. Korean Math. Soc. **57** (2020), No. 3, pp. 739–750
https://doi.org/10.4134/BKMS.b190458
pISSN: 1015-8634 / eISSN: 2234-3016

SEMISYMMETRIC CUBIC GRAPHS OF ORDER $34p^3$

Mohammad Reza Darafsheh and Mohsen Shahsavaran

Abstract. A simple graph is called semisymmetric if it is regular and edge transitive but not vertex transitive. Let p be a prime. Folkman proved [J. Folkman, *Regular line-symmetric graphs*, Journal of Combinatorial Theory **3** (1967), no. 3, 215–232] that no semisymmetric graph of order $2p$ or $2p^2$ exists. In this paper an extension of his result in the case of cubic graphs of order $34p^3$, $p \neq 17$, is obtained.

1. Introduction

In this paper all graphs are finite, undirected and simple, i.e., without loops or multiple edges. A graph is called semisymmetric if it is regular and edge transitive but not vertex transitive. The class of semisymmetric graphs was first studied by Folkman [6], who found several infinite families of such graphs and posed eight open problems.

An interesting research problem is to classify connected cubic semisymmetric graphs of different orders. In [6], Folkman proved that there are no semisymmetric graphs of order $2p$ or $2p^2$ for any prime p. For prime p, cubic semisymmetric graphs of order $2p^3$ were investigated in [11], in which they proved that there is no connected cubic semisymmetric graph of order $2p^3$ for any prime $p \neq 3$ and that for $p = 3$ the only such graph is the Gray graph. Also in [2] and [1] the authors proved that there is no connected cubic semisymmetric graph of order $4p^2$ and of order $8p^2$ respectively.

In this paper we investigate connected cubic semisymmetric graphs of order $34p^3$ for prime p, and try to classify them. This investigation actually leads to a proof of their nonexistence for all primes $p \neq 17$.

2. Preliminaries

In this paper, the cardinality of a finite set A, is denoted by $|A|$. The symmetric and alternating groups of degree n, the dihedral group of order $2n$ and the cyclic group of order n are respectively denoted by $\mathbb{S}_n$, $\mathbb{A}_n$, $\mathbb{D}_{2n}$, $\mathbb{Z}_n$. If

Received May 3, 2019; Accepted November 6, 2019.
2010 *Mathematics Subject Classification.* 05E18, 20D60, 05C25, 20B25.
Key words and phrases. Edge-transitive graph, vertex-transitive graph, semisymmetric graph, order of a graph, classification of cubic semisymmetric graphs.

G is a group and $H \leq G$, then $Aut(G)$, G', $Z(G)$, $C_G(H)$ and $N_G(H)$ denote respectively the group of automorphisms of G, the commutator subgroup of G, the center of G, the centralizer and the normalizer of H in G. We also write $H \trianglelefteq^c G$ to denote H is a characteristic subgroup of G. If $H \trianglelefteq^c K \trianglelefteq G$, then $H \trianglelefteq G$. For a prime p dividing the order of finite G, $O_p(G)$ will denote the largest normal p-subgroup of G. It is easy to verify that $O_p(G) \trianglelefteq^c G$. A function f acts on its argument from the left, i.e., we write $f(x)$. The composition, fg, of two functions f and g, is defined as $(fg)(x) = f(g(x))$. For a group G and a nonempty set Ω, an action of G on Ω is a function $(g, \omega) \to g.\omega$ from $G \times \Omega$ to Ω, where $1.\omega = \omega$ and $g.(h.\omega) = (gh).\omega$ for every $g, h \in G$ and every $\omega \in \Omega$. We write $g\omega$ instead of $g.\omega$, if there is no fear of ambiguity. For $\omega \in \Omega$, the stabilizer of ω in G is defined as $G_\omega = \{g \in G : g\omega = \omega\}$. The action is called *semiregular* if the stabilizer of each element in Ω is trivial; it is called *regular* if it is semiregular and transitive.

Let Γ be a graph. The vertex set, the edge set and the set of all automorphisms of Γ are respectively denoted by $V(\Gamma)$, $E(\Gamma)$ and $Aut(\Gamma)$. For two vertices u and v, we write $u \sim v$ to denote u is adjacent to v. The set of all vertices adjacent to u is denoted by $\Gamma(u)$. The degree or valency of a vertex u is $|\Gamma(u)|$. The graph Γ is called *regular* if all of its vertices have the same valency. If Γ is a graph and $N \trianglelefteq Aut(\Gamma)$, then Γ_N will denote a simple undirected graph whose vertices are the orbits of N in its action on $V(\Gamma)$, and where two vertices Nu and Nv are adjacent if and only if $u \sim nv$ in Γ for some $n \in N$.

Let Γ_c and Γ be two graphs. Then Γ_c is said to be a *covering graph* for Γ if there is a surjection $f : V(\Gamma_c) \to V(\Gamma)$ which preserves adjacency and for each $u \in V(\Gamma_c)$, the restricted function $f|_{\Gamma_c(u)} : \Gamma_c(u) \to \Gamma(f(u))$ is a one to one correspondence. The function f is called a *covering projection*. Clearly, if Γ is bipartite, then so is Γ_c. For each $u \in V(\Gamma)$, the *fibre* on u is defined as $fib_u = f^{-1}(u)$. The following important set is a subgroup of $Aut(\Gamma_c)$ and is called the *group of covering transformations* for f:

$$CT(f) = \{\sigma \in Aut(\Gamma_c) \mid \forall u \in V(\Gamma), \sigma(fib_u) = fib_u\}.$$

It is known that $K = CT(f)$ acts semiregularly on each fibre [9]. If this action is regular, then Γ_c is said to be a *regular K-cover* of Γ.

Let $X \leq Aut(\Gamma)$. We say Γ is *X-vertex transitive* or *X-edge transitive* if X acts transitively on $V(\Gamma)$ or $E(\Gamma)$ respectively. Also Γ is called *X-semisymmetric* if it is regular and X-edge transitive but not X-vertex transitive. For $X = Aut(\Gamma)$, we omit X and simply talk about Γ being edge transitive, vertex transitive or semisymmetric. An X-edge transitive but not X-vertex transitive graph is necessarily bipartite, where the two partites are the orbits of the action of X on $V(\Gamma)$. If Γ is regular, then the two partite sets have equal cardinality. So an X-semisymmetric graph is bipartite such that X is transitive on each partite but X carries no vertex from one partite set to the other. A census of all connected semisymmetric cubic graphs of orders up to 768 is given in [5].

Any minimal normal subgroup of a finite group, is the internal direct product of isomorphic copies of a simple group.

A finite simple group G is called a K_n-group if its order has exactly n distinct prime divisors, where $n \in \mathbb{N}$. The following two results determine all simple K_3-groups and K_4-groups [4, 8, 13, 17].

Theorem 2.1. (i) *If G is a simple K_3-group, then G is one of the following groups*: $\mathbb{A}_5$, $\mathbb{A}_6$, $L_2(7)$, $L_2(2^3)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

(ii) *If G is a simple K_4-group, then G is one of the following groups*:

(1) $\mathbb{A}_7$, $\mathbb{A}_8$, $\mathbb{A}_9$, $\mathbb{A}_{10}$, M_{11}, M_{12}, J_2, $L_2(2^4)$, $L_2(5^2)$, $L_2(7^2)$, $L_2(3^4)$, $L_2(97)$, $L_2(3^5)$, $L_2(577)$, $L_3(2^2)$, $L_3(5)$, $L_3(7)$, $L_3(2^3)$, $L_3(17)$, $L_4(3)$, $U_3(2^2)$, $U_3(5)$, $U_3(7)$, $U_3(2^3)$, $U_3(3^2)$, $U_4(3)$, $U_5(2)$, $S_4(2^2)$, $S_4(5)$, $S_4(7)$, $S_4(3^2)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $Sz(2^3)$, $Sz(2^5)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;

(2) *$L_2(r)$ where r is a prime, $r^2-1 = 2^a \cdot 3^b \cdot s$, $s > 3$ is a prime, $a, b \in \mathbb{N}$*;

(3) *$L_2(2^m)$ where m, $2^m - 1$, $\frac{2^m+1}{3}$ are primes greater than* 3;

(4) *$L_2(3^m)$ where m, $\frac{3^m+1}{4}$ and $\frac{3^m-1}{2}$ are odd primes.*

Theorem 2.2 ([14])**.** *If H is a subgroup of a group G, then $C_G(H) \trianglelefteq N_G(H)$ and $\frac{N_G(H)}{C_G(H)}$ is isomorphic to a subgroup of $Aut(H)$.*

Theorem 2.3 ([12])**.** *Let G be a finite group and p a prime. If G has an abelian Sylow p-subgroup, then p does not divide $|G' \cap Z(G)|$.*

An immediate consequence of the following theorem of Burnside is that the order of every nonabelian simple group is divisible by at least 3 distinct primes.

Theorem 2.4 ([14])**.** *For any two distinct primes p and q and any two non-negative integers a and b, every finite group of order $p^a q^b$ is solvable.*

Transitive permutation groups of prime degree are primitive. The following list can be seen e.g. in [3].

Proposition 2.5. *A transitive permutation group of degree* 17 *is one of the following groups*:

$$\mathbb{S}_{17},\ \mathbb{A}_{17},\ L_2(2^4),\ L_2(2^4) \rtimes \mathbb{Z}_2,\ P\Gamma L_2(2^4),$$
$$\mathbb{Z}_{17},\ \mathbb{Z}_{17} \rtimes \mathbb{Z}_2,\ \mathbb{Z}_{17} \rtimes \mathbb{Z}_4,\ \mathbb{Z}_{17} \rtimes \mathbb{Z}_8,\ \mathbb{Z}_{17} \rtimes \mathbb{Z}_{16}.$$

In the following theorem, by the inverse of a pair (a, b), we mean (b, a). Also for each i, A_i, B_i, C_i and D_i are noncyclic groups of order i with known structures. We will not need their structures.

Theorem 2.6 ([7])**.** *If Γ is a connected cubic X-semisymmetric graph, then the order of the stabilizer of any vertex is of the form $2^r \cdot 3$ for some $0 \le r \le 7$. More precisely if $\{u, v\}$ is any edge of Γ, then the pair (X_u, X_v) can only be one of the following fifteen pairs or their inverses*: $(\mathbb{Z}_3, \mathbb{Z}_3)$, $(\mathbb{S}_3, \mathbb{S}_3)$, $(\mathbb{S}_3, \mathbb{Z}_6)$, $(\mathbb{D}_{12}, \mathbb{D}_{12})$, $(\mathbb{D}_{12}, \mathbb{A}_4)$, $(\mathbb{S}_4, \mathbb{D}_{24})$, $(\mathbb{S}_4, \mathbb{Z}_3 \rtimes \mathbb{D}_8)$, $(\mathbb{A}_4 \times \mathbb{Z}_2, \mathbb{D}_{12} \times \mathbb{Z}_2)$, $(\mathbb{S}_4 \times \mathbb{Z}_2, \mathbb{D}_8 \times \mathbb{S}_3)$, $(\mathbb{S}_4, \mathbb{S}_4)$, $(\mathbb{S}_4 \times \mathbb{Z}_2, \mathbb{S}_4 \times \mathbb{Z}_2)$, (A_{96}, B_{96}), (A_{192}, B_{192}), (C_{192}, D_{192}), (A_{384}, B_{384}).

Proposition 2.7 ([15]). *Let Γ be a connected cubic X-semisymmetric graph for some $X \leq Aut(\Gamma)$ and let $N \trianglelefteq X$. If $|\frac{X}{N}|$ is not divisible by 3, then Γ is also N-semisymmetric.*

Proposition 2.8 ([11]). *Let Γ be a connected cubic X-semisymmetric graph for some $X \leq Aut(\Gamma)$; then either $\Gamma \simeq K_{3,3}$, the complete bipartite graph on 6 vertices, or X acts faithfully on each of the bipartition sets of Γ.*

The following Proposition is part of Proposition 2.4 of [11] which is stated as we need it in this paper.

Proposition 2.9 ([11]). *Let Γ be a connected cubic X-edge transitive graph for some $X \leq Aut(\Gamma)$; If u and v are two arbitrary adjacent vertices, then $X_u \cap X_v$ is a common Sylow 2-subgroup of X_u and X_v.*

Theorem 2.10 ([10]). *Let Γ be a connected cubic X-semisymmetric graph. Let $\{U, W\}$ be a bipartition for Γ and assume $N \trianglelefteq X$. If the actions of N on both U and W are intransitive, then N acts semiregularly on both U and W, Γ_N is $\frac{X}{N}$-semisymmetric, and Γ is a regular N-covering of Γ_N.*

This theorem has a nice result. For every normal subgroup $N \trianglelefteq X$ either N is transitive on at least one partite set or it is intransitive on both partite sets. In the former case, the order of N is divisible by $|U| = |W|$. In the latter case, according to Theorem 2.10, the induced action of N on both U and W is semiregular and hence the order of N divides $|U| = |W|$. So we have the following handy corollary.

Corollary 2.11. *If Γ is a connected cubic X-semisymmetric graph with $\{U, W\}$ as a bipartition and $N \trianglelefteq X$, then either $|N|$ divides $|U|$ or $|U|$ divides $|N|$.*

3. Main results

In this section, our goal is to prove the following important result:

Theorem 3.1. *Let p be a prime number other than 17. Then there is no connected cubic semisymmetric graph of order $34p^3$.*

This theorem may be read like this: If there is a connected cubic edge transitive graph of order $34p^3$, $p \neq 17$ prime, then it will also be vertex transitive.

To prove this theorem, we need some lemmas that we now state and prove.

Lemma 3.2. *Let p be a prime where $3 < p \neq 17$, and let $0 \leq i \leq 7$.*

(i) *There is no simple group of order $2^i \cdot 3 \cdot 17 \cdot p^j$ for $j = 2, 3$.*

(ii) *The group $L_2(2^4)$ is the only simple K_4-group whose order is of the form $2^i \cdot 3 \cdot 17 \cdot p$.*

Proof. The order of every group in sub-items (2), (3) and (4) of item (ii) of Theorem 2.1 cannot be divisible by the square of p. Also the only simple K_4-groups in sub-item (1) of item (ii) of Theorem 2.1 whose orders are divisible by 17, are $L_2(2^4)$ of order $2^4 \cdot 3 \cdot 5 \cdot 17$, $L_2(577)$ of order $2^6 \cdot 3^2 \cdot 577 \cdot 17^2$, $L_3(17)$ of order $2^9 \cdot 3^2 \cdot 307 \cdot 17^3$ and $S_4(2^2)$ of order $2^8 \cdot 3^2 \cdot 5^2 \cdot 17$. So part (i) follows. Also it follows that among the groups listed in sub-item (1) of item (ii) of Theorem 2.1, the only group whose order is of the form $2^i \cdot 3 \cdot 17 \cdot p$, is $L_2(2^4)$.

To complete the second part, first consider groups in sub-item (3) of item (ii) of Theorem 2.1. Let $L_2(2^m)$ be a group of order $2^i \cdot 3 \cdot 17 \cdot p$; then

$$2^m \cdot 3 \cdot (2^m - 1) \cdot (\frac{2^m + 1}{3}) = 2^i \cdot 3 \cdot 17 \cdot p,$$

where m, $2^m - 1$ and $\frac{2^m+1}{3}$ are all primes according to Theorem 2.1. This equation has no answer as neither $2^m - 1 = 17$ nor $\frac{2^m+1}{3} = 17$ has a solution for m.

Now let the group $L_2(r)$ in sub-item (2) be a candidate group. Then for odd prime r and for prime $s > 3$ we have $r^2 - 1 = 2^a \cdot 3^b \cdot s$ and

$$2^{a-1} \cdot 3^b \cdot s \cdot r = 2^i \cdot 3 \cdot 17 \cdot p.$$

From these we obtain $b = 1$, $0 \leq a - 1 \leq 7$ and either $s = 17$ or $r = 17$. The group $L_2(17)$ is a K_3 group, so $r \neq 17$. Also the equality $s = 17$ is not possible, since the equation $r^2 - 1 = 2^a \cdot 3 \cdot 17$ has no plausible solution for r when $a = 1, 2, \ldots, 8$.

Finally note that the order of a group in sub-item (4) is divisible by 3^m for $m > 1$. □

Lemma 3.3. *The group $\mathbb{Z}_{16} \times GL_2(7)$ does not have a subgroup isomorphic to $L_2(7)$.*

Proof. Firstly $GL_2(7)$ does not have a subgroup isomorphic to $L_2(7)$. Suppose on the contrary that $L_2(7) \simeq K \leq GL_2(7)$. As $SL_2(7) \trianglelefteq GL_2(7)$, we have $K \cap SL_2(7) \trianglelefteq K$ and so $K \cap SL_2(7) = 1$ or K since K is simple. If $K \cap SL_2(7) = 1$, then $SL_2(7)K$ is a subgroup of $GL_2(7)$ of order $|SL_2(7)| \cdot |L_2(7)|$. But this order is divisible by 7^2 whereas $|GL_2(7)|$ is not. Therefore $K \cap SL_2(7) = K$ and so $K \leq SL_2(7)$ implying $K \trianglelefteq SL_2(7)$ since $|K| = \frac{|SL_2(7)|}{2}$. Let $Z = Z(SL_2(7))$. Then

$$\frac{K}{K \cap Z} \simeq \frac{KZ}{Z} \trianglelefteq \frac{SL_2(7)}{Z} \simeq K.$$

Again because K is simple, this implies $\frac{K}{K \cap Z} = 1$ or $|\frac{K}{K \cap Z}| = |K|$. In the former case, $K \cap Z = K$ and so $K \leq Z$ which is impossible. In the latter case, $K \cap Z = 1$ and so KZ is a subgroup of $SL_2(7)$ of order $|K| \cdot |Z| = |SL_2(7)|$, implying that $SL_2(7) = KZ$. Now we get $SL_2(7)' = (KZ)' = K' = K$. By using the well-known fact that $SL_2(q)' = SL_2(q)$ for $q > 3$, we obtain $K = SL_2(7)$, a contradiction to $K \simeq L_2(7)$.

Now let $G = \mathbb{Z}_{16} \times GL_2(7) = N_1 N_2$ be an internal direct product of $N_1 \simeq \mathbb{Z}_{16}$ and $N_2 \simeq GL_2(7)$ and suppose on the contrary that $L_2(7) \simeq H \leq G$. Since $H \cap N_2 \trianglelefteq H$, either $H \cap N_2 = 1$ or $H \cap N_2 = H$. If $H \cap N_2 = H$, then $H \leq N_2$ which is not possible as we already showed that $GL_2(7)$ does not have subgroups isomorphic to $L_2(7)$. Accordingly $H \cap N_2 = 1$ and therefore the order of the subgroup $HN_2 \leq G$ is $|H||N_2|$ which should divide $|G| = |N_1||N_2|$. This requires $|N_1| = 16$ to be divisible by $|H|$ which is not the case. □

Lemma 3.4. *Let Γ be a connected cubic X-semisymmetric graph. Let $G \trianglelefteq X$ and suppose for every vertex u the stabilizer G_u is a 2-group. Then for each vertex u, $G_u = 1$.*

Proof. Since $G \trianglelefteq X$, for every vertex u, $G_u \trianglelefteq X_u$. Fix an arbitrary vertex u of Γ and take v to be an arbitrary neighbor of u. Also suppose $g \in G_u$ is arbitrary. By Sylow's Theorem, G_u is contained in a Sylow 2-subgroup of X_u. According to Proposition 2.9 $X_u \cap X_v$ is a Sylow 2-subgroup of X_u. Hence G_u is contained in a conjugate of $X_u \cap X_v$. So assume $G_u \leq x^{-1}(X_u \cap X_v)x$ for some $x \in X_u$. Then $G_u = xG_ux^{-1} \leq X_u \cap X_v$. This yields $g \in X_v$ and hence g also stabilizes v.

The conclusion is that if $g \in G$ stabilizes a vertex u, then it will also stabilize every neighbor of u. By connectedness of Γ, g will stabilize every vertex of Γ and so g is the identity automorphism. □

For every prime power q subgroups of $L_2(q)$ have been classified (see Chapter 3 of [14]). It can be verified that $L_2(17)$ has no *proper* subgroup of order $2^4 \cdot s$ for any integer $s > 1$.

Lemma 3.5. *There is no connected cubic semisymmetric graph of order* $34 \cdot 3^3$.

Proof. Suppose on the contrary that Γ is a connected cubic semisymmetric graph of order $34 \cdot 3^3$ with a bipartition $\{U, W\}$. Each of the bipartition sets has cardinality $17 \cdot 3^3$ and if $A = Aut(\Gamma)$, then $|A| = 2^r \cdot 3^4 \cdot 17$ for some $0 \leq r \leq 7$.

If $P \trianglelefteq A$ is of order 3^3, then P is intransitive on both U and W and hence according to Theorem 2.10 Γ_P is connected cubic $\frac{A}{P}$-semisymmetric with a bipartition $\{U_P, W_P\}$ where $|U_P| = |W_P| = 17$ and where $|\frac{A}{P}| = 2^r \cdot 3 \cdot 17$. Therefore $\frac{A}{P}$ is transitive on U_P and also by Proposition 2.8 the action of $\frac{A}{P}$ on U_P is faithful. So $\frac{A}{P}$ is a transitive permutation group of degree 17. All such groups are known and are listed in Proposition 2.5. The order of none of these groups is of the form $2^i \cdot 3 \cdot 17$. So $\frac{A}{P}$ cannot be a transitive permutation group of degree 17, a contradiction. Therefore in the rest of the proof, we assume that A does not have any normal subgroup of order 3^3.

Let $N \simeq T^k$ be a minimal normal subgroup of A, where T is simple. If T is nonabelian, then in view of Corollary 2.11 and Theorem 2.4, $|N|$ is divisible by $17 \cdot 3^3$. So T is a simple K_3-group. Since the power of 17 in $|A|$ is 1, we conclude that $k = 1$ and $N \simeq T$. Hence N is a simple group of order $2^i \cdot 3^j \cdot 17$ where $j = 3$ or 4. But there is no such group as the only simple K_3-group whose order is divisible by 17, is $L_2(17)$ of order $2^4 \cdot 3^2 \cdot 17$. Therefore T is abelian and hence N is elementary abelian. This means that $|N|$ is not divisible by $|U| = 17 \cdot 3^3$ and so by Corollary 2.11 $|N|$ divides $|U| = 17 \cdot 3^3$. Because we have assumed A does not have normal subgroups of order 3^3, it follows that $N \simeq \mathbb{Z}_{17}$ or $\mathbb{Z}_3^i$ for some $1 \leq i \leq 2$.

In the following we consider two general cases and discuss that both result in contradictions:

(a) Suppose A has at least one normal subgroup of order $17 \cdot 3^i$ for some $i \geq 0$. Let M be the largest such subgroup, i.e., $M \trianglelefteq A$, $|M| = 17 \cdot 3^j$ for some $j \geq 0$, and if $K \trianglelefteq A$ and $|K| = 17 \cdot 3^i$, then $|K| \leq |M|$.

Let P be a Sylow 3-subgroup of M. The number of Sylow 3-subgroups of M is 1 and so $P \trianglelefteq^c M \trianglelefteq A$ which leads to $P \trianglelefteq A$. Now according to Corollary 2.11 $|P|$ must divide $|U| = 17 \cdot 3^3$ which results in $j \leq 3$. Also since we have assumed A does not have any normal subgroup of order 3^3, it follows that $j < 3$. Now due to its order, M is intransitive on both U and W and hence according to Theorem 2.10 Γ_M is a connected cubic $\frac{A}{M}$-semisymmetric graph with a bipartition $\{U_M, W_M\}$ where $|U_M| = |W_M| = 3^{3-j}$ and where $|\frac{A}{M}| = 2^r \cdot 3^{4-j}$. If $\frac{K}{M}$ is a minimal normal subgroup of $\frac{A}{M}$, by Theorem 2.4 it is solvable and hence elementary abelian of order q^i for some prime q and some $i \geq 1$. By applying Corollary 2.11 to Γ_M, it follows that $q = 3$ and so $K \trianglelefteq A$ is of order $17 \cdot 3^{j+i}$, contradicting the choice of M.

(b) Now suppose A does not have any normal subgroup of order $17 \cdot 3^i$ for any $i \geq 0$. Let N be a minimal normal subgroup of A. As we showed earlier, $N \simeq \mathbb{Z}_{17}$ or $\mathbb{Z}_3^i$ for some $1 \leq i \leq 2$. In this case N cannot be isomorphic to $\mathbb{Z}_{17}$ since $|N| = 17 \cdot 3^0$. It follows that $O_3(A) \neq 1$. Corollary 2.11 implies that $|O_3(A)| \leq 3^3$. Also by our assumption, $|O_3(A)| \neq 3^3$; So $|O_3(A)| = 3^i$ for $i = 1$ or 2. Let $M = O_3(A)$. According to Theorem 2.10 Γ_M is a connected cubic $\frac{A}{M}$-semisymmetric graph of order $34 \cdot 3^{3-i}$ with a bipartition $\{U_M, W_M\}$ where $|U_M| = |W_M| = 17 \cdot 3^{3-i}$ and where $|\frac{A}{M}| = 2^r \cdot 3^{4-i} \cdot 17$. Let $\frac{K}{M}$ be a minimal normal subgroup of $\frac{A}{M}$.

If $\frac{K}{M}$ is solvable, it is elementary abelian and hence it follows from Corollary 2.11 that $|\frac{K}{M}|$ divides $|U_M| = |W_M| = 17 \cdot 3^{3-i}$. Consequently $|\frac{K}{M}| = 17$ or 3^j for some $1 \leq j \leq 3 - i$. If $|\frac{K}{M}| = 17$, then $K \trianglelefteq A$ is of order $17 \cdot 3^i$; but we have assumed A does not have any such normal subgroup. On the other hand, if $|\frac{K}{M}| = 3^j$, then $|K| = 3^{i+j} > 3^i$, contradicting the assumption that $|O_3(A)| = 3^i$.

Now let $G = \frac{K}{M}$ be unsolvable. Then it follows from Theorem 2.4 and Theorem 2.10 that G is transitive on at least one of the bipartition sets; Suppose G is transitive on U_M. So $|G|$ is divisible by $|U_M| = 17 \cdot 3^{3-i}$. The stabilizer G_u of a vertex $u \in U_M$ has cardinality $\frac{|G|}{|U_M|}$. Since the power of 17 in $|G|$ is 1, G must be a simple K_3-group. The only simple K_3-group whose order is divisible by 17, is $L_2(17)$ of order $2^4 \cdot 3^2 \cdot 17$. Therefore $G \simeq L_2(17)$.

If $i = 2$, then for a vertex $u \in U_M$ we have $|G_u| = \frac{|G|}{|U_M|} = \frac{2^4 \cdot 3^2 \cdot 17}{17 \cdot 3} = 2^4 \cdot 3$. But $L_2(17)$ has no subgroup of order $2^4 \cdot 3$. So the case $i = 2$ results in a contradiction.

If $i = 1$, then for every vertex $u \in U_M$ we have $|G_u| = \frac{2^4 \cdot 3^2 \cdot 17}{17 \cdot 3^2} = 2^4$. Now there are two possibilities; either G is also transitive on W_M or G is intransitive on W_M. In the former case, for each vertex $w \in W_M$ we have $|G_w| = \frac{|G|}{|W_M|} = 2^4$ and therefore we can invoke Lemma 3.4 to conclude that for each vertex v of the graph Γ_M the stabilizer size is $|G_v| = 1$ which is obviously a contradiction. Now consider the latter case, where G is not transitive on W_M. Since Γ_M is $\frac{A}{M}$-semisymmetric, $\frac{A}{M}$ is transitive on W_M and also according to Proposition 2.8, $\frac{A}{M}$ is faithful on W_M. Because $G \trianglelefteq \frac{A}{M}$, it follows from Proposition 6.3 and Proposition 7.1 of [16] that all the orbits of the action of G on W_M are of equal size t, and t divides $|W_M| = 17 \cdot 3^2$. Therefore $s = \frac{17 \cdot 3^2}{t}$ is an integer. Let Δ be an orbit of the action of G on W_M and let $w \in \Delta$. Since $\frac{A}{M}$ is faithful on W_M, we conclude that G_w is a proper subgroup of G (equivalently $t \neq 1$). Also since G is not transitive on W_M, we have $t < 17 \cdot 3^2$ and so $s > 1$. Now $|G_w| = \frac{|G|}{|\Delta|} = 2^4 \cdot s$; but $L_2(17)$ does not have any proper subgroup of such order. So the case $i = 1$ leads to a contradiction too. □

Lemma 3.6. *Let $p > 3$ be a prime number other than 17. If Γ is a connected cubic semisymmetric graph of order $34p^3$, then $Aut(\Gamma)$ has a normal Subgroup of order p^3.*

Proof. Let $A = Aut(\Gamma)$ and take $\{U, W\}$ to be a bipartition for Γ. Each of the two partite sets has cardinality $17p^3$ and since A is transitive on the partite sets, we have $|A| = 2^r \cdot 3 \cdot 17 \cdot p^3$ for some $0 \leq r \leq 7$. We prove the result by showing that the assumption $|O_p(A)| < p^3$ leads to a contradiction. Let $N \simeq T^k$ be a minimal normal subgroup of A, where T is simple.

If T is nonabelian, since the power of 3 and 17 in $|A|$ is 1, we should have $k = 1$ and $N \simeq T$ is nonabelian simple. In this case the order of N cannot divide $|U| = 17p^3$ according to Theorem 2.4. So according to Corollary 2.11 $|N|$ should be divisible by $|U| = 17p^3$. The order of every simple K_3-group, all listed in part (i) of Theorem 2.1, is divisible by 2 and 3. Therefore N cannot be a K_3-group and hence it must be a simple K_4-group whose order is of the form $2^i \cdot 3 \cdot 17 \cdot p^3$ for some $1 \leq i \leq 7$. But no such group exists according to Lemma 3.2

Therefore N should be elementary abelian and hence by Corollary 2.11 $|N|$ divides $17p^3$. As a result, $N \simeq \mathbb{Z}_{17}$ or $\mathbb{Z}_p^i$ for some $1 \leq i \leq 3$. In any cases, Γ_N would itself be a connected cubic $\frac{A}{N}$-semisymmetric graph of order $\frac{34p^3}{|N|}$.

Case 1. $O_p(A) = 1$. In this case, the minimal normal subgroup of A is $N \simeq \mathbb{Z}_{17}$ and Γ_N is $\frac{A}{N}$-semisymmetric of order $2p^3$. We have $|\frac{A}{N}| = 2^r \cdot 3 \cdot p^3$ and $|U_N| = |W_N| = p^3$ where $\{U_N, W_N\}$ is a bipartition for Γ_N. Let $\frac{M}{N}$ be a minimal normal subgroup of $\frac{A}{N}$. If $\frac{M}{N}$ is unsolvable, then it must be a simple $\{2, 3, p\}$-group whose order is divisible by p^3 according to Corollary 2.11. But no simple $\{2, 3, p\}$-group (all listed in Theorem 2.1) has order divisible by p^3. So $\frac{M}{N}$ is elementary abelian and hence $\frac{M}{N} \simeq \mathbb{Z}_p^i$ for $i = 1$, 2 or 3. Consequently

$M \trianglelefteq A$ is of order $17p^i$ for $i = 1$, 2 or 3. The number of Sylow p-subgroups of M divides 17 and hence must be 1 since $p \neq 2$. Now the Sylow p-subgroup of M is characteristic in M and hence normal in A, a contradiction.

Case 2. $|O_p(A)| = p$. Let $M = O_p(A)$. According to Theorem 2.10, Γ_M is a connected cubic $\frac{A}{M}$-semisymmetric graph with the bipartition $\{U_M, W_M\}$, where $|U_M| = |W_M| = 17p^2$ and $|\frac{A}{M}| = 2^r \cdot 3 \cdot 17 \cdot p^2$. Take $\frac{L}{M}$ to be a minimal normal subgroup of $\frac{A}{M}$.

If $\frac{L}{M}$ is unsolvable, then its order should be divisible by $|U_M| = 17p^2$ according to Corollary 2.11 and Theorem 2.4, and hence it is a simple $\{2, 3, 17, p\}$-group of order $2^i \cdot 3 \cdot 17 \cdot p^2$ for some $1 \leq i \leq 7$. But there is no such group according to Lemma 3.2.

Now assume $\frac{L}{M}$ is solvable. In this case, $\frac{L}{M}$ is elementary abelian and by Corollary 2.11 its order divides $|U_M| = 17p^2$. So $\frac{L}{M} \simeq \mathbb{Z}_{17}$ or $\mathbb{Z}_p^i$ for $i = 1$ or 2. The isomorphism $\frac{L}{M} \simeq \mathbb{Z}_p^i$ results in $|L| = p^{i+1}$ which contradicts the assumption that $|O_p(A)| = p$. Hence $\frac{L}{M} \simeq \mathbb{Z}_{17}$ and so $|L| = 17p$. The normal subgroup $L \trianglelefteq A$, is intransitive on both U and W due to its order. So we can consider the graph Γ_L which is connected cubic $\frac{A}{L}$-semisymmetric (Theorem 2.10) with the bipartition $\{U_L, W_L\}$, where $|U_L| = |W_L| = p^2$ and where $|\frac{A}{L}| = 2^r \cdot 3 \cdot p^2$.

Let $\frac{T}{L}$ be a minimal normal subgroup of $\frac{A}{L}$. If it is solvable, then it follows from Corollary 2.11 that $\frac{T}{L} \simeq \mathbb{Z}_p^j$ for $j = 1$ or 2, and therefore $|T| = 17p^{j+1}$. A Sylow p-subgroup of T is normal in A, contradicting our current assumption on $|O_p(A)|$. On the other hand, if $\frac{T}{L}$ is unsolvable, then by Corollary 2.11 its order is divisible by $|U_L| = p^2$ and hence it is a simple group whose order is of the form $2^i \cdot 3 \cdot p^2$ for some $1 \leq i \leq 7$. But there is no such simple K_3-group.

Therefore every assumption on $\frac{T}{L}$ and hence every assumption on $\frac{L}{M}$ results in a contradiction from which we conclude that $|O_p(A)| = p$ is not possible.

Case 3. $|O_p(A)| = p^2$. Let $M = O_p(A)$. According to Proposition 2.10, Γ_M is a connected cubic $\frac{A}{M}$-semisymmetric graph with the bipartition $\{U_M, W_M\}$, where $|U_M| = |W_M| = 17p$ and $|\frac{A}{M}| = 2^r \cdot 3 \cdot 17 \cdot p$. Take $\frac{L}{M}$ to be a minimal normal subgroup of $\frac{A}{M}$.

If $\frac{L}{M}$ is unsolvable, then it is a simple group whose order is divisible by $|U_M| = 17p$ and hence it is a simple group of order $2^i \cdot 3 \cdot 17 \cdot p$ for some $0 \leq i \leq 7$. According to Lemma 3.2 the only such group is $L_2(2^4)$ which of course yields $p = 5$. So assume $p = 5$ and $\frac{L}{M} \simeq L_2(2^4)$. Since 3 does not divide the order of $\frac{\frac{A}{M}}{\frac{L}{M}}$, according to Proposition 2.7, Γ_M is G-semisymmetric where $G = \frac{L}{M} \simeq L_2(2^4)$. Now G is transitive on U_M and on W_M, each with $17 \cdot 5$ points. So the stabilizer G_u is of order 48 for any vertex u of Γ_M. According to Proposition 2.6, if $\{u, w\}$ is an edge, then there are two possibilities for a pair (G_u, G_w) of stabilizers of size 48. One possibility is $(\mathbb{S}_4 \times \mathbb{Z}_2, \mathbb{D}_8 \times \mathbb{S}_3)$ or its inverse, and the other possibility is $(\mathbb{S}_4 \times \mathbb{Z}_2, \mathbb{S}_4 \times \mathbb{Z}_2)$ or its inverse. Therefore at least one of G_u and G_w must be isomorphic to $\mathbb{S}_4 \times \mathbb{Z}_2$ and hence $\mathbb{S}_4$ should be a subgroup of G. But according to ([14], Chapter 3) for a prime power q, the group $L_2(q)$ has a subgroup isomorphic to $\mathbb{S}_4$ only when $q^2 \equiv 1 \pmod{16}$ which obviously does not hold for $L_2(2^4)$.

Now assume $\frac{L}{M}$ is solvable. In this case, $\frac{L}{M}$ is elementary abelian and hence intransitive on both U_M and W_M. So $\frac{L}{M} \simeq \mathbb{Z}_{17}$ or $\mathbb{Z}_p$. The isomorphism $\frac{L}{M} \simeq \mathbb{Z}_p$ results in $|L| = p^3$ which contradicts the assumption that $|O_p(A)| = p^2$. Hence $\frac{L}{M} \simeq \mathbb{Z}_{17}$ and so $|L| = 17p^2$. The normal subgroup L, is intransitive on both U and W due to its order. So we can consider the graph Γ_L which is connected cubic $\frac{A}{L}$-semisymmetric (Proposition 2.10) with the bipartition $\{U_L, W_L\}$, where $|U_L| = |W_L| = p$ and where $|\frac{A}{L}| = 2^r \cdot 3 \cdot p$. Let $\frac{T}{L}$ be a minimal normal subgroup of $\frac{A}{L}$. If it is solvable, then $\frac{T}{L} \simeq \mathbb{Z}_p$ and therefore $|T| = 17p^3$. A Sylow p-subgroup of T is characteristic in T and hence normal in A, contradicting our current assumption on $|O_p(A)|$. On the other hand, if $\frac{T}{L}$ is unsolvable, then it is a simple $\{2, 3, p\}$-group and hence $\frac{T}{L} \simeq \mathbb{A}_5, L_2(7)$. In the following we show that these two cases will result in contradiction.

(1) If $\frac{T}{L} \simeq \mathbb{A}_5$, then $p = 5$ and Γ_L is also $\frac{T}{L}$-semisymmetric according to Proposition 2.7, as 3 does not divide the order of $\frac{\frac{A}{L}}{\frac{T}{L}}$. So $G = \frac{T}{L}$ is transitive on U_L and on W_L, each with 5 points. For any vertices $u \in U_L$ and $w \in W_L$, the stabilizers G_u and G_w are of order 12 and hence both are isomorphic to $\mathbb{A}_4$, the only subgroup of $\mathbb{A}_5$ of order 12. But the pair $(G_u, G_w) = (\mathbb{A}_4, \mathbb{A}_4)$ is not possible for an edge $\{u, w\}$ of a cubic G-semisymmetric graph according to Theorem 2.6.

(2) If $\frac{T}{L} \simeq L_2(7)$, then $p = 7$ and Γ is also T-semisymmetric according to Proposition 2.7 as the order of $\frac{A}{T} \simeq \frac{\frac{A}{L}}{\frac{T}{L}}$ is not divisible by 3. In this case $|L| = 17 \cdot 7^2$. The number of Sylow 17-subgroups of L divides 7^2 and hence equals 1. Similarly the number of Sylow 7-subgroups of L is 1. So if P and Q are respectively the Sylow 17-subgroup and the Sylow 7-subgroup of L, then $L \simeq P \times Q$ and hence L is abelian. Now T is nonabelian and L is a maximal normal subgroup of T since $\frac{T}{L}$ is nonabelian simple. We have $L \trianglelefteq C_T(L) \trianglelefteq N_T(L) = T$ which implies $C_T(L) = L$ or T.

If $C_T(L) = L$, then by Theorem 2.2, $\frac{T}{L} \leq Aut(L) \simeq Aut(P) \times Aut(Q) \simeq \mathbb{Z}_{16} \times Aut(Q)$. Now $|Q| = 7^2$ implies either $Q \simeq \mathbb{Z}_{7^2}$ or $Q \simeq \mathbb{Z}_7 \times \mathbb{Z}_7$. If $Q \simeq \mathbb{Z}_{7^2}$, then $Aut(Q) \simeq \mathbb{U}_{7^2}$, the multiplicative group of integers modulo 7^2 which is abelian of order $\varphi(7^2) = 42$. In this case $\frac{T}{L}$ should be isomorphic to a subgroup of $\mathbb{Z}_{16} \times \mathbb{U}_{7^2}$ which is not possible since $\frac{T}{L}$ is nonabelian . On the other hand if $Q \simeq \mathbb{Z}_7 \times \mathbb{Z}_7$, then $Aut(Q) \simeq GL_2(7)$ and $\frac{T}{L}$ should be isomorphic to a subgroup of $\mathbb{Z}_{16} \times GL_2(7)$ which is not possible according to Lemma 3.3.

If $C_T(L) = T$, then $L \leq Z(T)$ and so either $Z(T) = L$ or $Z(T) = T$. As T is not abelian, we should have $Z(T) = L$. Since $|T| = |L||L_2(7)| = 2^3 \cdot 3 \cdot 17 \cdot 7^3$, Sylow 17-subgroups of T are abelian and so according to Theorem 2.3 $|T' \cap$

$Z(T)|$ is not divisible by 17. By taking $|Z(T)| = 17 \cdot 7^2$ into consideration, this yields $T' \cap Z(T) = 1$, 7 or 7^2. Again since $Z(T) = L$ is maximal normal in T, the relations $Z(T) \le T'Z(T) \trianglelefteq T$ imply either $T'Z(T) = Z(T)$ or $T'Z(T) = T$. The equality $T'Z(T) = Z(T)$ yields $T' \le Z(T)$ and so $T' = T' \cap Z(T) = 1$, 7 or 7^2. Since T is not abelian, $T' = 1$ does not hold. If $|T'| = 7^i$ for $i = 1$ or 2, then $\frac{T}{T'}$ is abelian of order $2^3 \cdot 3 \cdot 17 \cdot 7^{3-i}$. So all the subgroups of $\frac{T}{T'}$ are normal in it. If P_3 and P_{17} are the Sylow 3-subgroup and the Sylow 17-subgroup of $\frac{T}{T'}$ respectively, then $P_3P_{17} \trianglelefteq \frac{T}{T'}$ is of order 21 and so there exists some $K \trianglelefteq T$ with $\frac{K}{T'} = P_3P_{17}$. The order of K is $3 \cdot 17 \cdot 7^i$. According to Corollary 2.11, one of $|U| = 17 \cdot 7^3$ and $|T|$ should divide the other. But since $i < 3$, none of them divides the other, a contradiction.

Now suppose $T'Z(T) = T$. By taking cardinalities, we obtain $2^3 \cdot 3 \cdot 17 \cdot 7^3 = \frac{|T'| \cdot (17 \cdot 7^2)}{7^i}$ where here $i = 0$, 1 or 2. So $|T'| = 2^3 \cdot 3 \cdot 7^{1+i}$. With this order $|T'|$ neither divides $|U| = 17 \cdot 7^3$ nor is divisible by it, a contradiction.

We conclude that $\frac{T}{L}$ cannot be isomorphic to $L_2(7)$ as it leads to a contradiction. So the case $|O_p(A)| = p^2$ is impossible. □

Proof of Theorem 3.1. The result for $p = 2$ follows from [5] and for $p = 3$ follows from Lemma 3.5. Now for any prime $3 < p \neq 17$, if Γ is a connected cubic semisymmetric graph of order $34p^3$ with the bipartition $\{U, W\}$, then by Lemma 3.6 $A = Aut(\Gamma)$ has a normal subgroup P of order p^3 which is obviously intransitive on both U and W. Therefore according to Theorem 2.10 Γ_P must be a connected cubic G-semisymmetric graph of order 34 with the bipartition $\{U_P, W_P\}$, where $G = \frac{A}{P}$ and $|U_P| = |W_P| = 17$. We have $|G| = 2^r \cdot 3 \cdot 17$ and G is transitive on both U_P and W_P. Also according to Proposition 2.8 the action of G on each of U_P and W_P is faithful. So G is a transitive permutation group of degree 17. All such groups are listed in Proposition 2.5. But the order of any of the groups $\mathbb{S}_{17}$, $\mathbb{A}_{17}$, $L_2(2^4)$, $L_2(2^4) \rtimes \mathbb{Z}_2$, $P\Gamma L_2(2^4)$ or $\mathbb{Z}_{17} \rtimes \mathbb{Z}_n$ for $n = 1, 2, 4, 8, 16$, is not of the form $2^i \cdot 3 \cdot 17$. This contradiction proves that there is no connected semisymmetric cubic graph of order $34p^3$ for any prime $p \neq 17$. □

References

[1] M. Alaeiyan and M. Ghasemi, *Cubic edge-transitive graphs of order* $8p^2$, Bull. Aust. Math. Soc. **77** (2008), no. 2, 315–323. https://doi.org/10.1017/S0004972708000361

[2] M. Alaeiyan and B. N. Onagh, *Cubic edge-transitive graphs of order* $4p^2$, Acta Math. Univ. Comenian. (N.S.) **78** (2009), no. 2, 183–186.

[3] F. Buekenhout and D. Leemans, *On the list of finite primitive permutation groups of degree* ≤ 50, J. Symbolic Comput. **22** (1996), no. 2, 215–225. https://doi.org/10.1006/jsco.1996.0049

[4] Y. Bugeaud, Z. Cao, and M. Mignotte, *On simple K_4-groups*, J. Algebra **241** (2001), no. 2, 658–668. https://doi.org/10.1006/jabr.2000.8742

[5] M. Conder, A. Malnič, D. Marušič, and P. Potočnik, *A census of semisymmetric cubic graphs on up to 768 vertices*, J. Algebraic Combin. **23** (2006), no. 3, 255–294. https://doi.org/10.1007/s10801-006-7397-3

[6] J. Folkman, *Regular line-symmetric graphs*, J. Combinatorial Theory **3** (1967), 215–232.

[7] D. M. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. of Math. (2) **111** (1980), no. 2, 377–406. https://doi.org/10.2307/1971203
[8] M. Herzog, *On finite simple groups of order divisible by three primes only*, J. Algebra **10** (1968), 383–388. https://doi.org/10.1016/0021-8693(68)90088-4
[9] J. H. Kwak and R. Nedela, *Graphs and their coverings*, Lecture Notes Series, **17**, 2007.
[10] Z. Lu, C. Wang, and M. Xu, *On semisymmetric cubic graphs of order* $6p^2$, Sci. China Ser. A **47** (2004), no. 1, 1–17. https://doi.org/10.1360/02ys0241
[11] A. Malnič, D. Marušič, and C. Wang, *Cubic edge-transitive graphs of order* $2p^3$, Discrete Math. **274** (2004), no. 1-3, 187–198. https://doi.org/10.1016/S0012-365X(03)00088-8
[12] J. S. Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge, 1978.
[13] W. J. Shi, *On simple* K_4*-groups*, Chinese science Bulletin, **36** (1991), no. 17, 1281–1283.
[14] M. Suzuki, *Group theory. II*, translated from the Japanese, Grundlehren der Mathematischen Wissenschaften, **248**, Springer-Verlag, New York, 1986. https://doi.org/10.1007/978-3-642-86885-6
[15] C. Q. Wang and T. S. Chen, *Semisymmetric cubic graphs as regular covers of* $K_{3,3}$, Acta Math. Sin. (Engl. Ser.) **24** (2008), no. 3, 405–416. https://doi.org/10.1007/s10114-007-0998-5
[16] H. Wielandt, *Finite permutation groups*, Translated from the German by R. Bercov, Academic Press, New York, 1964.
[17] S. Zhang and W. J. Shi, *Revisiting the number of simple* K_4*-groups*, arXiv:1307.8079v1 [math.NT], 2013.

Mohammad Reza Darafsheh
School of Mathematics, Statistics, and Computer Science
College of Science
University of Tehran, Tehran, Iran
Email address: darafsheh@ut.ac.ir

Mohsen Shahsavaran
School of Mathematics, Statistics, and Computer Science
College of Science
University of Tehran, Tehran, Iran
Email address: m.shahsavaran@ut.ac.ir