FUZZY AND LEVEL SUBALGEBRAS
OF BCK(BCI)-ALGEBRAS

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The concept of fuzzy sets was introduced by Zadeh in [6]. Das [1] and
Rosenfeld [5] applied it to the fundamental theory of groups. Ougen
[4] applied the concept of fuzzy set to BCIF-algebra, and he got some
results. In this paper, we study fuzzy subalgebras of a BCK(BCI)-
algebra. We also define level subalgebra and study their properties.

DEFINITION 1. A BCI-algebra is a non-empty set $X$ with a binary
operation $\ast$ and a constant $0$ satisfying the axioms:

1. $\{(x \ast y) \ast (x \ast z)\} \ast (z \ast y) = 0,$
2. $\{x \ast (x \ast y)\} \ast y = 0,$
3. $x \ast x = 0,$
4. $x \ast y = 0$ and $y \ast x = 0$ imply that $x = y,$
5. $x \ast 0 = 0$ implies that $x = 0,$

for all $x, y, z \in X$. If (5) is replaced by $0 \ast x = 0$, then the algebra is
called a BCK-algebra.

A non-empty subset $A$ of a BCK(BCI)-algebra $X$ is called a subal-
gebra of $X$ if $x, y \in A$ implies $x \ast y \in A$.

DEFINITION 2. ([1]). Let $X$ be a set. A fuzzy set in $X$ is a function
$\mu : X \rightarrow [0, 1]$.

DEFINITION 3. ([4], [5]). Let $X$ be a BCK(BCI)-algebra. A fuzzy
set $\mu$ in $X$ is called a fuzzy subalgebra of $X$ if, for all $x, y \in X$,

$$\mu(x \ast y) \geq \min(\mu(x), \mu(y)).$$

LEMMA 4. ([4]). Let $X$ be a BCK(BCI)-algebra. If $\mu : X \rightarrow [0, 1]$ is
a fuzzy subalgebra of $X$, then $\mu(x) \leq \mu(0)$ for all $x \in X.$

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THEOREM 5. Let $\mu$ be a fuzzy set in a $\text{BCK(BCI)}$-algebra $X$. If $\mu$ is a fuzzy subalgebra of $X$, then the set

$$A := \{x \in X : \mu(x) = \mu(0)\}$$

is a subalgebra of $X$.

Proof. Let $x, y \in A$. Then $\mu(x) = \mu(0) = \mu(y)$. Since $\mu$ is a fuzzy subalgebra, it follows that

$$\mu(x \ast y) \geq \min(\mu(x), \mu(y)) = \mu(0).$$

On the other hand, by Lemma 4, we have $\mu(x \ast y) \leq \mu(0)$. Hence $\mu(x \ast y) = \mu(0)$, and $x \ast y \in A$.

THEOREM 6. The intersection of any set of fuzzy subalgebras of a $\text{BCK(BCI)}$-algebra is a fuzzy subalgebra.

Proof. Let $\{\mu_i\}$ be a family of fuzzy subalgebras of a $\text{BCK(BCI)}$-algebra $X$. Then, for any $x, y \in X$,

$$(\cap \mu_i)(x \ast y) = \inf(\mu_i(x \ast y))$$

$$\geq \inf(\min(\mu_i(x), \mu_i(y)))$$

$$= \min(\inf \mu_i(x), \inf \mu_i(y))$$

$$= \min((\cap \mu_i)(x), (\cap \mu_i)(y)),$$

which completes the proof.

REMARK 7. The union of any set of fuzzy subalgebras need not be a fuzzy subalgebra. Indeed, if $\mu$ and $\nu$ are fuzzy subalgebras of $\text{BCK(BCI)}$-algebra $X$ such that $\mu(x) > \mu(y) > \nu(x) > \nu(y)$ for all $x, y \in X$, then the inequality

$$(\mu \cup \nu)(x \ast y) \geq \min((\mu \cup \nu)(x), (\mu \cup \nu)(y))$$

does not hold.
THEOREM 8. Let $X$ be a BCK(BCI)-algebra, $A$ a subset of $X$, and let $\mu$ be a fuzzy set in $X$ such that $\mu$ is into $\{0,1\}$, so that $\mu$ is the characteristic function of $A$. Then $\mu$ is a fuzzy subalgebra of $X$ if and only if $A$ is a subalgebra of $X$.

Proof. Assume that $\mu$ is a fuzzy subalgebra of $X$. Since $\mu$ is the characteristic function of $A$, therefore $\mu(x) = 1 = \mu(y)$ for all $x, y \in A$. Then

$$\mu(x \ast y) \geq \min(\mu(x), \mu(y)) = 1,$$

and hence $\mu(x \ast y) = 1$. This implies that $x \ast y \in A$, and that $A$ is a subalgebra of $X$. Conversely, suppose that $A$ is a subalgebra of $X$. Let $x, y \in X$. If $x, y \in A$, then $\mu(x) = 1 = \mu(y)$ and $x \ast y \in A$. Thus $\mu(x \ast y) = 1 = \min(\mu(x), \mu(y))$. If $x, y \in X - A$, then $\mu(x) = 0 = \mu(y)$. Thus $\mu(x \ast y) \geq 0 = \min(\mu(x), \mu(y))$. If $x \in A$ and $y \in X - A$, then $\mu(x) = 1$ and $\mu(y) = 0$. Hence $\mu(x \ast y) \geq 0 = \min(\mu(x), \mu(y))$. Similarly, for $x \in X - A$ and $y \in A$, we have $\mu(x \ast y) \geq \min(\mu(x), \mu(y))$. This completes the proof.

DEFINITION 9. Let $X$ and $X'$ be BCK(BCI)-algebras. A mapping $f : X \to X'$ is called a homomorphism if, for any $x, y \in X$,

$$f(x \ast y) = f(x) \ast f(y).$$

DEFINITION 10. ([5]). Let $f$ be a mapping defined on a set $X$. If $\mu$ is a fuzzy set in $X$, then the fuzzy set $\nu$ in $f(X)$ defined by

$$\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x)$$

for all $y \in f(X)$ is called the image of $\mu$ under $f$. Similarly, if $\nu$ is a fuzzy set in $f(X)$, then the fuzzy set $\mu = \nu \circ f$ in $X$ (i.e., the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in X'$) is called the preimage of $\nu$ under $f$.

THEOREM 11. An onto homomorphic preimage of a fuzzy subalgebra is a fuzzy subalgebra.

Proof. Let $f : X \to X'$ be an onto homomorphism of BCK(BCI)-algebras, $\nu$ a fuzzy subalgebra of $X'$, and $\mu$ the preimage of $\nu$ under $f$. 

Then
\[
\mu(x \ast y) = \nu(f(x \ast y)) \\
= \nu(f(x) \ast f(y)) \\
\geq \min(\nu(f(x)), \nu(f(y))) \\
= \min(\mu(x), \mu(y))
\]
for all \(x, y \in X\). Hence \(\mu\) is a fuzzy subalgebra of \(X\).

**DEFINITION 12. ([5]).** A fuzzy set \(\mu\) in \(X\) has sup property if, for any subset \(T \subseteq X\), there exists \(t_0 \in T\) such that
\[
\mu(t_0) = \sup_{t \in T} \mu(t).
\]

**THEOREM 13.** An onto homomorphic image of a fuzzy subalgebra with sup property is a fuzzy subalgebra.

**Proof.** Let \(f : X \rightarrow X'\) be an onto homomorphism of BCK(BCI)-algebras, \(\mu\) a fuzzy subalgebra of \(X\) with sup property, and \(\nu\) the image of \(\mu\) under \(f\). Given \(x', y' \in X'\), let \(x_0 \in f^{-1}(x')\), \(y_0 \in f^{-1}(y')\) be such that
\[
\mu(x_0) = \sup_{t \in f^{-1}(x')} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}(y')} \mu(t),
\]
respectively. Then
\[
\nu(x' \ast y') = \sup_{z \in f^{-1}(x' \ast y')} \mu(z) \\
\geq \min(\mu(x_0), \mu(y_0)) \\
= \min(\sup_{t \in f^{-1}(x')} \mu(t), \sup_{t \in f^{-1}(y')} \mu(t)) \\
= \min(\nu(x'), \nu(y')).
\]
Hence \(\nu\) is a fuzzy subalgebra of \(X'\).

**DEFINITION 14 ([1]).** Let \(\mu\) be a fuzzy set in a set \(X\). For \(t \in [0, 1]\), the set
\[
\mu_t := \{x \in X : \mu(x) \geq t\}
\]
is called a level subset of \(\mu\).

Note that \(\mu_t\) is a subset of \(X\) in the ordinary sense. The terminology "level set" was introduced by Zadeh.
PROPOSITION 15 ([4]). Let $X$ be a $BCK(BCI)$-algebra and $\mu$ a fuzzy subalgebra of $X$. Then the level subset $\mu_t$, for $t \in [0, 1]$, $t \leq \mu(0)$, is a subalgebra of $X$, where "0" in $\mu(0)$ is the constant of $X$.

PROPOSITION 16 ([4]). Let $X$ be a $BCK(BCI)$-algebra and let $\mu$ be a fuzzy set in $X$ such that $\mu_t$ is a subalgebra of $X$ for all $t \in [0, 1]$, $t \leq \mu(0)$. Then $\mu$ is a fuzzy subalgebra of $X$.

DEFINITION 17. Let $X$ be a $BCK(BCI)$-algebra and let $\mu$ be a fuzzy subalgebra of $X$. The subalgebras $\mu_t$, $t \in [0, 1]$ and $t \leq \mu(0)$, are called level subalgebras of $\mu$.

Note that if $X$ is a finite $BCK(BCI)$-algebra, then the number of subalgebras of $X$ is finite whereas the number of level subalgebras of a fuzzy subalgebra $\mu$ appears to be infinite. But, since every level subalgebra is indeed a subalgebra of $X$, not all these level subalgebras are distinct. The next theorem characterises this aspect.

THEOREM 18. Let $\mu$ be a fuzzy subalgebra of a $BCK(BCI)$-algebra $X$. Two level subalgebras $\mu_{t_1}, \mu_{t_2}$ (with $t_1 < t_2$) of $\mu$ are equal if and only if there is no $x \in X$ such that $t_1 < \mu(x) < t_2$.

Proof. Assume that $\mu_{t_1} = \mu_{t_2}$ for $t_1 < t_2$ and that there exists $x \in X$ such that $t_1 < \mu(x) < t_2$. Then $\mu_{t_2} \subseteq \mu_{t_1}$ and $\mu_{t_1} \neq \mu_{t_2}$, which contradicts the hypothesis. Conversely suppose that there is no $x \in X$ such that $t_1 < \mu(x) < t_2$. Since $t_1 < t_2$, we have $\mu_{t_2} \subseteq \mu_{t_1}$. Let $x \in \mu_{t_1}$. Then $\mu(x) \geq t_1$, and hence $\mu(x) \geq t_2$, because $\mu(x)$ does not lie between $t_1$ and $t_2$. Hence $x \in \mu_{t_2}$, which implies that $\mu_{t_1} \subseteq \mu_{t_2}$. This completes the proof.

References

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