REMARKS ON DENJOY-DUNFORD AND DENJOY-PETTIS INTEGRALS

CHUN-KEE PARK

ABSTRACT. In this paper we generalize some results of R. A. Gordon ([4]) and J. L. Gamez and J. Mendoza ([3]) and prove some convergence theorems for Denjoy-Dunford and Denjoy-Pettis integrable functions.

1. Introduction

In 1989 Gordon ([4]) introduced the concepts of Denjoy-Dunford and Denjoy-Pettis integrals for Banach-valued functions and proved some properties of those integrals. Gamez and Mendoza improved some results of Gordon. Gordon ([5]) also obtained some convergence theorems for Denjoy integrable real-valued functions. In this paper we generalize some results of Gordon ([4]) and Gamez and Mendoza ([3]) and obtain some convergence theorems for Denjoy-Dunford and Denjoy-Pettis integrable functions.

2. Preliminaries

Throughout this paper $X$ will denote a real Banach space and $X^*$ its dual.

DEFINITION 2.1 ([4]). Let $F : [a, b] \to X$ and let $E$ be a subset of $[a, b]$.

(a) The function $F$ is BV on $E$ if $\sup \left\{ \sum_{i} \|F(d_i) - F(c_i)\| \right\}$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i]\}$ of
nonoverlapping intervals that have endpoints in $E$.

(b) The function $F$ is AC on $E$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in $E$ and satisfy $\sum_i (d_i - c_i) < \delta$.

(c) The function $F$ is BVG on $E$ if $E$ can be expressed as a countable union of sets on each of which $F$ is BV.

(d) The function $F$ is ACG on $E$ if $F$ is continuous on $E$ and if $E$ can be expressed as a countable union of sets on each of which $F$ is AC.

**Definition 2.2 ([4]).** Let $\{F_\alpha\}$ be a family of functions from $[a, b]$ to $X$ and let $E$ be a subset of $[a, b]$. The family $\{F_\alpha\}$ is uniformly BVG (ACG) on $E$ if each $F_\alpha$ is BVG (ACG) on $E$ and if each perfect set in $E$ contains a portion on which every $F_\alpha$ is BV (AC).

**Definition 2.3 ([4]).** Let $F : [a, b] \to X$ and let $t \in (a, b)$. A vector $z$ in $X$ is the approximate derivative of $F$ at $t$ if there exists a measurable set $E \subset [a, b]$ that has $t$ as a point of density such that $\lim_{s \to t, s \in E} \frac{F(s) - F(t)}{s - t} = z$. We will write $F_{ap}'(t) = z$.

A function $f : [a, b] \to \mathbb{R}$ is Denjoy integrable on $[a, b]$ if there exists an ACG function $F : [a, b] \to \mathbb{R}$ such that $F_{ap}' = f$ almost everywhere on $[a, b]$. The function $f$ is Denjoy integrable on the set $E \subset [a, b]$ if $f \chi_E$ is Denjoy integrable on $[a, b]$.

**Definition 2.4 ([4]).** (a) A function $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$ if for each $x^* \in X^*$ the function $x^* f$ is Denjoy integrable on $[a, b]$ and if for every interval $I$ in $[a, b]$ there exists a vector $x^*_I$ in $X^{**}$ such that $x^*_I(z) = \int_I x^* f$ for all $x^* \in X^*$.

(b) A function $f : [a, b] \to X$ is Denjoy-Pettis integrable on $[a, b]$ if $f$ is Denjoy-Dunford integrable on $[a, b]$ and if $x^*_I \in X$ for every interval $I$ in $[a, b]$.

Throughout this paper $(DD) \int_a^b f$ and $(DP) \int_a^b f$ will denote the Denjoy-Dunford integral and the Denjoy-Pettis integral of $f$ on $[a, b]$, respectively.
3. Denjoy-Dunford and Denjoy-Pettis Integrability

In this section we obtain some properties of Denjoy-Dunford and Denjoy-Pettis integrable functions.

**Theorem 3.1.** (a) If $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$, then $f$ is weakly measurable.

(b) If $f : [a, b] \to X$ is bounded and Denjoy-Dunford integrable on $[a, b]$, then $f$ is Dunford integrable on $[a, b]$.

**Proof.** (a) If $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$, then $x^* f : [a, b] \to \mathbb{R}$ is Denjoy integrable on $[a, b]$ for all $x^* \in X^*$. Hence $x^* f$ is measurable for all $x^* \in X^*$ ([4, Theorem 12 (a)]). Therefore $f$ is weakly measurable.

(b) If $f : [a, b] \to X$ is bounded and Denjoy-Dunford integrable on $[a, b]$, then $x^* f : [a, b] \to \mathbb{R}$ is bounded and Denjoy integrable on $[a, b]$ for all $x^* \in X^*$. Hence $x^* f$ is Lebesgue integrable on $[a, b]$ for all $x^* \in X^*$ ([5, Theorem 15.9]). Therefore $f$ is Dunford integrable on $[a, b]$. \qed

It follows immediately from Pettis Measurability Theorem and Theorem 3.1 that if $X$ is a separable Banach space and $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$ then $f$ is measurable.

**Theorem 3.2** ([3]). A function $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$ if and only if $x^* f$ is Denjoy integrable on $[a, b]$ for all $x^* \in X^*$.

**Theorem 3.3.** Suppose that $f : [a, b] \to X$ is Denjoy-Dunford integrable on each interval $[c, d] \subset (a, b)$. If $\lim_{c \to a^+} \lim_{d \to b^-} (DD) \int_c^d f$ exists in norm in $X^{**}$, then $f$ is Denjoy-Dunford integrable on $[a, b]$ and $(DD) \int_a^b f = \lim_{c \to a^+} \lim_{d \to b^-} (DD) \int_c^d f$.

**Proof.** Let $\lim_{c \to a^+} \lim_{d \to b^-} (DD) \int_c^d f = x_0^{**}$, where $x_0^{**} \in X^{**}$. By hypothesis, for each $x^* \in X^*$, $x^* f : [a, b] \to \mathbb{R}$ is Denjoy integrable on each
interval \([c, d] \subset (a, b)\) and

\[
(x^*, x_0^{**}) = \lim_{\substack{c \to a^- \\
 d \to b^+}} \left( x^*, \int_{c}^{d} f \right) = \lim_{\substack{c \to a^- \\
 d \to b^+}} \int_{c}^{d} x^* f.
\]

Hence for each \(x^* \in X^*\), \(x^* f\) is Denjoy integrable on \([a, b]\) and

\[
\int_{a}^{b} x^* f = \lim_{\substack{c \to a^+ \\
 d \to b^-}} \int_{c}^{d} x^* f \quad (\text{[5, Theorem 15.12]})
\]

Thus \(f\) is Denjoy-Dunford integrable on \([a, b]\) by Theorem 3.2 and

\[
(x^*, x_0^{**}) = \lim_{\substack{c \to a^+ \\
 d \to b^-}} \int_{c}^{d} x^* f = \int_{a}^{b} x^* f = \left( x^*, \int_{a}^{b} f \right)
\]

for all \(x^* \in X^*\). Hence \((DD) \int_{a}^{b} f = x_0^{**} = \lim_{\substack{c \to a^+ \\
 d \to b^-}} (DD) \int_{c}^{d} f\). \(\square\)

**Definition 3.4.** Let \(\{f_\alpha\}\) be a family of Denjoy-Dunford integrable functions from \([a, b]\) to \(X\). The family \(\{f_\alpha\}\) is uniformly Denjoy-Dunford integrable on \([a, b]\) if for each perfect set \(E \subset [a, b]\) there exists an interval \([c, d] \subset [a, b]\) with \(c, d \in E\) and \(E \cap (c, d) \neq \emptyset\) such that every \(f_\alpha\) is Dunford integrable on \(E \cap [c, d]\) and for every \(\alpha\) the series

\[
\sum_{n} \left\| (DD) \int_{c_n}^{d_n} f_\alpha \right\|
\]

converges where \([c, d] - E = \cup_{n} (c_n, d_n)\).

**Theorem 3.5.** Let \(\{f_\alpha\}\) be a family of Denjoy-Dunford integrable functions from \([a, b]\) to \(X\) and let \(F_\alpha(t) = (DD) \int_{a}^{t} f_\alpha\) for each \(\alpha\). If the family \(\{F_\alpha\}\) is uniformly ACG on \([a, b]\), then the family \(\{f_\alpha\}\) is uniformly Denjoy-Dunford integrable on \([a, b]\).

**Proof.** Suppose that the family \(\{F_\alpha\}\) is uniformly ACG on \([a, b]\) and let \(E\) be a perfect set in \([a, b]\). Then there exists an interval \([c, d] \subset [a, b]\) with \(c, d \in E\) and \(E \cap (c, d) \neq \emptyset\) such that every \(F_\alpha\) is AC on
for each $x^* \in X^*$ the function $F_\alpha x^*$ is also AC on $E \cap [c, d]$ and $(x^*, F_\alpha(t)) = \int_a^t x^* f_\alpha, t \in [a, b]$. For each $x^* \in X^*$ let $G_{\alpha, x^*} : [c, d] \to \mathbb{R}$ be the function that equals $F_\alpha x^*$ on $E$ and is linear on the intervals contiguous to $E$. Then the function $G_{\alpha, x^*}$ is AC on $[c, d]$ for each $x^* \in X^*$ ([4, Theorem 3]). Hence $G'_{\alpha, x^*}$ exists almost everywhere on $[c, d]$ and is Lebesgue integrable on $[c, d]$ for each $x^* \in X^*$. Since $G'_{\alpha, x^*} = (F_\alpha x^*)' = x^* f_\alpha$ almost everywhere on $E \cap [c, d]$ for each $x^* \in X^*$, $x^* f_\alpha$ is Lebesgue integrable on $E \cap [c, d]$ for each $x^* \in X^*$. Thus $f_\alpha$ is Dunford integrable on $E \cap [c, d]$. Since $F_\alpha$ is BV on $E \cap [c, d]$, the series $\sum_n ||F_\alpha(d_n) - F_\alpha(c_n)|| = \sum DD \int_{c_n}^{d_n} f_\alpha$ converges where $[c, d] - E = \cup_n (c_n, d_n)$. Since this is valid for each $\alpha$, the family $\{f_\alpha\}$ is uniformly Denjoy-Dunford integrable on $[a, b]$.

**Theorem 3.6 ([5]).** Let $E$ be a bounded, closed subset of $\mathbb{R}$ with bounds $a$ and $b$ and let $((a_n, b_n))$ be the sequence of intervals contiguous to $E$ in $[a, b]$. Suppose that $f : [a, b] \to \mathbb{R}$ is Denjoy integrable on $E$ and on each interval $[a_k, b_k]$. If $\lim_{k \to \infty} \omega \left( \int_{a_k}^{b_k} f, [a_k, b_k] \right) = 0$ and the series $\sum_{k=1}^{\infty} \left| \int_{a_k}^{b_k} f \right|$ converges, then $f$ is Denjoy integrable on $[a, b]$ and

$$\int_a^b f = \int_a^b f \chi_E + \sum_{k=1}^{\infty} \int_{a_k}^{b_k} f.$$  

**Theorem 3.7.** Let $E$ be a bounded, closed subset of $\mathbb{R}$ with bounds $a$ and $b$ and let $((a_k, b_k))$ be the sequence of intervals contiguous to $E$ in $[a, b]$. Suppose that $f : [a, b] \to X$ is Denjoy-Dunford integrable on $E$ and on each interval $[a_k, b_k]$. If $\lim_{k \to \infty} \omega \left( (DD) \int_{a_k}^{b_k} f, [a_k, b_k] \right) = 0$ and the series $\sum_{k=1}^{\infty} \left| (DD) \int_{a_k}^{b_k} f \right|$ converges, then $f$ is Denjoy-Dunford integrable on $[a, b]$ and

$$(DD) \int_a^b f = (DD) \int_a^b f \chi_E + \sum_{k=1}^{\infty} (DD) \int_{a_k}^{b_k} f.$$
Proof. For each $x^* \in X^*$, $x^* f$ satisfies the hypothesis of Theorem 3.6. Hence by Theorem 3.6, for each $x^* \in X^*$, $x^* f$ is Denjoy integrable on $[a, b]$ and

$$\int_a^b x^* f = \int_a^b x^* f_{XE} + \sum_{k=1}^{\infty} \int_{a_k}^{b_k} x^* f.$$ 

By Theorem 3.2, $f$ is Denjoy-Dunford integrable on $[a, b]$ and

$$\left< x^*, (DD) \int_{a}^{b} f \right> = \left< x^*, (DD) \int_{a}^{b} f_{XE} \right> + \sum_{k=1}^{\infty} \left< x^*, (DD) \int_{a_k}^{b_k} f \right>$$

for each $x^* \in X^*$. Since $\sum_{k=1}^{\infty} \left\| (DD) \int_{a_k}^{b_k} f \right\|$ converges, we have

$$\sum_{k=1}^{\infty} \left< x^*, (DD) \int_{a_k}^{b_k} f \right> = \left< x^*, \sum_{k=1}^{\infty} (DD) \int_{a_k}^{b_k} f \right>$$

for each $x^* \in X^*$. Hence we have $(DD) \int_{a}^{b} f = (DD) \int_{a}^{b} f_{XE} + \sum_{k=1}^{\infty} (DD) \int_{a_k}^{b_k} f$. \hfill \square

Definition 3.8. Let $\{F_\alpha\}$ be a family of functions from $[a, b]$ to $X$ and let $E$ be a subset of $[a, b]$. The family $\{F_\alpha\}$ is equi AC on $E$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i} \|F_\alpha(d_i) - F_\alpha(c_i)\| < \epsilon$ for all $\alpha$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in $E$ and satisfy $\sum_{i} (d_i - c_i) < \delta$. 
Definition 3.9. Let \( \{ F_\alpha \} \) be a family of functions from \([a, b]\) to \(X\) and let \(E\) be a closed subset of \([a, b]\) with its bounds \(c\) and \(d\). The family \( \{ F_\alpha \} \) is equi BV on \(E\) if each \( F_\alpha \) is BV on \(E\) and for each \(\epsilon > 0\) there exists a positive integer \(N\) such that 
\[
\sum_{n=N}^{\infty} \| F_\alpha(d_n) - F_\alpha(c_n) \| < \epsilon \quad \text{for all } \alpha \text{ where } [c, d] - E = \bigcup_{n=1}^{\infty} (c_n, d_n).
\]

Lemma 3.10. Let \( \{ F_\alpha \} \) be a family of functions from \([a, b]\) to \(X\) and let \(E\) be a closed subset of \([a, b]\) with its bounds \(c\) and \(d\). If \( \{ F_\alpha \} \) is equi AC on \(E\), then \( \{ F_\alpha \} \) is equi BV on \(E\).

Proof. Suppose that \( \{ F_\alpha \} \) is equi AC on \(E\). Then each \( F_\alpha \) is BV on \(E\). Let \(\epsilon > 0\) be given and let \([c, d] - E = \bigcup_{i=1}^{\infty} (c_i, d_i)\). Since \( \{ F_\alpha \} \) is equi AC on \(E\), there exists \(\delta > 0\) such that 
\[
\sum_{i=1}^{\infty} \| F_\alpha(d_i') - F_\alpha(c_i') \| < \epsilon/2 \quad \text{for all } \alpha \text{ whenever } \{[c_i', d_i']\} \text{ is a finite collection of nonoverlapping intervals that have endpoints in } E \text{ and satisfy } \sum_{i=1}^{\infty} (d_i' - c_i') < \delta.
\]

Since \(\sum_{i=1}^{\infty} (d_i - c_i) < \infty\), there exists a positive integer \(N\) such that 
\[
\sum_{i=N}^{\infty} (d_i - c_i) < \delta.
\]
Hence we have
\[
n \geq N \Rightarrow \sum_{i=N}^{n} \| F_\alpha(d_i) - F_\alpha(c_i) \| < \frac{\epsilon}{2}
\]
for all \(\alpha\). Letting \(n \to \infty\), we have
\[
\sum_{i=N}^{\infty} \| F_\alpha(d_i) - F_\alpha(c_i) \| \leq \frac{\epsilon}{2} < \epsilon
\]
for all \(\alpha\). Therefore \( \{ F_\alpha \} \) is equi BV on \(E\). \(\square\)

Definition 3.11. Let \( \{ F_\alpha \} \) be a family of functions from \([a, b]\) to \(X\). The family \( \{ F_\alpha \} \) is equi ACG on a subset \(E\) of \([a, b]\) if each \( F_\alpha \) is ACG on \(E\) and if each perfect set in \(E\) contains a portion on which the family \( \{ F_\alpha \} \) is equi AC.
THEOREM 3.12. Let \( \{f_\alpha\} \) be a family of Denjoy-Dunford integrable functions from \([a, b]\) to \(X\) and let \( F_\alpha(t) = (DD) \int_a^t f_\alpha \) for each \( \alpha \). If the family \( \{F_\alpha\} \) is equi ACG on \([a, b]\), then for each perfect set \( E \subset [a, b] \) there exists a portion \( E \cap (c, d) \) of \( E \) such that every \( f_\alpha \) is Dunford integrable on \( E \cap [c, d] \) and \( \sum_n \left\| (DD) \int_{c_n}^{d_n} f_\alpha \right\| \) converges uniformly on \( \alpha \) where \([c, d] - E = \cup_n (c_n, d_n)\).

PROOF. Suppose that \( \{F_\alpha\} \) is equi ACG on \([a, b]\) and let \( E \subset [a, b] \) be a perfect set. Then \( \{F_\alpha\} \) is uniformly ACG on \([a, b]\). By Theorem 3.5, there exists a portion \( E \cap (c', d') \neq \emptyset \) of \( E \) with \( c', d' \in E \) such that every \( f_\alpha \) is Dunford integrable on \( E \cap [c', d'] \). Since \( \{F_\alpha\} \) is equi ACG on \([a, b]\), for the perfect set \( E \cap [c', d'] \) there exists a portion \( E \cap (c, d) \neq \emptyset \) of \( E \cap [c', d'] \) with \( c, d \in E \) such that \( \{F_\alpha\} \) is equi AC on \( E \cap [c, d] \). Each \( f_\alpha \) is also Dunford integrable on \( E \cap [c, d] \). By Lemma 3.10, \( \{F_\alpha\} \) is equi BV on \( E \cap [c, d] \). Hence for each \( \epsilon > 0 \) there exists a positive integer \( N \) such that \( \sum_{n=N}^{\infty} \|F_\alpha(d_n) - F_\alpha(c_n)\| < \epsilon \) for all \( \alpha \) where \([c, d] - E = \cup_n (c_n, d_n)\).

Therefore \( \sum_n \left\| (DD) \int_{c_n}^{d_n} f_\alpha \right\| \) converges uniformly on \( \alpha \) where \([c, d] - E = \cup_n (c_n, d_n)\). \( \square \)

4. Convergence Theorems

In this section we obtain some results of the convergence of Denjoy-Dunford and Denjoy-Pettis integrable functions.

THEOREM 4.1 ([5]). Let \( (f_n) \) be a sequence of Denjoy integrable functions from \([a, b]\) to \(\mathbb{R}\), and let \( F_n(t) = \int_a^t f_n \) for each \( n \), and suppose that \( (f_n) \) converges pointwise to \( f \) on \([a, b]\). If \( (F_n) \) is equicontinuous and equi ACG on \([a, b]\), then \( f \) is Denjoy integrable on \([a, b]\) and \( \int_a^b f = \lim_{n \to \infty} \int_a^b f_n \).
Theorem 4.2. Let \((f_n)\) be a sequence of Denjoy-Dunford integrable functions from \([a, b]\) to \(X\), and let \(F_n(t) = (DD) \int_a^t f_n\) for each \(n\), and suppose that \((f_n)\) converges pointwise to \(f\) on \([a, b]\). If \((F_n)\) is equicontinuous and equi ACG on \([a, b]\), then \(f\) is Denjoy-Dunford integrable on \([a, b]\) and \((DD) \int_a^b f = \lim_{n \to \infty} (DD) \int_a^b f_n\) in the weak* topology of \(X^{**}\).

Proof. We note that \((x^* f_n)\) and \((x^* F_n)\) satisfy the hypothesis of Theorem 4.1 for every \(x^* \in X^*\). Hence \(x^* f\) is Denjoy integrable on \([a, b]\) and \(\int_a^b x^* f = \lim_{n \to \infty} \int_a^b x^* f_n\) for every \(x^* \in X^*\). By Theorem 3.2, \(f\) is Denjoy-Dunford integrable on \([a, b]\) and \(\left\langle x^*, (DD) \int_a^b f \right\rangle = \lim_{n \to \infty} \left\langle x^*, (DD) \int_a^b f_n \right\rangle\) for every \(x^* \in X^*\). Hence \((DD) \int_a^b f = \lim_{n \to \infty} (DD) \int_a^b f_n\) in the weak* topology of \(X^{**}\). \(\Box\)

Theorem 4.3 ([4]). Let \(X\) be weakly sequentially complete and let \(f : [a, b] \to X\) be Denjoy-Dunford integrable on \([a, b]\). If \(f\) is measurable, then \(f\) is Denjoy-Pettis integrable on \([a, b]\).

Theorem 4.4. Let \(X\) be weakly sequentially complete, and let \((f_n)\) be a sequence of measurable Denjoy-Dunford integrable functions from \([a, b]\) to \(X\), and let \(F_n(t) = (DD) \int_a^t f_n\) for each \(n\), and suppose that \((f_n)\) converges pointwise to \(f\) on \([a, b]\). If \((F_n)\) is equicontinuous and equi ACG on \([a, b]\), then \(f\) is Denjoy-Pettis integrable on \([a, b]\) and \((DP) \int_a^b f = \lim_{n \to \infty} (DP) \int_a^b f_n\) in the weak topology of \(X\).

Proof. By Theorem 4.2, \(f\) is Denjoy-Dunford integrable on \([a, b]\) and \(\left\langle x^*, (DD) \int_a^b f \right\rangle = \lim_{n \to \infty} \left\langle x^*, (DD) \int_a^b f_n \right\rangle\) for every \(x^* \in X^*\). By Theorem 4.3, \(f_n\) is Denjoy-Pettis integrable on \([a, b]\) for each \(n\). Since each \(f_n\) is measurable and \((f_n)\) converges pointwise to \(f\) on \([a, b]\), \(f\) is also
measurable on $[a, b]$. By Theorem 4.3, $f$ is also Denjoy-Pettis integrable on $[a, b]$ and $(DP) \int_a^b f = \lim_{n \to \infty} (DP) \int_a^b f_n$ in the weak topology of $X$. \hfill \square

**Theorem 4.5.** Let $(f_n)$ be a sequence of Denjoy-Dunford integrable functions from $[a, b]$ to a reflexive Banach space $X$, and let $F_n(t) = (DD) \int_a^t f_n$ for each $n$, and suppose that $(f_n)$ converges pointwise to $f$ on $[a, b]$. If $(F_n)$ is equicontinuous and equi ACG on $[a, b]$, then $f$ is Denjoy-Dunford integrable on $[a, b]$ and there is a sequence $(g_n)$ with $g_n \in \text{co}\{f_n | n = 1, 2, 3, \ldots \}$ such that $(DD) \int_a^b f = \lim_{n \to \infty} (DD) \int_a^b g_n$ in norm.

**Proof.** By Theorem 4.2, $f$ is Denjoy-Dunford integrable on $[a, b]$ and $(DD) \int_a^b f = \lim_{n \to \infty} (DD) \int_a^b f_n$ in the weak* topology of $X^{**}$. Since $X$ is reflexive, $(DD) \int_a^b f = \lim_{n \to \infty} (DD) \int_a^b f_n$ weakly in $X^{**}$. Thus

$$\lim_{n \to \infty} \left( (DD) \int_a^b f_n - (DD) \int_a^b f \right) = 0 \text{ weakly in } X^{**}.$$

By Corollary 2[1, p11], there is a sequence $(x_n^{**})$ of convex combinations of the $(DD) \int_a^b f_n - (DD) \int_a^b f$ such that $\lim_{n \to \infty} \|x_n^{**}\| = 0$. For each $n$, let $x_n^{**} = \sum_{i=1}^{k(n)} \alpha_{n_i}$

$$\left( (DD) \int_a^b f_{n_i} - (DD) \int_a^b f \right)$$

where $\alpha_{n_i} \geq 0$ for each $i$ and $\sum_{i=1}^{k(n)} \alpha_{n_i} = 1$. Then

$$\lim_{n \to \infty} \|x_n^{**}\| = \lim_{n \to \infty} \left\| \sum_{i=1}^{k(n)} \alpha_{n_i} \left( (DD) \int_a^b f_{n_i} - (DD) \int_a^b f \right) \right\|$$

$$= \lim_{n \to \infty} \left\| (DD) \int_a^b \left( \sum_{i=1}^{k(n)} \alpha_{n_i} f_{n_i} \right) - (DD) \int_a^b f \right\|$$

$$= 0.$$
For each $n$, let $g_n = \sum_{i=1}^{k(n)} \alpha_n f_{n_i}$. Then for each $n$, $g_n \in \text{co}\{f_n | n = 1, 2, 3, \ldots \}$ and $(DD) \int_a^b f = \lim_{n \to \infty} (DD) \int_a^b g_n$ in norm. \hfill \Box

**Theorem 4.6.** Let $X$ be weakly sequentially complete, and let $(f_n)$ is a sequence of measurable Denjoy-Dunford integrable functions from $[a, b]$ to $X$, and let $F_n(t) = (DD) \int_a^t f_n$ for each $n$, and suppose that $(f_n)$ converges pointwise to $f$ on $[a, b]$. If $(F_n)$ is equicontinuous and equi ACG on $[a, b]$, then $f$ is Denjoy-Pettis integrable on $[a, b]$ and there is a sequence $(g_n)$ with $g_n \in \text{co}\{f_n | n = 1, 2, 3, \ldots \}$ such that $(DP) \int_a^b f = \lim_{n \to \infty} (DP) \int_a^b g_n$ in norm.

**Proof.** By Theorem 4.4, $f$ is Denjoy-Pettis integrable on $[a, b]$ and $(DP) \int_a^b f = \lim_{n \to \infty} (DP) \int_a^b f_n$ weakly in $X$. By Corollary 2 ([1, p. 11]), there is a sequence $(x_n)$ of convex combinations of the $(DP) \int_a^b f_n - (DP) \int_a^b f$ such that $\lim_{n \to \infty} \|x_n\| = 0$. Using the same method in the proof of Theorem 4.5, we obtain a sequence $(g_n)$ with $g_n \in \text{co}\{f_n | n = 1, 2, 3, \ldots \}$ such that $(DP) \int_a^b f = \lim_{n \to \infty} (DP) \int_a^b g_n$ in norm. \hfill \Box

**References**


Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea

*E-mail:* ckpark@cc.kangwon.ac.kr