FUZZY SEMI–INNER–PRODUCT SPACE

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In this paper, we defined fuzzy semi-inner-product space and investigated some properties of fuzzy semi product space.

1. Preliminaries

Definition 1.1 [1]. A fuzzy real number $\zeta$ is a nonascending, left continuous function from $\mathbb{R}$ into $I = [0, 1]$ with $\zeta(-\infty^+) = 1$ and $\zeta(+\infty^-) = 0$. The set of all fuzzy real numbers will be denoted by $R(I)$. The partial ordering $\geq$ on $R(I)$ is the natural ordering of real functions. The set of all reals $R$ is canonically embedded in $R(I)$ in the following fashion, for every $r \in R$, we associated the fuzzy real number $\bar{r} \in R(I)$ which is defined by

$$\bar{r} = \begin{cases} 1 & \text{if } t \leq r \\ 0 & \text{if } t > r \end{cases}$$

The set $R^*(I)$ of all nonnegative real numbers is defined by

$$R^*(I) = \{ \zeta \in R(I) : \zeta \geq 0 \}.$$
Definition 1.2 [5]. Let $\zeta$, $\xi$ be two fuzzy real numbers in $R(I)$, and let $s$ be any real number. Then

(i) Addition of fuzzy real numbers $\oplus$ is defined on $R(I)$ by

$$(\xi \oplus \zeta)(s) = \sup\{\xi(t) \land \zeta(s - t) : t \in R\}.$$ 

(ii) Scalar multiplication by a nonnegative $r \in R$ is defined on $R(I)$ by

$$(r\xi)(s) = \begin{cases} 0 & \text{if } r = 0 \\ \xi(\frac{s}{r}) & \text{if } r > 0 \end{cases}$$

It is well known that the above two operations are well defined on $R(I)$ and the canonical embedding of $R$ in $R(I)$ preserves these two operations. The results in the following proposition are well known.

Proposition 1.3 [4].

(i) Addition and scalar multiplication preserve the order $\geq$ on $R(I)$.

(ii) $R(I)$ is closed under these two operations.

(iii) For $\eta$, $\zeta$ and $\xi \in R(I)$, we have

$$\eta \oplus \xi \geq \zeta \oplus \xi \text{iff } \eta \geq \zeta.$$ 

Definition 1.4 [6]. Multiplication of two nonnegative fuzzy real numbers $\eta$, $\zeta \in R^*(I)$ is defined by

$$(\eta\zeta)(s) = \begin{cases} 1 & \text{if } s \leq 0 \\ \sup\{\eta(b) \land \zeta(\frac{s}{b}) : b > 0\} & \text{if } s > 0, \end{cases}$$

where $s \in R$. It is shown in [6] that $R^*(I)$ is closed under multiplication.
Definition 1.5 [2]. A fuzzy pseudo-norm on a real of complex space $X$ is a function $\| \| : X \to R^*(I)$ which satisfies the following two conditions; for $x, y \in X$ and $s$ in the field

(i) $\|sx\| = |s|\|x\|,$
(ii) $\|x + y\| \leq \|x\| \oplus \|y\|.$

The algebraic properties of addition and nonnegative scalar multiplication on $R^*(I)$ enable us to embed $R^*(I)$ in the smallest real vector space $M(I)$ as follow.

Definition 1.6 [3]. The set $M(I)$ is the cartesian product $R^*(I) \times R^*(I)$ modulo the equivalence relation $\sim$ defined by

$$(\eta, \zeta) \sim (\xi, \lambda) \iff \eta \oplus \lambda = \zeta \oplus \xi.$$ 

The partial order $\geq$ on $M(I)$ is defined by

$$(\eta, \zeta) \geq (\xi, \lambda) \iff \eta \oplus \lambda \geq \zeta \oplus \xi.$$ 

The set $M^*(I)$ is defined by

$$M^*(I) = \{(\eta, \zeta) \in M(I) : (\eta, \zeta) \geq 0\} = \{(\eta, \zeta) \in M(I) : \eta \geq \zeta\}$$ 

$R^*(I)$ is canonically embedded in $M(I)$ by representing each $\eta \in R^*(I)$ as $(\eta, 0) \in M(I)$. Also $R$ is embedded in $M(I)$ as follows: for $r \in R$, $r$ is identified with $(\bar{r}, 0) \in M(I)$ if $r \geq 0$ and with $(0, (-r)) \in M(I)$ if $r < 0$.

Addition $\oplus$ and real scalar multiplication are defined on $M(I)$ by:

\begin{align*}
(\eta, \zeta) \oplus (\xi, \lambda) &= (\eta \oplus \xi, \zeta \oplus \lambda), \quad (i) \\
t(\eta, \zeta) &= \begin{cases} 
(\eta, t\zeta) & \text{if } t \geq 0 \\
(|t|\zeta, |t|\eta) & \text{if } t < 0.
\end{cases} \quad (ii)
\end{align*}
Theorem 1.7 [3]. The above addition and scalar multiplication are well defined on \( M(I) \). Under these two operations, \( M(I) \) is the smallest real vector space including \( R^*(I) \). In particular, the canonical embedding of \( R^*(I) \) into \( M(I) \) preserves addition and nonnegative scalar multiplication, while the canonical embedding of \( R \) into \( M(I) \) is a vector space embedding.

Definition 1.8 [3]. The \( N \)-Euclidean norm on \( M(I) \) is the fuzzy pseudo-norm \( \| (\eta, \zeta) \| \) defined by: for \( (\eta, \zeta) \in M(I) \),

\[
\| (\eta, \zeta) \| = \inf\{ \xi \in R^*(I) : \xi \geq (\zeta, \eta) \text{ and } \xi \geq (\eta, \zeta) \}
\]

\[
= \inf\{ \xi \in R^*(I) : \xi \oplus \zeta \geq \eta \text{ and } \xi \oplus \eta \geq \zeta \},
\]

where \( \xi = (\xi, \bar{0}) \) according to the embedding of \( R^*(I) \) in \( M(I) \).

Definition 1.9 [4]. A real algebra \( X \) with a fuzzy pseudo-norm \( \| \cdot \| \) on \( X \) will be called a fuzzy pseudo-norm algebra if for all \( x, y \in X \),

\[
\| xy \| \leq \| x \| \| y \|,
\]

where multiplication in the right-hand side is the fuzzy multiplication on \( R^*(I) \).

Definition 1.10 [4]. Multiplication on \( M(I) \) is defined by:

for \( (\eta, \zeta), (\xi, \lambda) \in M(I) \),

\[
(\eta, \zeta)(\xi, \lambda) = (\eta \xi \oplus \zeta \lambda, \eta \lambda \oplus \zeta \xi).
\]

Theorem 1.11 [4].

(i) Multiplication on \( M(I) \) is well defined.

(ii) The canonical embeddings of \( R^*(I) \) and \( R \) into \( M(I) \) preserve multiplication.

(iii) Under addition, scalar multiplication, \( M(I) \) is a real associative and commutative algebra with unit element \( \bar{1} = (\bar{1}, \bar{0}) \).
(iv) $M(I)$ is not an integral domain.

(v) $(M(I), || \cdot ||)$ is a fuzzy pseudo-normed vector space and is a fuzzy pseudo-normed algebra under its multiplication.

**Definition 1.12.** Let $U$ be a fuzzy subset of a universe $X$ and let $\alpha \in I_1 = [0, 1)$. The $\alpha$-cut of $U$ is the crisp subset of $X$

$$U^{(\alpha)} = \{x \in X : U(x) > \alpha\}.$$

Fuzzy real numbers in $R^*(I)$ can be considered as fuzzy subsets of the set $R^*$ of all nonnegative reals. Therefore, for each $\eta \in R(I)$, its $\alpha$-cut $\eta^{(\alpha)} = [0, t)$ or $= [0, t]$, where $t = \vee \{x \in R : \eta(x) > \alpha\}$ is uniquely identified with the number $t$. It is obvious that $\alpha$-cuts preserve the three operations on $R^*(I)$ and order on $R^*(I)$ in the following sense: for every $\eta, \zeta \in R^*(I), \alpha \in I_1$, and $r \geq 0$ we have

(i) $$(\eta \oplus \zeta)^{(\alpha)} = \eta^{(\alpha)} + \zeta^{(\alpha)}$$

(ii) $$(r\eta)^{(\alpha)} = r\eta^{(\alpha)}$$

(iii) $$(\eta\zeta)^{(\alpha)} = \eta^{(\alpha)}\zeta^{(\alpha)}$$

$$\eta \leq \zeta \text{ iff } \eta^{(\alpha)} \leq \zeta^{(\alpha)}, \forall \alpha \in I_1.$$  

**Proposition 1.13 [4].**

(i) For $(\eta, \zeta) \in R^*(I)$, $\eta^2 < \zeta^2$ iff $\eta < \zeta$.

(ii) For every $\eta \in R^*(I)$, there exists a unique square root $\xi$ in $R^*(I)$ such that $\xi^2 = \eta$.

(iii) For $(\eta, \zeta) \in M(I)$ we have $(\eta, \zeta)^2 \in M^*(I)$. 
(iv) For \((\eta, \zeta) \in M(I)\) we have \(||(\eta, \zeta)^2|| = ||(\eta, \zeta)||^2||.

2. Fuzzy Semi-Inner-Product Space

In this section, we will define the fuzzy semi-inner-product and establish some properties that goes with it. First we introduce the notation of the \(\alpha\)-cuts of the fuzzy real numbers to \(M(I)\).

**Definition 2.1.** Let \((\eta, \zeta) \in M(I)\) and \(\alpha \in I_1\). We define the \(\alpha\)-cut of \((\eta, \zeta)\) to be the real number

\[
(\eta, \zeta)^{(\alpha)} = \eta^{(\alpha)} - \zeta^{(\alpha)}.
\]

**Proposition 2.2 [7].**

(i) The \(\alpha\)-cut \((\eta, \zeta)^{(\alpha)}\) is well-defined on \(M(I)\).

(ii) \((\eta, \zeta) = (\xi, \lambda)\) in \(M(I)\) iff they have the same indexed family of \(\alpha\)-cuts,

(iii) \((\eta, \zeta) \in M^*(I)\) iff \(\forall \alpha \in I_1, (\eta, \zeta)^{(\alpha)} \geq 0\).

**Proposition 2.3 [7].** For each fixed \(\alpha \in I_1\), taking \(\alpha\)-cuts is an order preserving real algebra homomorphism from \(M(I)\) onto \(R\).

**Definition 2.4.** A fuzzy semi-inner-product on a unitary \(M(I)\)-modulo \(X\) is a function \(\cdot : X \times X \to M(I)\) which satisfies the following three axioms;

\((F_1)\) \(\cdot\) is linear in one component only.

\((F_2)\) \(x \cdot x > 0\) for every nonzero \(x \in X\).

\((F_3)\) \(||x \cdot y||^2 \leq (x \cdot x)(y \cdot y)\) for every \(x, y \in X\)

The pair \((X, \cdot)\) is called a fuzzy semi-inner-product space. The fuzzy pseudo-norm \(|| \cdot ||\) associated with \((X, \cdot)\) is the function \(|| \cdot || : X \to R^*(I)\) defined for all \(x \in X\) by \(||x|| = ||x \cdot x||^{\frac{1}{2}}\) with values in \(R^*(I)\). We write \((X, \cdot, || \cdot ||)\) to show that the norm \(|| \cdot ||\) is the function thus derived from the fuzzy semi-inner-product.
Let us now define the real quadratic form \( <,>_{\alpha}: X \times X \to R \) for every \( x,y \in X, \alpha \in I_1 \) fixed, by \( <x,y>_{\alpha} = (x \cdot y)^{(\alpha)} \).

**Lemma 2.5.** If for \( \alpha \in I_1 \) and \( x \in X, (x \cdot x)^{(\alpha)} = 0 \), then \( (x,y)^{(\alpha)} = 0 \) for all \( y \in X \).

**Proof.** By \((F_3)\), we obtain \(| <x,y>_{\alpha} |^2 \leq <x,x>_{\alpha} <y,y>_{\alpha} \). This yields \(((x \cdot y)^{(\alpha)})^2 \leq (x \cdot x)^{(\alpha)}(y \cdot y)^{(\alpha)}\). If \((x \cdot x)^{(\alpha)} = 0\) for some \( \alpha \in I_1 \), then \((x \cdot y)^{(\alpha)} = 0\) for all \( y \in X \).

**Proposition 2.6.** If \((\eta, \zeta) \geq (\xi, \lambda)\) in \( M^*(I) \), then \( ||(\eta \zeta)|| \geq ||(\xi, \lambda)|| \) in \( R^*(I) \).

**Proof.** Since \((\eta, \zeta) \in M^*(I)\), then \( \theta \geq (\zeta, \eta) \) for all \( \theta \in R^*(I) \). From the properties of the infimum and Definition 1.8, we have

\[
||(\eta \zeta)|| = \inf\{\theta \in R^*(I) : \theta \geq (\eta, \zeta)\} \\
\geq \inf\{\theta \in R^*(I) : \theta \geq (\xi, \lambda)\} \\
= ||(\xi, \lambda)||.
\]

In the following proposition, we will denote the elements of the ring \( M(I) \) by just a single letter.

**Proposition 2.7.** Let \( x,y \in M^*(I) \) be such that \( y^{(\alpha)} = 0 \) whenever \( \alpha \in I_1 \) satisfies \( x^{(\alpha)} = 0 \). Then for all \( z \in M^*(I) \) we have \( xz \geq xy \iff z \geq y \).

**Proof.** \( xz \geq xy \iff (xz)^{(\alpha)} \geq (x,y)^{(\alpha)}, \forall \alpha \in I_1 \). Using the properties of \( \alpha \)-cuts on \( M(I) \), we have \( x^{(\alpha)}, y^{(\alpha)}, z^{(\alpha)} \geq 0 \), and \( z^{(\alpha)} \geq x^{(\alpha)} y^{(\alpha)} \). If \( z^{(\alpha)} = 0 \), then \( z^{(\alpha)} \geq y^{(\alpha)} \). If \( x^{(\alpha)} = 0 \), then \( y^{(\alpha)} = 0 \), and hence \( z^{(\alpha)} \geq y^{(\alpha)} \). Since this holds for all \( \alpha \in I_1 \), then we conclude that \( z \geq y \).

**Theorem 2.8.** Let \( (x, \cdot, || \cdot ||) \) be a fuzzy semi-inner-product space. Then, considering \( X \) as a real vector space, \( || \cdot || \) is indeed a fuzzy pseudo-norm on \( X \). It also satisfies:
(i) \( \|x\| > 0 \) for every nonzero \( x \in X \), and
(ii) \( \|(\eta, \zeta)x\| \leq \|((\eta, \zeta))\| \|x\| \) for all \( x \in X \) and \( (\eta, \zeta) \in M(I) \).

**Proof.** \( \|tx\|^2 = \|tx \cdot tx\| = t \|x \cdot tx\| \leq t \|x\| \|tx\| \). Thus \( \|tx\| \leq |t| \|x\| \). For \( \lambda \neq 0 \),
\[ \|x\| = \|\frac{1}{\lambda}tx\| \leq \frac{1}{|\lambda|} \|tx\|, \quad |t| \|x\| \leq \|tx\|. \]
Therefore \( \|tx\| = |t| \|x\| \), \( \forall x \in X, t \in R \). Also,
\[ \|x + y\|^2 = \|((x + y) \cdot (x + y))\| \\
= \|x \cdot (x + y)\| + \|y \cdot (x + y)\| \\
\leq \|x\| \|x + y\| + \|y\| \|x + y\| \\
= (\|x\| + \|y\|) \|x + y\|. \]

Then \( \|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X \)

Hence \( \| \| \) is a fuzzy pseudo-norm on \( X \).

To prove (i), let \( x \in X \) be a nonzero element in \( X \). Then \( x \cdot x > 0 \) in \( M(I) \) and hence \( \|x\| = \|x \cdot x\|^\frac{1}{2} > 0 \) in \( R^*(I) \).

To prove (ii), let \( (\eta, \zeta) \in M(I) \), then
\[ \|(\eta, \zeta)x\|^2 = \|((\eta, \zeta)x \cdot (\eta, \zeta)x)\| \\
= \|((\eta, \zeta)^2(x, x))\|. \]

But, since \( (M(I), \| \|) \) is a fuzzy pseudo-normed algebra(Theorem 1.11(v)),
then
\[ \|((\eta, \zeta)^2(x \cdot x))\| \leq \|((\eta, \zeta)^2\|) \|((x \cdot x))\| \\
= \|((\eta, \zeta))^2\| \|((x \cdot x))\| \\
= \|((\eta, \zeta))^2\| \|x\|^2, \]
where the first equality is true by Proposition 1.13 (iv).
Due to the fact that the square roots exist and are unique in $R^*(I)$, we obtain by Proposition 1.13 (i),

$$||((\eta, \zeta)x)|| \leq ||(\eta, \zeta)|| \cdot |x|.$$ 

**Proposition 2.9.** Let $\{(X_i, \cdot, || \cdot ||) : i = 1, 2, \cdots, n\}$ be a finite indexed family of fuzzy semi-inner-product spaces, and let $X = \prod X_i = \{(x_1, x_2, \cdots, x_n) : x_1 \in X_1, x_2 \in X_2, \cdots, x_n \in X_n\}$ be the product module of the $X_i$'s(under coordinate-wise operations). Define the function $\cdot : X \times X \to M(I)$ by for $x = (x_i)$ and $y = (y_i)$ in $X$,

$$x \cdot y = \bigoplus_{i=1}^{n} x_i \cdot y_i.$$ 

Then this function $\cdot$ is a fuzzy semi-inner-product on $X$.

**Proof.** The proof is straightforward from the properties of each fuzzy semi-inner-product $\cdot_i$ and the properties of fuzzy summation.

**References**


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