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ON w-COPURE FLAT MODULES AND DIMENSION

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Dedicated to Professor Fanggui Wang for the introduction and development of the w-operation theory

Abstract. Let R be a commutative ring. An R-module M is said to be w-flat if $\mathrm{Tor}_1^R(M,N)$ is GV-torsion for any R-module N. It is known that every flat module is w-flat, but the converse is not true in general. The w-flat dimension of a module is defined in terms of w-flat resolutions. In this paper, we study the w-flat dimension of an injective w-module. To do so, we introduce and study the so-called w-copure (resp., strongly w-copure) flat modules and the w-copure flat dimensions for modules and rings. The relations between the introduced dimensions and other (classical) homological dimensions are discussed. We also study change of rings theorems for the w-copure flat dimension in various contexts. Finally some illustrative examples regarding the introduced concepts are given.

1. Introduction

Throughout, all rings considered are commutative with unity and all modules are unital. Let R be a ring and M be an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of M, and $\mathrm{wdim}(R)$ and $\mathrm{gldim}(R)$ to denote, respectively, the weak and global homological dimensions of R.

Enochs and Jenda [8,9] introduced the notion of Gorenstein projective modules (G-projective modules for short), as an extension of the same notion to modules that are not necessarily finitely generated, and the Gorenstein injective modules (G-injective modules for short) as a dual notion of Gorenstein projective modules. To complete the analogy with the classical modules in the homological theory, Enochs, Jenda, and Torrecillas [10] introduced the Gorenstein flat modules (G-flat modules for short). Recall that an R-module M is called *Gorenstein flat*, if there exists an exact sequence of flat R-modules $\mathbf{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ such that $M \cong \mathrm{Im}\,(F_0 \to F^0)$ and such

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that the functor $-\otimes_R I$ leaves $\mathbf{F}$ exact whenever I is an injective R-module. The complex $\mathbf{F}$ is called a *complete flat resolution*. The *Gorenstein flat dimension* is defined in terms of Gorenstein flat resolutions, and denoted by $\mathrm{Gfd}\,(-)$ [14].

In [2], the authors defined *the weak Gorenstein global dimension* of a ring R to be

$$\mathrm{wGgldim}(R)=\sup\{\mathrm{Gfd}\,_R(M)\mid M \text{ is an } R\text{-module}\}.$$

It is proved (in [17, Theorem 2.12]) that, for any ring R, we have

$$\mathrm{wGgldim}(R)=\sup\{\mathrm{fd}\,_R(E)\mid E \text{ is an injective } R\text{-module}\}.$$

The weak Gorenstein global dimension of a ring R is well studied, by N. Ding and J. Chen (1993), in [5] under the name $IFD(R)$. Another approach of the $IFD(R)$ was also done by considering the (strongly) copure flat modules. Recall that an R-module M is called *copure flat* if $\mathrm{Tor}\,_1^R(E,M)=0$ for any injective R-module E, and M is said to be *strongly copure flat* if $\mathrm{Tor}\,_i^R(E,M)=0$ for any injective R-modules E and any $i\geq 1$ [7]. Copure flat modules and strongly copure flat modules were discovered when studying flat preenvelopes. Let M be an R-module. The *copure flat dimension* of M [7], denoted by $\mathrm{cfd}_R(M)$, is the smallest integer $n>0$ such that $\mathrm{Tor}\,_{n+i}^R(E,M)=0$ for any injective R-module E and any integer $i\geq 1$. If no such n exists, then $\mathrm{cfd}_R(M)=\infty$. The *copure flat dimension of a ring* R [11], denoted by $\mathrm{cfD}(R)$, is defined to be the supremum of copure flat dimensions of R-modules. It is proved that $\mathrm{cfD}(R)=\mathrm{IFD}(R)$ ([11, Theorem 3.8]).

Since the concept of semi-divisorial modules, which generalizes both divisorial modules and injective modules, was introduced by Glaz and Vasconcelos ([13]) and was modified to allow the semi-divisorial closure (or w-closure) by the second author, the so-called w-operation has proved to be useful in the study of multiplicative ideal theory and module theory. The introduction of the w-operation in the class of flat modules has been successful, see for instance [1, 16, 19, 21, 23, 25]. The notion of w-flat modules appeared first in [19] when R is a domain and was extended to arbitrary commutative rings in [16].

Let J be an ideal of R. Following [30], J is called a *Glaz-Vasconcelos ideal* (a GV-ideal for short) if J is finitely generated and the natural homomorphism $\varphi: R\to J^*=\mathrm{Hom}_R(J,R)$ is an isomorphism. Let M be an R-module and define

$$\mathrm{tor}_{GV}(M)=\{x\in M\mid Jx=0 \text{ for some } J\in GV(R)\},$$

where $GV(R)$ is the set of GV-ideals of R. It is clear that $\mathrm{tor}_{GV}(M)$ is a submodule of M. Now M is said to be *GV-torsion* (resp., *GV-torsion-free*) if $\mathrm{tor}_{GV}(M)=M$ (resp., $\mathrm{tor}_{GV}(M)=0$). A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}_R^1(R/J,M)=0$ for any $J\in GV(R)$. Projective modules and reflexive modules are w-modules. In the recent paper [31], it was shown that flat modules are w-modules. The notion of w-modules was introduced firstly over a domain [24] in the study of Strong Mori domains and was extended to commutative rings with zero divisors in [30]. Let w-Max(R) denote the set of maximal w-ideals of R, i.e., w-ideals of R maximal among proper integral w-ideals of R. Following [30, Proposition 3.8], every maximal w-ideal is prime.

For any GV-torsion-free module M,

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the *w-envelope* of M, where $E(M)$ denotes the injective hull of M. It is clear that a GV-torsion-free module M is a w-module if and only if $M_w = M$.

Let M and N be R-modules and $f : M \to N$ be a homomorphism. Following [20], f is called a *w-monomorphism* (resp., *w-epimorphism*, *w-isomorphism*) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism) for all $\mathfrak{m} \in w$-Max(R). An R-module M is called a *w-flat module* if the induced map $1 \otimes f : M \otimes A \to M \otimes B$ is a w-monomorphism for any w-monomorphism $f : A \to B$. Certainly flat modules are w-flat, but the converse implication is not true in general. Recently, modules of this type have received attention in several papers in the literature (see for example [16, 21, 25]). Characterizations of w-flat modules are given in [16, Theorem 3.3].

Lemma 1.1. *Let R be a ring and M be an R-module. Then the following are equivalent.*

(1) *M is w-flat.*
(2) *$M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for all $\mathfrak{m} \in w$-Max(R).*
(3) *$\mathrm{Tor}_1^R(M, N)$ is GV-torsion for all R-modules N.*
(4) *$\mathrm{Tor}_n^R(M, N)$ is GV-torsion for all R-modules N and all $n \geq 1$.*

In [25], the authors introduced and investigated the w-flat dimensions of modules and rings. Let R be a ring and n be a nonnegative integer. We say that an R-module M has *w-flat dimension less or equal to n*, denoted by w-fd$_R(M) \leq n$, if $\mathrm{Tor}_{n+1}^R(M, N)$ is a GV-torsion R-module for all R-modules N. Hence, the *w-weak global dimension of R* is defined to be

$$w\text{-wdim}(R) = \sup\{w\text{-fd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

In Section 2, we introduce and characterize the (strongly) w-copure flat modules and the w-copure flat dimension of modules. We show then that the w-copure flat dimension is a refinement of the w-flat dimension. Section 3 deals with the w-copure flat dimension of rings. In Proposition 3.2, it is proved that the w-copure flat dimension of a ring R is equal to the supremum of w-flat dimensions of injective w-modules. The relations between the w-copure flat dimension and the weak (Gorenstein) global dimension of a ring are discussed. In Section 4, we also study change of rings theorems for the w-copure flat dimension in various contexts. Some illustrative examples are given.

Any undefined terminology or notation is standard, as in [3, 13, 23].

2. The w-copure flat dimension of modules

We begin this section by introducing key concepts which will be used throughout the paper.

Definition 2.1. Let R be a ring and M be an R-module.

(1) M is called *w-copure flat* if $\mathrm{Tor}\,_1^R(E, M)$ is a GV-torsion R-module for any injective w-module E, and M is said to be *strongly w-copure flat* if $\mathrm{Tor}\,_n^R(E, M)$ is a GV-torsion R-module for any injective w-module E and any $n \geq 1$.

(2) The *w-copure flat dimension* of M, denoted by $w\text{-cfd}_R(M)$, is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Tor}\,_{n+i}^R(E, M)$ is GV-torsion for any injective w-module E and any $i \geq 1$. If there is no such n, set $w\text{-cfd}_R(M) = \infty$.

Obviously by Lemma 1.1, every w-flat module is strongly w-copure flat (and so w-copure flat) and every copure flat (resp., strongly copure flat) is w-copure flat (resp., strongly w-copure flat). Recall that a ring R is called a *DW-ring* if every ideal of R is a w-ideal, or equivalently $GV(R) = \{R\}$ [18]. In this case, the only GV-torsion module is (0), and so over such a ring, copure flat (resp., strongly copure flat) module and w-copure flat (resp., strongly w-copure flat) coincide. Examples of DW-rings are Prüfer domains, domains with Krull dimension one, and rings with Krull dimension zero.

Next we give an example of a strongly w-copure flat module which is not w-flat.

Example 2.2. Let R be a QF-ring, but not semisimple. For example, $R := k[X]/(X^2)$, where k is a field. Then every flat module is projective and every injective module is flat, and so every R-module is a strongly copure flat module. Since R is not a semisimple ring, there exists an R-module which is not flat. Thus every strongly copure flat R-module is not necessarily flat. Since $\dim(R) = 0$, R is a DW-ring. Thus every strongly w-copure flat R-module is not necessarily w-flat.

Proposition 2.3. *Let R be a ring and $0 \to A \to B \to C \to 0$ be a w-exact sequence of R-modules, where C is strongly w-copure flat. Then A is strongly w-copure flat if and only if B is strongly w-copure flat.*

Proof. Let E be an injective w-module and let $i \geq 1$. By [23, Theorem 6.6.2], the induced homomorphism $\mathrm{Tor}\,_i^R(E, A) \to \mathrm{Tor}\,_i^R(E, B)$ is a w-isomorphism. Thus the assertion follows. □

Corollary 2.4. *Let R be a ring and M be a GV-torsion-free R-module. Then M is strongly w-copure flat if and only if M_w is strongly w-copure flat.*

Proof. By [23, Proposition 6.2.5], M_w/M is GV-torsion, and so strongly w-copure flat. Hence applying Proposition 2.3 to the short exact sequence $0 \to M \to M_w \to M_w/M \to 0$ gives the result. □

Proposition 2.5. *The following statements are equivalent for an R-module M.*

(1) *M is strongly w-copure flat.*
(2) $\mathrm{Ext}_R^i(L, \mathrm{Hom}_R(M, E)) = 0$ *for any injective w-modules E and L, and any integer $i \geq 1$.*
(3) *$M \otimes F$ is strongly w-copure flat for any w-flat R-module F.*
(4) *$M \otimes F$ is strongly w-copure flat for any flat R-module F.*
(5) *If $\xi : \cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is w-exact with each F_i w-flat, then $E \otimes \xi$ remains w-exact for any injective w-module E.*
(6) *There exists a w-exact sequence $\xi : \cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i w-flat such that $E \otimes \xi$ remains w-exact for any injective w-module E.*
(7) *If $\xi : \cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is w-exact with each F_i flat, then $E \otimes \xi$ remains w-exact for any injective w-module E.*
(8) *There exists a w-exact sequence $\xi : \cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i flat such that $E \otimes \xi$ remains w-exact for any injective w-module E.*

Proof. Let E be an injective w-module and A and B be any two R-modules. By [12, Theorem 1.1.8], we have the following isomorphism

$$\mathrm{Ext}_R^i(A, \mathrm{Hom}_R(B, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_i^R(A, B), E) \tag{2.1}$$

for any $i \geq 1$.

(1) $\Rightarrow$ (2) Let L be an arbitrary injective w-module. Since $\mathrm{Tor}_n^R(L, M)$ is GV-torsion, we have $\mathrm{Hom}_R(\mathrm{Tor}_i^R(L, M), E) = 0$. By the isomorphism (2.1), we have $\mathrm{Ext}_R^i(L, \mathrm{Hom}_R(B, E)) = 0$.

(2) $\Rightarrow$ (1) Again by the isomorphism (2.1), we have $\mathrm{Hom}_R(\mathrm{Tor}_i^R(L, M), E) = 0$. By [23, Exercise 6.22], $\mathrm{Tor}_i^R(L, M)$ is GV-torsion. Thus M is strongly w-copure flat.

(1) $\Rightarrow$ (4) Let E be an injective w-module. By [23, Theorem 3.4.10], we have the natural isomorphism $\mathrm{Tor}_i^R(E, M \otimes_R F) \cong \mathrm{Tor}_i^R(E, M) \otimes_R F$ for any $i \geq 1$. Since $\mathrm{Tor}_i^R(E, M)$ is a GV-torsion module for any $i \geq 1$, so is $\mathrm{Tor}_i^R(E, M \otimes_R F)$ for any $i \geq 1$. Thus $M \otimes F$ is strongly w-copure flat.

(4) $\Rightarrow$ (3) Let E be an injective w-module. By Lemma 1.1, $F_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module, and so a flat R-module, for all $\mathfrak{m} \in w\text{-Max}(R)$. Thus $\mathrm{Tor}_i^R(E, M \otimes_R F_{\mathfrak{m}})$ is a GV-torsion R-module. By [23, Proposition 6.2.18], $\mathrm{Tor}_i^R(E, M \otimes_R F_{\mathfrak{m}})$ is also a w-module. Thus $\mathrm{Tor}_i^R(E, M \otimes_R F_{\mathfrak{m}}) = 0$. By the natural isomorphism $\mathrm{Tor}_i^R(E, M \otimes_R F)_{\mathfrak{m}} \cong \mathrm{Tor}_i^R(E, M \otimes_R F_{\mathfrak{m}}) = 0$, we get that $\mathrm{Tor}_i^R(E, M \otimes_R F)$ is GV-torsion. Thus $M \otimes F$ is strongly w-copure flat.

(3) $\Rightarrow$ (1) This follows immediately by taking $F := R$.

(1) $\Rightarrow$ (5) Let $\xi : \cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ be a w-exact sequence with each F_i flat. Set $K_0 := \ker(F_0 \to M)$ and $K_i := \ker(F_i \to F_{i-1})$ for any $i \geq 1$. Then $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to K_i \to F_i \to K_{i-1} \to 0$ for any $i \geq 1$ are w-exact. Let E be an injective w-module over R. Since

$\mathrm{Tor}\,_1^R(E, M)$ is GV-torsion, $E \otimes_R -$ leaves the sequence $0 \to K_0 \to F_0 \to M \to 0$ w-exact. It follows from Proposition 2.3 that K_0 is strongly w-copure flat since F_0 is w-flat. Then we can show that $E \otimes \xi$ leaves the sequence ξ w-exact by repeating the same procedure.

(5) $\Rightarrow$ (7) This is trivial.

(7) $\Rightarrow$ (8) This follows immediately by taking a flat resolution of M.

(8) $\Rightarrow$ (6) This is trivial.

(6) $\Rightarrow$ (1) Let E be an injective w-module over R and consider the w-exact sequence $\xi : \cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ of w-flat modules F_i such that $E \otimes \xi$ remains w-exact. Set $K_0 := \ker(F_0 \to M)$. Then there is a w-exact sequence $0 \to K_0 \to F_0 \to M \to 0$. Clearly $E \otimes_R -$ preserves the w-exactness of this sequence. Hence $\mathrm{Tor}\,_1^R(E, M)$ is GV-torsion. Now by induction, $\mathrm{Tor}\,_i^R(E, M)$ is GV-torsion for any $i \geq 1$. So M is strongly w-copure flat. □

For the w-copure flat module, we can have the corresponding result. But we omit its proof since it is easy and similar.

Proposition 2.6. *The following statements are equivalent for an R-module M.*

(1) *M is w-copure flat.*
(2) $\mathrm{Ext}\,_R^1(L, \mathrm{Hom}\,_R(M, E)) = 0$ *for any injective w-modules E and L.*
(3) *$M \otimes F$ is w-copure flat for any w-flat R-module F.*
(4) *$M \otimes F$ is w-copure flat for any flat R-module F.*
(5) *If $\xi : 0 \to A \to F \to M \to 0$ is w-exact with F w-flat, then $E \otimes \xi$ remains w-exact for any injective w-module E.*
(6) *There exists a w-exact sequence $\xi : 0 \to A \to F \to M \to 0$ with F w-flat such that $E \otimes \xi$ remains w-exact for any injective w-module E.*
(7) *If $\xi : 0 \to A \to F \to M \to 0$ is w-exact with F flat, then $E \otimes \xi$ remains w-exact for any injective w-module E.*
(8) *There exists a w-exact sequence $\xi : 0 \to A \to F \to M \to 0$ with F flat such that $E \otimes \xi$ remains w-exact for any injective w-module E.*

Recall that an R-module M is called a *strong w-module* if $\mathrm{Ext}\,_R^i(C, M) = 0$ for any GV-torsion module C and any $i \geq 1$. See [26] for more results of strong w-modules.

Proposition 2.7. *The following statements are equivalent for an R-module M.*

(1) *M is strongly w-copure flat.*
(2) *If $0 \to N \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$ is w-exact with each E_i injective, then $\mathrm{Tor}\,_i^R(N, M)$ is GV-torsion for any injective w-module M and any $i \geq 1$.*
(3) *If N is a strong w-module with $\mathrm{id}_R(N) < \infty$, then $\mathrm{Tor}\,_i^R(N, M)$ is GV-torsion for any $i \geq 1$.*

Proof. (1) $\Rightarrow$ (2) Let $A := \ker(E_{n-1} \to E_n)$. Then $0 \to A \to E_{n-1} \to E_n$ is a w-exact sequence. By [23, Theorem 6.7.2], we have a w-exact sequence

$$\mathrm{Tor}_{i+1}^R(E_n, M) \to \mathrm{Tor}_i^R(A, M) \to \mathrm{Tor}_i^R(E_{n-1}, M).$$

Thus we get that $\mathrm{Tor}_i^R(A, M)$ is GV-torsion. Then the assertion follows by applying [23, Theorem 6.7.2] repeatedly.

(2) $\Rightarrow$ (3) Since N is a strong w-module with $\mathrm{id}_R(N) < \infty$, by [26, Proposition 2.2] there exists an exact sequence

$$0 \to N \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$$

such that each E_i is an injective w-module. Now the assertion follows by hypothesis.

(3) $\Rightarrow$ (1) This is trivial since every injective w-module is a strong w-module. □

Proposition 2.8. *Let R be a ring. Then the class of all w-copure flat* (*resp., strongly w-copure flat*) *R-modules is closed under direct sums and direct summands.*

Proof. As seen in [22, Lemma 0.1], it is clear that the set of GV-torsion modules is closed under direct sums and direct summands. Hence our result follows immediately from the definition of w-copure flat (resp., strongly w-copure flat) modules. □

Before giving a functorial description of the w-copure flat dimension of modules, we need the following lemma, which is similar to [25, Lemma 2.2].

Lemma 2.9. *Let E be an injective w-module over R and $0 \to A \to F \to C \to 0$ be a w-exact sequence of R-modules with F a strongly w-copure flat module. Then the induced map $\mathrm{Tor}_{i+1}^R(E, C) \to \mathrm{Tor}_i^R(E, A)$ is a w-isomorphism for any $i \geq 1$. Hence for any $i \geq 1$, $\mathrm{Tor}_{i+1}^R(E, C)$ is GV-torsion if and only if so is $\mathrm{Tor}_i^R(E, A)$.*

Proposition 2.10. *Let R be a ring, M be an R-module, and n be a nonnegative integer. Then the following statements are equivalent.*

(1) *w-$\mathrm{cfd}_R(M) \leq n$.*
(2) *$\mathrm{Tor}_{n+i}^R(E, M)$ is GV-torsion for any injective w-module E and any $i \geq 1$.*
(3) *$\mathrm{Tor}_{n+i}^R(N, M)$ is GV-torsion for any strong w-module N with $\mathrm{id}_R(N) < \infty$ and any $i \geq 1$.*
(4) *If a sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ is exact with $F_0, \ldots, F_{n-1}$ strongly w-copure flat, then F_n is also strongly w-copure flat.*
(5) *If a sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ is exact with $F_0, \ldots, F_{n-1}$ flat, then F_n is strongly w-copure flat.*

(6) *If a sequence* $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ *is* w-*exact with* $F_0, \ldots, F_{n-1}$ *strongly* w-*copure flat, then* F_n *is also strongly* w-*copure flat.*

(7) *If a sequence* $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ *is* w-*exact with* $F_0, \ldots, F_{n-1}$ *flat, then* F_n *is strongly* w-*copure flat.*

Proof. (1) $\Leftrightarrow$ (2) This is just the definition of the w-copure flat dimension of modules.

(1) $\Leftrightarrow$ (3) This follows from Proposition 2.7.

(2) $\Leftrightarrow$ (5) Let $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be any exact sequence with $F_0, \ldots, F_{n-1}$ flat. Then F_n is strongly w-copure flat if and only if $\operatorname{Tor}^R_{n+i}(E, M) \cong \operatorname{Tor}^R_i(E, F_n)$ is GV-torsion for any injective module E and any $i \geq 1$.

(6) $\Rightarrow$ (7) $\Rightarrow$ (5) and (6) $\Rightarrow$ (4) $\Rightarrow$ (5) These are trivial.

(2) $\Rightarrow$ (6) Let E be an injective w-module over R. Set $L_n := F_n$ and $L_i := \operatorname{Im}(F_i \to F_{i-1})$ for $i = 1, \ldots, n-1$. Then both $0 \to L_{i+1} \to F_i \to L_i \to 0$ and $0 \to L_1 \to F_0 \to M \to 0$ are w-exact. By using Lemma 2.9 repeatedly, we see that for any $k \geq 1$, $\operatorname{Tor}^R_k(E, F_n)$ is GV-torsion if and only if $\operatorname{Tor}^R_{n+k}(E, M)$ is GV-torsion. By hypothesis, $\operatorname{Tor}^R_k(E, F_n)$ is GV-torsion for any $k \geq 1$. Thus F_n is strongly w-copure flat. □

Proposition 2.11. *Let* R *be a ring and* M *be an* R-*module. Then*:

(1) $w\text{-cfd}_R(M) \leq w\text{-fd}_R(M)$ *with equality if* $w\text{-fd}_R(M) < \infty$.

(2) *If* M *is a* w-*copure flat* R-*module and* $w\text{-fd}_R(M) \leq 1$, *then* M *is* w-*flat.*

Proof. (1) It suffices to prove that if $n := w\text{-fd}_R(M) < \infty$, then every strongly w-copure flat module is w-flat, i.e., $n = 0$. Let M be a strongly w-copure flat module. Assume on the contrary that $n > 0$. Then there exists an R-module N such that $\operatorname{Tor}^R_n(N, M)$ is not GV-torsion. Without loss of generality, we assume that N is GV-torsion free. Thus $E := E(N)$ is an injective w-module. Hence the exact sequence

$$0 \to N \to E \to E/N \to 0$$

gives rise to the exactness of the sequence

$$\operatorname{Tor}^R_{n+1}(E/N, M) \to \operatorname{Tor}^R_n(N, M) \to \operatorname{Tor}^R_n(E, M). \tag{2.2}$$

The left term of (2.2) is a GV-torsion module since $w\text{-fd}_R(M) = n$, and the right term of (2.2) is a GV-torsion module since M is strongly w-copure flat. Hence by [22, Lemma 0.1], $\operatorname{Tor}^R_n(N, M)$ is a GV-torsion module, a contradiction. Thus $n = 0$.

(2) Since $w\text{-fd}_R(M) \leq 1$, M is a strongly w-copure flat module. Now the assertion follows by (1). □

The next example shows that every strongly w-copure flat module is not necessarily strongly copure flat.

Example 2.12. Let $(R, \mathfrak{m})$ be a regular local ring with $\mathrm{gldim}(R) = 2$. Then $\mathrm{fd}_R(R/\mathfrak{m}) = 2$, and hence $\mathrm{fd}_R(\mathfrak{m}) = 1$. By [25], every strongly copure flat module is a flat module. Thus $\mathrm{cfd}_R(\mathfrak{m}) = 1$. Since R is a UFD, and hence is a PvMD. By [25], w-$\mathrm{wdim}(R) \geq 1$. By Proposition 2.11, every strongly w-copure flat module is w-flat, and so w-$\mathrm{cfd}_R(\mathfrak{m}) = 0$. Therefore every strongly w-copure flat module is not necessarily strongly copure flat.

The proof of the next result is standard homological algebra. Thus we omit its proof.

Proposition 2.13. *Let R be a ring and $0 \to A \to B \to C \to 0$ be a w-exact sequence of R-modules. If two of w-$\mathrm{cfd}_R(A)$, w-$\mathrm{cfd}_R(B)$, and w-$\mathrm{cfd}_R(C)$ are finite, so is the third. Moreover,*

(1) w-$\mathrm{cfd}_R(B) \leq \sup\{w\text{-}\mathrm{cfd}_R(A), w\text{-}\mathrm{cfd}_R(C)\}$.
(2) w-$\mathrm{cfd}_R(A) \leq \sup\{w\text{-}\mathrm{cfd}_R(B), w\text{-}\mathrm{cfd}_R(C) - 1\}$.
(3) w-$\mathrm{cfd}_R(C) \leq \sup\{w\text{-}\mathrm{cfd}_R(B), w\text{-}\mathrm{cfd}_R(A) + 1\}$.

Corollary 2.14. *Let R be a ring and $0 \to A \to B \to C \to 0$ be a w-exact sequence of R-modules, where B is strongly w-copure flat. Then either the three modules are strongly w-copure flat or w-$\mathrm{cfd}_R(C) = w$-$\mathrm{cfd}_R(A) + 1$.*

The next result characterizes coherent rings with finite weak Gorenstein global dimension over which every strongly w-copure flat module is strongly copure flat.

Proposition 2.15. *Let R be a coherent ring with finite weak Gorenstein global dimension. Then the following statements are equivalent.*

(1) *Every strongly w-copure flat R-module is strongly copure flat.*
(2) *R is a DW-ring.*

Proof. (1) $\Rightarrow$ (2) Let J be a GV-ideal. Then by [23, Theorem 6.2.6] R/J is a GV-torsion R-module, and so by [23, Corollary 6.7.4(1)] R/J is a w-flat R-module. Hence by hypothesis, R/J is strongly copure flat. Then by [11, Theorem 2.12], R/J is Gorenstein flat. Thus R/J is a submodule of a flat R-module, and so it is a GV-torsion free R-module (as a submodule of a w-module). Then $R/J = (0)$, and so $R = J$. Consequently R is a DW-ring.

(2) $\Rightarrow$ (1) This is clear. □

3. The w-copure flat dimension of rings

We start with the following definition.

Definition 3.1. Let R be a ring. The w-copure flat dimension of R, denoted by w-$\mathrm{cfD}(R)$, is defined as the supremum of the w-copure flat dimensions of R-modules.

Proposition 3.2. *Let R be a ring and n be a nonnegative integer. Then the following statements are equivalent.*

(1) w-cfD $(R) \leq n$.
(2) *For any R-module M and any exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with $P_0, P_1, \ldots, P_{n-1}$ projective, we have P_n is a w-copure flat module.*
(3) w-fd $_R(E) \leq n$ *for any injective w-module E.*
(4) *Every GV-torsion-free R-module can be embedded in an R-module with w-flat dimension at most n.*
(5) *Every strong w-module can be embedded in an R-module with w-flat dimension at most n.*
(6) w-fd $_R(\mathrm{Hom}_R(F,E)) \leq n$ *for any w-flat R-module F and any injective w-module E.*
(7) w-fd $_R(\mathrm{Hom}_R(F,E)) \leq n$ *for any flat R-module F and any injective w-module E.*
(8) id $_R(\mathrm{Hom}_R(L,E)) \leq n$ *for any injective w-modules E and L.*

Proof. (1) $\Rightarrow$ (2) This follows from Proposition 2.10.

(2) $\Rightarrow$ (3) Let E be an injective w-module. Consider an arbitrary R-module M and pick an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_0, \ldots, P_{n-1}$ are projective. By hypothesis, P_n is w-copure flat, and then $\mathrm{Tor}^R_{n+1}(E,M) \cong \mathrm{Tor}^R_1(E,P_n)$ is a GV-torsion R-module. Hence by [25, Proposition 2.3], we have w-fd $_R(E) \leq n$.

(3) $\Rightarrow$ (1) Let M be an R-module and E be any injective w-module. Since w-fd $_R(E) \leq n$, we obtain that $\mathrm{Tor}^R_{n+1}(E,M)$ is a GV-torsion R-module. By Proposition 2.10, w-cfd$_R(M) \leq n$. Thus w-cfD $(R) \leq n$.

(3) $\Rightarrow$ (4) Let N be a GV-torsion-free module. Then $E(N)$ is an injective w-module. Now the assertion follows immediately.

(4) $\Rightarrow$ (5) This is trivial.

(5) $\Rightarrow$ (3) Let E be an injective w-module. By hypothesis, E can be embedded in an R-module L with w-fd $_R(L) \leq n$. Thus we have $L = E \oplus B$, where B is a submodule of L. By using [25, Proposition 2.3] and the fact that the set of GV-torsion R-modules is closed under direct sums and direct summands, we can prove easily that for any two R-modules A and B, we have w-fd $_R(A \oplus B) = \sup\{w\text{-fd}_R(A), w\text{-fd}_R(B)\}$. Thus we have w-fd $_R(E) \leq n$.

(3) $\Rightarrow$ (6) Let A be an R-module. Since F is a w-flat module, $\mathrm{Tor}^R_1(A,F)$ is GV-torsion. By the isomorphism (2.1), we have that $\mathrm{Ext}^1_R(A, \mathrm{Hom}_R(F,E)) = 0$. Hence $\mathrm{Hom}_R(F,E)$ is an injective module. Therefore it follows from the hypothesis that w-fd $_R(\mathrm{Hom}_R(F,E)) \leq n$.

(6) $\Rightarrow$ (7) This is trivial.

(7) $\Rightarrow$ (3) This follows by letting $F = R$.

(1) $\Rightarrow$ (8) Let M be any R-module and E, L be injective w-modules. Thus by using the isomorphism (2.1), Proposition 2.10, and [22, Lemma 0.1], we have

$$\mathrm{Ext}^{n+i}_R(M, \mathrm{Hom}_R(L,E)) \cong \mathrm{Hom}_R\left(\mathrm{Tor}^R_{n+i}(M,L), E\right) = 0$$

for any $i \geq 1$, and so id $_R(\mathrm{Hom}_R(E,L)) \leq n$.

(8) $\Rightarrow$ (1) Let M be an R-module and E, L be injective w-modules. Since $\mathrm{id}_R(\mathrm{Hom}_R(L,E)) \leq n$, it follows from the isomorphism (2.1) that

$$\mathrm{Hom}_R(\mathrm{Tor}^R_{n+i}(L,M),E) \cong \mathrm{Ext}^{n+i}_R(M,\mathrm{Hom}_R(L,E)) = 0$$

for any $i \geq 1$. By [23, Exercise 6.22], $\mathrm{Tor}^R_{n+i}(L,M)$ is a GV-torsion module for any $i \geq 1$. Thus w-$\mathrm{cfd}_R(M) \leq n$. Hence w-$\mathrm{cfD}(R) \leq n$. □

Corollary 3.3. *Let R be a ring. Then*

$$w\text{-}\mathrm{cfD}(R) = \sup\{w\text{-}\mathrm{fd}_R(E) \mid E \textit{ is an injective } w\textit{-module}\}.$$

Proof. This follows immediately from Proposition 3.2. □

Corollary 3.4. *Let R be a ring. Then*

$$\sup\{w\text{-}\mathrm{fd}_R(M) \mid w\text{-}\mathrm{fd}_R(M) < \infty\} \leq w\text{-}\mathrm{cfD}(R) \leq w\text{-}\mathrm{wdim}(R).$$

Moreover,

(1) *If w-$\mathrm{wdim}(R) < \infty$, then the two above inequalities become equalities.*
(2) *If w-$\mathrm{cfD}(R) < \infty$, then*

$$\sup\{w\text{-}\mathrm{fd}_R(M) \mid w\text{-}\mathrm{fd}_R(M) < \infty\} = w\text{-}\mathrm{cfD}(R).$$

Proof. (1) This follows from Proposition 2.11.

(2) By Corollary 3.3, there is an injective w-module E such that w-$\mathrm{fd}_R(E) = w$-$\mathrm{cfD}(R) < \infty$. Then w-$\mathrm{cfD}(R) \leq \sup\{w\text{-}\mathrm{fd}_R(M) \mid w\text{-}\mathrm{fd}_R(M) < \infty\}$, and so we have the desired equality. □

Following [4], we say that a ring R is an *IF ring* if every injective R-module is flat. In order to characterize the rings of w-copure flat dimension 0, we introduce a new class of rings, which is a w-version of IF rings.

Definition 3.5. A ring R is called a *w-IF-ring* if every injective w-module is w-flat.

Corollary 3.6. *Let R be a ring. Then R is a w-IF-ring if and only if w-$\mathrm{cfD}(R)$ $= 0$.*

Proposition 3.7. *Let R be a ring. Then* $\mathrm{wGgldim}(R) \leq 1$ *if and only if R is DW and w-$\mathrm{cfD}(R) \leq 1$.*

Proof. As seen in [17, Theorem 2.12] and Proposition 3.2, it is clear that w-$\mathrm{cfD}(R) \leq \mathrm{wGgldim}(R)$ with equality if R is a DW-ring. Hence it suffices to prove that if $\mathrm{wGgldim}(R) \leq 1$, then R is DW. Let $J \in GV(R)$. Since $\mathrm{wGgldim}(R) \leq 1$, J is Gorenstein flat. Then there exists a short exact sequence $0 \to J \to F \to G \to 0$ where F is flat and G is Gorenstein flat. Since F is a w-module and G is a GV-torsion-free module (as a submodule of a flat module), it follows from [15, Lemma 3.2] that J is a w-module. Hence $J_w = J$. On the other hand, by [30, Proposition 3.5] $J_w = R$. Thus $GV(R) = \{R\}$, which means that R is a DW-ring. □

Corollary 3.8. *A ring R is IF if and only if it is a DW-ring and every module is (strongly) w-copure flat.*

Proof. Recall that R is IF if and only if $\mathrm{wGgldim}\,(R) = 0$. Hence the assertion follows from Proposition 3.7. □

Proposition 3.9. *A ring R is von Neumann regular if and only if every R-module is (strongly) w-copure flat and w-$\mathrm{wdim}\,(R) < \infty$.*

Proof. If R is von Neumann regular, then w-$\mathrm{wdim}\,(R) = 0$ and every R-module is flat (and so strongly w-copure flat). Conversely, if every R-module is w-copure flat, then by the equivalence of (1) and (2) in Proposition 3.2 we have w-$\mathrm{cfD}\,(R) = 0$. Now, since w-$\mathrm{wdim}\,(R) < \infty$, we get w-$\mathrm{wdim}\,(R) = 0$ (by Corollary 3.4). Then by [21, Theorem 4.4], R is von Neumann regular. □

Proposition 3.10. *The following statements are equivalent for a domain R.*

(1) *R is a field.*
(2) *Every R-module is strongly w-copure flat.*
(3) *Every R-module is w-copure flat.*
(4) *Every cyclic R-module is strongly w-copure flat.*
(5) *Every cyclic R-module is w-copure flat.*
(6) *w-$\mathrm{cfD}\,(R) = 0$.*

Proof. (1) ⇒ (2) ⇒ (4) These are clear.

(4) ⇒ (1) Let a be a nonzero element of R. Clearly $w\text{-fd}_R(R/aR) \leq \mathrm{fd}_R(R/aR) \leq \mathrm{pd}_R(R/aR) \leq 1$ since R is a domain. Thus R/aR is w-flat (by Proposition 2.11). By applying [21, Proposition 2.2] to the exact sequence $0 \to aR \to R \to R/aR \to 0$, we obtain that $(a)_w = (a^2)_w$. Hence $a \in (a^2)_w$. Thus by [21, Theorem 4.4], R is von Neumann regular. Consequently R is a field since it is a domain.

(1) ⇒ (3) ⇒ (5) ⇒ (1) These are similar to the "strongly w-copure flat dimension" case.

(1) ⇔ (6) This follows immediately by the definition of the w-coupre flat dimension of a ring. □

Next we characterize the rings of w-copure flat dimension at most one.

Proposition 3.11. *The following conditions are equivalent for a ring R.*

(1) *w-$\mathrm{cfD}\,(R) \leq 1$.*
(2) *Every submodule of a w-copure flat R-module is strongly w-copure flat.*
(3) *Every submodule of a w-copure flat R-module is w-copure flat.*

Proof. (1) ⇒ (2) Let M be a w-copure flat module and let N be a submodule of M. Let E be an injective w-module and $n \geq 1$ be an integer. We have the following exact sequence of R-modules:

$$\mathrm{Tor}\,^R_{i+1}(E, M/N) \to \mathrm{Tor}\,^R_i(E, N) \to \mathrm{Tor}\,^R_i(E, M). \tag{3.1}$$

The right term of (3.1) is always GV-torsion since M is w-flat and $w\text{-fd}\,(E) \leq 1$ by Proposition 2.11. Also the left term of (3.1) is GV-torsion since $w\text{-fd}\,(E) \leq 1$. Thus $\text{Tor}\,_i^R(E, N)$ is GV-torsion for any $i \geq 1$, which means that N is strongly w-copure flat.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) By hypothesis, every submodule of a projective R-module is w-copure flat. Hence the assertion follows from (2) $\Rightarrow$ (1) in Proposition 3.2. $\square$

Proposition 3.12. *Let R be a coherent ring. Then $w\text{-cfD}\,(R) \leq 1$ if and only if every w-copure flat is strongly w-copure flat.*

Proof. ($\Rightarrow$) This follows from Proposition 3.11.

($\Leftarrow$) Let M be an R-module. Pick a short exact sequence $0 \to K \to P \to M \to 0$ with P projective. Note that K has a flat preenvelope $f : K \to F$ since R is coherent [29, Theorem 2.5.1]. So f is a monomorphism, and we get an exact sequence $0 \to K \to F \to L \to 0$, where L is a copure flat module (by [6, Corollary 3.2]). Then L is strongly w-copure flat, and so is K. Then $w\text{-cfd}\,_R(M) \leq 1$, and so $w\text{-cfD}\,(R) \leq 1$. $\square$

Now we consider the w-copure flat dimension and w-weak global dimension of finite product rings.

Proposition 3.13. *Let R_1 and R_2 be two rings, M_1 and M_2 be an R_1-module and an R_2-module respectively. Set $R = R_1 \times R_2$ and $M = M_1 \times M_2$.*

(1) *Every R_1-module (resp., R_2-module) which is GV-torsion as an R-module is a GV-torsion R_1-module (resp., R_2-module). If M_1 (resp., M_2) is a GV-torsion R_1-module (resp., R_2-module), then M is a GV-torsion R-module.*

(2) $\text{Tor}\,_{n+i}^{R_1}(M_1, E) \cong \text{Tor}\,_{n+i}^{R_1}(M \otimes_R R_1, E) \cong \text{Tor}\,_{n+i}^{R}(M, E)$ *for any $i \geq 1$. In particular, M is a strongly w-copure flat R-module if and only if M_1 and M_2 are strongly w-copure flat R_1-module and R_2-module, respectively.*

(3) $w\text{-cfD}\,(R) = \sup\{w\text{-cfD}\,(R_1), w\text{-cfD}\,(R_2)\}$.

(4) $w\text{-fd}\,_R(M) = \sup\{w\text{-fd}\,_{R_1}(M_1), w\text{-fd}\,_{R_2}(M_2)\}$ *and* $w\text{-wdim}\,(R) = \sup\{w\text{-wdim}\,(R_1), w\text{-wdim}\,(R_2)\}$.

Proof. Note that any R-module N has a decomposition $N = N_1 \times N_2$, where N_1 (resp., N_2) is an R_1-module (resp., R_2-module). Now the assertion follows from the facts that

(a) $\text{Hom}\,_R(M, N) \cong \text{Hom}\,_{R_1}(M_1, N_1) \times \text{Hom}\,_{R_2}(M_2, N_2)$,

(b) $\text{Ext}\,_R^k(M, N) \cong \text{Ext}\,_{R_1}^k(M_1, N_1) \times \text{Ext}\,_{R_2}^k(M_2, N_2)$ for all $k \geq 1$,

(c) $M \otimes_R N \cong (M_1 \otimes_{R_1} N_1) \times (M_2 \otimes_{R_2} N_2)$, and

(d) $\text{Tor}\,_k^R(M, N) \cong \text{Tor}\,_k^{R_1}(M_1, N_1) \times \text{Tor}\,_k^{R_2}(M_2, N_2)$ for all $k \geq 1$. $\square$

Example 3.14. Let $(R, \mathfrak{m})$ be a regular local ring with $\text{gldim}(R) = 2$ and R' be an IF ring which is not von Neumann regular (take for example the ring $\mathbb{Z}/4\mathbb{Z}$). Then

(1) $1 = w\text{-wdim}\,(R) = w\text{-cfD}\,(R) < \text{wdim}\,(R) = \text{wGgldim}\,(R) = 2$.
(2) $w\text{-cfD}\,(R') = \text{wGgldim}\,(R') = 0$ and $w\text{-wdim}\,(R') = \text{wdim}\,(R') = \infty$.
(3) $w\text{-cfD}\,(R\times R') = 1 < \text{wGgldim}\,(R) = 2$ and $w\text{-wdim}\,(R) = \text{wdim}\,(R) = \infty$.

Proof. (1) It is known that R is a Krull domain, and so a $PvMD$. Thus by [25, Theorem 3.5] and [21, Theorem 4.4], $w\text{-wdim}\,(R) = 1$. Hence by using Proposition 3.4, $w\text{-cfD}\,(R) = 1$. Moreover, $\text{wGgldim}\,(R) = IFD(R) = \text{wdim}\,(R) = 2$ since $\text{wdim}\,(R) = 2 < \infty$ (by [5, Corollary 3.3]).

(2) By [5, Corollary 3.3], $\text{wdim}\,(R') = \infty$ and $\text{wGgldim}\,(R') = IFD(R) = 0$. On the other hand, $w\text{-cfD}\,(R') = \text{wGgldim}\,(R') = 0$ and $w\text{-wdim}\,(R') = \text{wdim}\,(R') = \infty$. Now by Corollary 3.8, $w\text{-cfD}\,(R) = 0$ and R is DW, and then $w\text{-wdim}\,(R') = \infty$.

(3) This follows from Proposition 3.13 and [17, Theorem 3.1]. □

4. Change of rings theorems for the w-copure flat dimension

In this section, we study change of rings theorems for the w-copure flat dimension in various contexts. Although some results of this section are analogous to those in [28], we need a new concept to give their proofs as follows.

Definition 4.1. (1) Let $\phi : R \to T$ be a ring homomorphism. Then T is said to *have property* (B_ϕ) if the following property is satisfied:
(B_ϕ) Let N be a T-module. If N is a GV-torsion R-module, then N is also a GV-torsion T-module.
(2) Let $R \subseteq T$ be a ring extension. Use $\phi : R \to T$ to denote the embedding map. Then this ring extension said to *have property* (B) if T has property (B_ϕ).
(3) Let $a \in R$. Use $\phi : R \to R/aR$ to denote the natural homomorphism. Then R is said to *have property* (B_a) if R/aR has property (B_ϕ).

Proposition 4.2. *Let $\phi : R \to T$ be a ring homomorphism such that T has property B_ϕ and let N be a T-module. Then*
(1) *If N is a GV-torsion-free T-module, then N is a GV-torsion-free R-module.*
(2) *If N is an injective w-module over T, then N is a strong w-module over R.*

Proof. (1) Set $A := \{x \in N \mid \text{there exists } J \in GV(R) \text{ such that } Jx = 0\}$. Then it is easy to see that A is a T-submodule of N. Since A is a GV-torsion R-module, it follows from property (B_ϕ) that A is a GV-torsion T-module. Since N is a GV-torsion-free T-module, we have $A = 0$. Therefore N is also a GV-torsion-free R-module.

(2) Let C be a GV-torsion R-module. Then $T \otimes_R C$ is a GV-torsion R-module. By hypothesis, $T \otimes_R C$ is a GV-torsion T-module. By [23, Theorem 3.4.11], $\text{Ext}\,^k_R(C, N) \cong \text{Hom}\,_T(\text{Tor}\,^R_k(T, C), E) = 0$. Thus N is a strong w-module over R. □

Recall that a ring extension $R \subseteq T$ is said to be *w-linked* if T as an R-module is a w-module. For example, the polynomial extension $R \subset R[X]$ is w-linked. Let $a \in R$. Write $\overline{R} = R/aR$. Let J be an ideal of R. Write $\overline{J} = \{\bar{b} \mid b \in J\}$, i.e., $\overline{J} = (J + aR)/aR$.

Proposition 4.3. (1) *If a ring extension $R \subseteq T$ is w-linked, then T has property (B).*

(2) *Let S be a multiplicative subset of a ring R. Use $\phi : R \to R_S$ to denote the natural homomorphism. Then R_S has property (B_ϕ).*

(3) *Let $a \in R$. If $\overline{J} \in GV(\overline{R})$ for any $J \in GV(R)$, then R has property (B_a).*

Proof. (1) Let N be a T-module, which is also a GV-torsion R-module. Let $x \in N$. Then there exists $J \in GV(R)$ such that $Jx = 0$. Hence $JTx = 0$. By [27, Lemma 3.3], $JT \in GV(T)$. Then N is a GV-torsion T-module. Therefore T has property (B).

(2) Let N be an R_S-module, which is also a GV-torsion R-module. Let $x \in N$. Then there exists $J \in GV(R)$ such that $Jx = 0$. By [23, Theorem 6.8.31], $J_S \in GV(R_S)$. Hence $J_S x = 0$. Then N is a GV-torsion R_S-module. Therefore R_S has property (B_ϕ).

(3) Let N be an $\overline{R}$-module, which is also a GV-torsion R-module. Let $x \in N$. Then there exists $J \in GV(R)$ such that $Jx = 0$. Hence $\overline{J}x = 0$. By hypothesis, $\overline{J} \in GV(\overline{R})$. Then N is a GV-torsion $\overline{R}$-module. Therefore R has property (B_a). □

Next we state the main theorem of this section.

Theorem 4.4. *Let $\phi : R \to T$ be a ring homomorphism such that $\mathrm{fd}_R T < \infty$ and T has property (B_ϕ). If M is a strongly w-copure flat R-module and $\mathrm{Tor}_k^R(T, M) = 0$ for any $k \geq 1$, then $T \otimes_R M$ is a strongly w-copure flat T-module.*

Proof. Let N be an injective w-module over T and let $0 \to A \to F \to M \to 0$ be an R-exact sequence, where F is a flat R-module. Since $\mathrm{Tor}_1^R(T, M) = 0$, we have the following exact sequence:

$$0 \to T \otimes_R A \to T \otimes_R F \to T \otimes_R M \to 0.$$

Thus we have the following commutative diagram with two exact rows:

$$\begin{array}{ccccccc}
0 \to X \to & \mathrm{Tor}_1^R(N, M) & \longrightarrow & N \otimes_R A & \longrightarrow & N \otimes_R F \\
& \downarrow{\scriptstyle\theta} & & \downarrow & & \downarrow \\
0 \to & \mathrm{Tor}_1^T(N, T \otimes_R M) & \longrightarrow & N \otimes_T (T \otimes_R A) & \longrightarrow & N \otimes_T (T \otimes_R F)
\end{array}$$

Thus θ is an epimorphism. By [23, Exercise 3.15], $\mathrm{id}_R N < \infty$. By Proposition 4.2, N is a strong w-module. Thus it follows from Proposition 2.7 that $\mathrm{Tor}_1^R(N, M)$ is a GV-torsion R-module. Now it follows from property (B_ϕ) that

$\operatorname{Tor}_1^T(N, T \otimes_R M)$ is a GV-torsion T-module. Note that A is also a strongly w-pure flat R-module by Proposition 2.3. By "dimension shifting", we have $\operatorname{Tor}_k^T(N, T \otimes_R M)$ is a GV-torsion T-module for any $k > 1$. Therefore $T \otimes_R M$ is a strongly w-copure flat T-module. □

Let R be a ring, M be an R-module, and $a \in R$. Then we say that M is *a-torsion-free* if $am = 0$ for $m \in M$ implies that $m = 0$.

Corollary 4.5. *Let M be a strongly w-copure flat R-module. Then*

(1) *$M[X]$ is a strongly w-copure flat $R[X]$-module.*
(2) *If S is a multiplicative subset of R, then M_S is a strongly w-copure flat R_S-module.*
(3) *Let $a \in R$, which is not a zero-divisor of both R and M. If R has property (B_a), then M/aM is a strongly w-copure flat R/aR-module.*

Proof. (1) and (2) follow by setting $T := R[X]$ and $T := R_S$ respectively in Theorem 4.4.

(3) Since M is a-torsion-free, we get that $\operatorname{Tor}_1^R(R/aR, M) = 0$ by [28, Lemma 1]. Now the assertion follows by applying Theorem 4.4. □

Set $N_w := \{f \in R[X] \mid c(f)_w = R\}$, where $c(f)$ denotes the content of f. Let $R\{X\} := R[X]_{N_w}$ and $M\{X\} := M[X]_{N_w}$ be the w-Nagata ring of R and the w-Nagata module of M respectively.

Proposition 4.6. *Let M be a strongly w-copure flat R-module. Then*

(1) *$M\{X\}$ is a strongly copure flat $R\{X\}$-module.*
(2) *$M_{\mathfrak{m}}$ is a strongly copure flat $R_{\mathfrak{m}}$-module for any maximal w-ideal $\mathfrak{m}$ of R.*

Proof. (1) Let M be a strongly w-copure flat R-module. By Corollary 4.5(1), $M[X]$ is a strongly w-copure flat $R[X]$-module. By Corollary 4.5(2), $M\{X\}$ is a strongly w-copure flat $R\{X\}$-module. Since $R\{X\}$ is a DW-ring, $M\{X\}$ is a strongly copure flat $R\{X\}$-module.

(2) Let E be an injective $R_{\mathfrak{m}}$-module. Then E is an injective w-module over R. Thus $\operatorname{Tor}_i^R(E, M)$ is a GV-torsion R-module for any $i \geq 1$. Since $\operatorname{Tor}_i^R(E, M)$ is also an $R_{\mathfrak{m}}$-module, it follows from [23, Proposition 6.2.8] that

$$\operatorname{Tor}_i^{R_{\mathfrak{m}}}(E_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong \operatorname{Tor}_i^R(E, M)_{\mathfrak{m}} \cong \operatorname{Tor}_i^R(E, M) = 0.$$

Now the assertion follows. □

Proposition 4.7. *Let S be a multiplicative subset of a ring R. Then w-$\operatorname{cfD}(R_S) \leq w$-$\operatorname{cfD}(R)$.*

Proof. Without loss of generality, we may assume that $m := w$-$\operatorname{cfD}(R) < \infty$. Let M be an R_S-module. Then by Proposition 2.10, there is an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is a strongly w-copure flat R-module. Thus we have an exact sequence

$$0 \to (P_m)_S \to (P_{m-1})_S \to \cdots \to (P_1)_S \to (P_0)_S \to M \to 0,$$

where each $(P_i)_S$ is a strongly w-copure flat R_S-module by Corollary 4.5(2). Therefore, $w\text{-cfD}\,(R_S) \leq m$. $\square$

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