REPRESENTATION OF OPERATOR SEMI-STABLE DISTRIBUTIONS

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ABSTRACT. For a linear operator $Q$ from $\mathbb{R}^d$ into $\mathbb{R}^d$, $\alpha > 0$ and $0 < b < 1$, the $(Q, b, \alpha)$-semi-stability and the operator semi-stability of probability measures on $\mathbb{R}^d$ are defined. Characterization of $(Q, b, \alpha)$-semi-stable Gaussian distribution is obtained and the relationship between the class of $(Q, b, \alpha)$-semi-stable non-Gaussian distributions and that of operator semi-stable distributions is discussed.

1. Introduction

Let $\mathbb{R}^d$ be the $d$-dimensional Euclidean space. In the paper [2], we studied operator semi-stable processes on $\mathbb{R}^d$, which are Lévy processes associated with operator semi-stable distributions. Under the condition of fullness, descriptions of operator semi-stable distributions on $\mathbb{R}^d$ were obtained by R. Jajte [4,5], W. Krakowiak [6], A. L uczak [7,8], V. Chorny [3] and others. Here fullness means that the support of the distribution is not contained in any $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$.

Let $\text{Aut}(\mathbb{R}^d)$ be the set of invertible linear operators from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Let \{\(Y_n : n = 1, 2, \cdots\)\} be a sequence of i.i.d. (=independent identically distributed) random variables on $\mathbb{R}^d$. In [4], R. Jajte investigated the weak limit of distributions of

\[(1.1) \quad A_n(Y_1 + Y_2 + \cdots + Y_{k_n}) + b_n,\]

where $A_n \in \text{Aut}(\mathbb{R}^d)$, $b_n \in \mathbb{R}^d$ and $\frac{k_{n+1}}{k_n} \rightarrow r$ with some $r \in [1, \infty)$. The limit distribution $\mu$ of (1.1) is called an operator semi-stable distribution. When the convergence of (1.1) holds with $b_n = 0$, we call $\mu$
a strictly operator semi-stable distribution. In this paper, we consider all operator semi-stable distributions on $\mathbb{R}^d$ without the assumption of fullness. Let $\text{End}(\mathbb{R}^d)$ be the set of linear operators from $\mathbb{R}^d$ into $\mathbb{R}^d$. The identity operator is denoted by $I$. For $B \in \text{End}(\mathbb{R}^d)$ and $r > 0$, we define $r^B = \exp\{B \log r\} = \sum (B \log r)^n / n!$. For $T \in \text{End}(\mathbb{R}^d)$, we write $(T \mu)(E) = \mu(T^{-1}(E))$. We denote the $b$-th convolution power of $\mu$ by $\mu^b$. Let $M_+(\mathbb{R}^d)$ be the class of linear operators on $\mathbb{R}^d$ all of whose eigenvalues have positive real parts.

Fix $\alpha > 0$ and $Q \in M_+(\mathbb{R}^d)$. An infinitely divisible distribution $\mu$ on $\mathbb{R}^d$ is called operator semi-stable with exponent $(Q, \alpha)$ if there are a number $b \in (0, 1)$ and a vector $c(b) \in \mathbb{R}^d$ such that

$$\mu^b = b^Q \mu * \delta_{c(b)}.$$  

(1.2)

Here $\delta_{c(b)}$ is the delta distribution at $c(b)$. When (1.2) is satisfied, we call $\mu$ $(Q, b, \alpha)$-semi-stable. It is called strictly operator semi-stable with exponent $(Q, \alpha)$ if there is $b \in (0, 1)$ such that

$$\mu^b = b^Q \mu.$$  

(1.3)

When (1.3) is satisfied, we call $\mu$ strictly $(Q, b, \alpha)$-semi-stable. The above definition of $(Q, b, \alpha)$-semi-stable distribution is described without the assumption that $\mu$ is full. If $\mu$ is a $(Q, b, \alpha)$-semi-stable distribution on $\mathbb{R}^d$, then $\mu$ is an operator semi-stable distribution on $\mathbb{R}^d$. But the converse is not true. The counterexamples are given at the end of this paper. The $(Q, b, \alpha)$ of a distribution satisfying (1.2) is not uniquely determined by $\mu$. If $\mu$ is semi-stable with exponent $\alpha$ in the sense of [1], then $\mu$ is an operator semi-stable distribution with exponent $(I, \alpha)$. We note that $\mu$ is $(Q, b, \alpha)$-semi-stable if and only if $\mu$ is $(\alpha^{-1}Q, b^\alpha, 1)$-semi-stable. The distribution satisfying (1.2) for every $b \in (0, \infty)$ is operator stable, which was introduced by M. Sharpe [13]. It is $(Q, \alpha)$-stable in the sense of [12]. By introducing the terminology $(Q, b, \alpha)$, the relations between operator semi-stable distributions and semi-stable distributions become clearer. The characterization of full operator semi-stable distributions on $\mathbb{R}^d$ is investigated by many authors. But they did not treat the whole structure of Gaussian operator semi-stable distributions.

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The main purpose of this paper is to obtain necessary and sufficient conditions for \((Q, b, \alpha)\)-semi-stable Gaussian distributions. The descriptions for full operator stable distribution were developed by many authors, but the complete characterization of Gaussian operator stable distributions is done by K. Sato [10] and K. Sato-M. Yamazato [12]. Our description of \((Q, b, \alpha)\)-semi-stable Gaussian distributions in this paper is an extension of the results in \((Q, \alpha)\)-stable case in [10, 12] to \((Q, b, \alpha)\)-semi-stable case.

In Section 2, we write some results and lemmas we use in the subsequent sections. In Section 3, we characterize \((Q, b, \alpha)\)-semi-stable Gaussian distributions, and in Section 4, we rewrite a necessary and sufficient condition for \((Q, b, \alpha)\)-semi-stable purely non-Gaussian distributions on \(R^d\). Its necessity part is similar to that of [3]. Relations between \((Q, b, \alpha)\)-semi-stable distributions and operator semi-stable distributions are given in Section 5.

2. Preliminaries

For \(x, y \in R^d\), we denote the Euclidean inner product of \(x\) and \(y\) by \(\langle x, y \rangle\) and the Euclidean norm of \(x\) by \(|x|\). Lévy shows that a distribution \(\mu\) on \(R^d\) with characteristic function \(\hat{\mu}(z)\) is infinitely divisible if and only if \(\hat{\mu}(z)\) has form

\[
\hat{\mu}(z) = \exp \left\{ i \langle \gamma, z \rangle - \frac{1}{2} (Az, z) + \int_{R^d} G(z, x) \nu(dx) \right\},
\]

where \(G(z, x) = e^{\langle z, x \rangle - i \langle z, x \rangle (1 + |x|^2)^{-1}}\), \(\gamma\) is a vector in \(R^d\), \(A\) is a symmetric nonnegative definite operator and \(\nu\) is a measure (called Lévy measure) on \(R^d\) satisfying \(\nu(\{0\}) = 0\) and \(\int |x|^2 (1 + |x|^2)^{-1} \nu(dx) < \infty\). This representation is unique and called the Lévy representation \((\gamma, A, \nu)\). We call \(\mu\) a purely non-Gaussian in the case of \(A = 0\). If \(\gamma = 0\) and \(A = 0\), then we call \(\mu\) a centered purely non-Gaussian. If \(\gamma = 0\) and \(\nu = 0\), then \(\mu\) is called a centered Gaussian. We denote the adjoint of a linear operator \(T\) by \(T^*\).

**Proposition 2.1.** Fix \(b \in (0, 1)\), \(\alpha > 0\) and \(Q \in M_+(R^d)\). Let \(\mu\) be \((Q, b, \alpha)\)-semi-stable on \(R^d\). If \(T \in Aut(R^d)\), then \(T \mu\) is \((TQT^{-1}, b, \alpha)\)-semi-stable on \(R^d\).
Proof. From the fact that $T^{-1}bQT = bT^{-1}QT$, we see that
\[
\hat{T}_\mu(z)b^\alpha = \hat{\mu}(bQ'z)e^{i(\langle c(b), T'z \rangle)} = \hat{\mu}(T'\hat{b}(TQT^{-1})'z)e^{i(\langle c(b), T'z \rangle)}
\]
\[
= \hat{T}_\mu(b^{(TQT^{-1})'z})e^{i(Tc(b), z)}.
\]
\[
\square
\]

We fix $Q \in M_+(R^d)$. Let $\mu$ be an operator semi-stable distribution with exponent $(Q, \alpha)$. For a real symmetric nonnegative definite operator $A$, $\phi_A(z)$ stands for $\langle Az, z \rangle$ for $z \in C^d$. Here $(\cdot)$ denotes the Hermitian inner product on $C^d$. We write $(b^nQ\nu)(E) = \nu(b^{-n}Q\nu)$.

**Lemma 2.2.** Fix $b \in (0,1)$, $Q \in M_+(R^d)$ and $\alpha > 0$. Let $\mu$ be infinitely divisible on $R^d$ with the Lévy representation $(\gamma, A, \nu)$. Then a necessary and sufficient condition for $\mu$ to be $(Q, b^\alpha, \alpha)$-semi-stable is that, for any integer $n$,
\[
\phi_A(b^nQ'z) = b^{n\alpha}\phi_A(z) \quad \text{for } z \in C^d
\]
and
\[
(b^nQ\nu)(E) = b^{n\alpha}\nu(E) \quad \text{for } E \in \mathcal{B}(R^d).
\]

**Proof.** If $\mu$ is $(Q, b^\alpha, \alpha)$-semi-stable, then, iterating (1.2), we get, for any positive integer $m$,
\[
\mu^{b^m} = b^{mQ}\mu * \delta(c(b^m)),
\]
where $c(b^m) = b^\alpha c(b^{(m-1)}) + b^{(m-1)}Qc(b)$. Hence a necessary and sufficient condition for $\mu$ to be $(Q, b^\alpha, \alpha)$-semi-stable is that, for any positive integer $m$,
\[
\phi_A(b^mQ'z) = b^{m\alpha}\phi_A(z) \quad \text{and} \quad (b^mQ\nu)(E) = b^{m\alpha}\nu(E).
\]
From the facts that $\phi_A(z) = \phi_A((bb^{-1})^Q'z) = b^\alpha\phi_A(b^{-Q'}z)$ and $\nu(E) = (bb^{-1})Q\nu(E) = b^\alpha b^{-Q}\nu(E)$, we see that
\[
\phi_A(b^{-Q'}z) = b^{-\alpha}\phi_A(z) \quad \text{and} \quad (b^{-Q}\nu)(E) = b^{-\alpha}\nu(E),
\]
which implies that, for any positive integer $m$,
\[
\phi_A(b^{-mQ'}z) = b^{-m\alpha}\phi_A(z) \quad \text{and} \quad (b^{-mQ}\nu)(E) = b^{-m\alpha}\nu(E).
\]
\[
\square
\]
The following lemmas are known. Proofs are omitted.
Lemmas 2.3 and 2.4 (Lemma 5.7 in [10]). Let $z_0 \in C^d$. If $A$ is real symmetric nonnegative definite and $\phi_A(z_0) = 0$, then $A z_0 = 0$.

3. Gaussian operator semi-stable distributions

In the following Theorem 3.1, we obtain the characterization of $(Q, b, \alpha)$-semi-stable Gaussian distributions on $R^d$. An example which shows that the class of Gaussian operator semi-stable distributions is strictly bigger than that of Gaussian operator stable distributions is given in a recent paper [9]. For $Q \in M_e(R^d)$, we write $B = bQ$. For $x \in C^d$, $\overline{x}$ stands for the complex conjugate of $x$, that is, each component of $\overline{x}$ is the complex conjugate of the corresponding component of $x$. Let $\vartheta_1, \ldots, \vartheta_p$, be all distinct eigenvalues of $bQ$. Let $f(\xi)$ be the minimal polynomial of $bQ$ with $f(\xi) = f_1(\xi)^{m_1} \cdots f_p(\xi)^{m_p}$, where $f_j(\xi) = \xi - \vartheta_j$ for $1 \leq j \leq p$. We denote the kernel of $(Q - \vartheta_j)^{m_j}$ in $C^d$ by $E_j$, that is, $E_j$ is the eigenspace of $bQ$ in the wide sense associated with the eigenvalue $\vartheta_j$ for $1 \leq j \leq p$. Denote by $P_j$ the projector onto $E_j$ in the decomposition

\begin{equation}
C^d = E_1 \oplus \cdots \oplus E_p.
\end{equation}

Let

\[ E'_j = \text{Kernel}(B' - \overline{\vartheta}_j I)^{m_j} \text{ in } C^d \text{ for } 1 \leq j \leq p. \]

Then we have

\begin{equation}
C^d = E'_1 \oplus \cdots \oplus E'_p.
\end{equation}

We see that $E'_j$ and $E_k$ are orthogonal for $j \neq k$ and $P'_j$ is the projector of $C^d$ onto $E'_j$ in the decomposition (3.2). Let $J = \{j : 1 \leq j \leq p, |\vartheta_j|^2 = b^\alpha\}$.  

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THEOREM 3.1. Fix $b \in (0,1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let $\mu$ be infinitely divisible on $R^d$ with the Lévy representation $(\gamma, A, 0)$. Then a necessary and sufficient condition for $\mu$ to be $(Q, b, \alpha)$-semi-stable is that

\begin{align}
(3.3) & \quad A P_j' = 0 \quad \text{for all} \quad j \notin J, \\
(3.4) & \quad (B - \bar{\vartheta}_j) A P_j' = 0 \quad \text{for all} \quad j \in J.
\end{align}

We will use the following lemma in the proof of Theorem 3.1. The proof is given in [10].

LEMMA 3.2 (Lemma 6.4 in [10] and Remark 3.1 in [12]). Let $A$ be real symmetric nonnegative definite. Then

\begin{align}
(B - \bar{\vartheta}_j) AP_j' = 0 \quad \text{for} \quad 1 \leq j \leq p
\end{align}

if and only if

\begin{align}
P_k AP_j' = 0 \quad \text{for} \quad j \neq k, \\
A(B' - \bar{\vartheta}_j) P_j' = 0 \quad \text{for} \quad 1 \leq j \leq p.
\end{align}

Proof of Theorem 3.1. Suppose that $\mu$ is a $(Q, b, \alpha)$-semi-stable distribution with Lévy representation $(\gamma, A, 0)$. Then we assert that, for any positive integer $m$ and $z_0 \in C^d$,

\begin{align}
(B' - \bar{\vartheta}_j)^m z_0 = 0 \quad \text{implies} \quad A(B' - \bar{\vartheta}_j) z_0 = 0.
\end{align}

For the proof of (3.7), we use induction in $m$. For $m = 1$, (3.7) is trivial. Suppose that (3.7) is true for $m - 1$ in place of $m$, and assume $(B' - \bar{\vartheta}_j)^m z_0 = 0$. Let us write $\bar{\vartheta}_j^{k-1} (B' - \bar{\vartheta}_j)^k z_0 = z_k$ for each nonnegative integer $k$. Since $(B' - \bar{\vartheta}_j)^m z_k = (B' - \bar{\vartheta}_j)^m-1(B' - \bar{\vartheta}_j) z_k = 0$ for $k \geq 0$, we have $A(B' - \bar{\vartheta}_j)^2 z_k = 0$ for $k \geq 0$ by the induction hypothesis. Hence we see that, for $n = 1, 2, \cdots$,

\begin{align}
AB^m z_0 = \bar{\vartheta}_j^{n} A[z_0 + nz_1]
\end{align}

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and
\[ \phi_A(B^m z_0) = |\theta_j|^{2n}[\phi_A(z_0) + 2n \text{Re}(Az_0, z_1) + n^2 \phi_A(z_1)]. \]

We write
\[ \Delta(n) = 2n \text{Re}(Az_0, z_1) + n^2 \phi_A(z_1). \]

Noticing that by Lemma 2.2
\[ \phi_A(B^m z_0) = b^m \phi_A(z_0), \]
we see that \( b^{\alpha n} \phi_A(z_0) = |\theta_j|^{2n}[\phi_A(z_0) + \Delta(n)] \). We consider three cases:

1. \( b^\alpha = |\theta_j|^2 \), \( b^\alpha < |\theta_j|^2 \) and \( b^\alpha > |\theta_j|^2 \).
   - In this case, we have that \( \Delta(n) = 0 \). If \( \phi_A(z_1) \neq 0 \), then we have that \( \Delta(n) \to \infty \) as \( n \to \infty \), which is a contradiction. Thus, \( \phi_A(z_1) = 0 \), from which follows \( Az_1 = 0 \) by Lemma 2.3.
   - \( b^\alpha < |\theta_j|^2 \). In this case, we have that
     \[ \left( \frac{b^\alpha}{|\theta_j|^2} \right)^n \phi_A(z_0) = \phi_A(z_0) + \Delta(n). \]
     Letting \( n \to \infty \), we get that \( \left( \frac{b^\alpha}{|\theta_j|^2} \right)^n \to 0 \). This leads to the fact that \( \Delta(n) \to -\phi_A(z_0) \) as \( n \to \infty \). But we have \( \Delta(n) \to \infty \) as \( n \to \infty \) if \( \phi_A(z_1) \neq 0 \). Hence, we see that \( Az_1 = 0 \).
   - \( b^\alpha > |\theta_j|^2 \). In this case, we have that
     \[ \left( \frac{|\theta_j|^2}{b^\alpha} \right)^n [\phi_A(z_0) + \Delta(n)] = \phi_A(z_0). \]
     Since \( \left( \frac{|\theta_j|^2}{b^\alpha} \right)^n \to 0 \) as \( n \to \infty \), we see that \( \phi_A(z_0) = 0 \). Hence, we have that \( \Delta(n) = 0 \). Thus, by the same method as (1), we see that \( Az_1 = 0 \).

Now we have proved that (3.7) is true. Let \( z \in E'_j \). From (3.7) we see that
\[ \phi_A(B'z) = \langle A B' z, B' z \rangle = \langle A \theta_j z, B' z \rangle = \theta_j \langle A z, B' z \rangle = \theta_j |\theta_j|^2 \phi_A(z). \]
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If \( j \notin J \), then, by (2.1), \( \phi_A(z) = 0 \), which is (3.3).

Suppose that \( z \in E'_{j} \), \( w \in E'_k \) and \( j \neq k \). If \( j \notin J \) or \( k \notin J \), then \( \langle Az, w \rangle = 0 \) by (3.3). Let us show that \( \langle Az, w \rangle = 0 \) when \( j \in J \) and \( k \in J \). Using (2.1) and (3.7), we get

\[
\phi_A(B'^n(z + w)) = b'^n \phi_A(z + w) = b'^n \phi_A(z) + b'^n \phi_A(w) + 2b'^n \text{Re} \langle Az, w \rangle
\]

and

\[
\phi_A(B'^n(z + w)) = b'^n \phi_A(z) + b'^n \phi_A(w) + 2\text{Re} \bar{\vartheta}_j^{-n} \vartheta_k^{-n} \langle Az, w \rangle.
\]

Hence, we see that \( \text{Re} \bar{\vartheta}_j^{-n} \vartheta_k^{-n} \langle Az, w \rangle = b'^n \text{Re} \langle Az, w \rangle \) for \( n = 1, 2, \ldots \)
Thus, we get \( \text{Re} \langle Az, w \rangle = 0 \). We also get \( \text{Im} \langle Az, w \rangle = \text{Re} \langle Az, w \rangle = 0 \).

Hence \( \langle Az, w \rangle = 0 \). Now we have (3.4). In fact, if \( z \in E'_{j} \), \( j \in J \), and \( w \in C^d \), then

\[
\langle (B - \vartheta_j)Az, w \rangle = \langle Az, (B' - \bar{\vartheta}_j)w \rangle = \langle Az, (B' - \bar{\vartheta}_j)P'_jw \rangle
\]

\[
= \langle z, A(B' - \bar{\vartheta}_j)P'_jw \rangle = 0
\]

by (3.7).

Conversely, suppose that \( A \) satisfies (3.3) and (3.4). From Lemma 3.2, we see that (3.5) and (3.6) hold. Thus by (3.3) and (3.5), we see that

\[
\phi_A(B'z) = \phi_A \left( \sum_{j=1}^{p} P'_j B'z \right) = \sum_{j=1}^{p} \phi_A(P'_j B'z) = \sum_{j \in J} \phi_A(P'_j B'z).
\]

By (3.6) and by \( B'P'_j = P'_j B'P'_j \), we have that

\[
\phi_A(P'_j B'z) = \phi_A(B'P'_j z) = |\vartheta_j|^2 \phi_A(P'_j z) = b^a \phi_A(P'_j z)
\]

for \( j \in J \). Hence \( \phi_A(B'z) = b^a \phi_A(z) \). The proof is complete. \( \Box \)
4. Purely non-Gaussian operator semi-stable distributions

We begin with some notation which follows [10, 11, 12]. We fix an arbitrary \( Q \in M_{+}(\mathbb{R}^d) \). Let \( \sigma_j = \alpha_j + i \beta_j, 1 \leq j \leq q + 2r \), be all distinct eigenvalues of \( Q \), where \( \alpha_j \) and \( \beta_j \) are real numbers such that \( \beta_j = 0 \) for \( 1 \leq j \leq q, \beta_j \neq 0 \) for \( q + 1 \leq j \leq q + 2r \), and \( \alpha_j + i \beta_j = \alpha_{j+r} - i \beta_{j+r} \) for \( q + 1 \leq j \leq q + r \). Here \( q \) and \( r \) are allowed to be zero. We note that \( p \leq q + 2r \) and the set \( \{ \vartheta_1, \ldots, \vartheta_p \} \) coincides with the set \( \{ b^{\vartheta_1}, \ldots, b^{\vartheta_{q+2r}} \} \), where \( b^{\vartheta_j} = b^{\beta_j} \) if \( \beta_j = \beta_k + 2n\pi \) with some integer \( n \). Let \( g(\xi) \) be the minimal polynomial of \( Q \) with \( g(\xi) = g_1(\xi)^{n_1} \cdots g_{q+r}(\xi)^{n_{q+r}} \), where \( g_j(\xi) = \xi - \alpha_j \) for \( 1 \leq j \leq q \), \( g_j(\xi) = (\xi - \alpha_j)^2 + \beta_j^2 \) for \( q + 1 \leq j \leq q + r \) and \( n_j, 1 \leq j \leq q + r \) are positive integers with \( \sum_{j=1}^{q+r} n_j \leq d \). Let \( W_j \) be the kernel of \( g_j(Q)^{n_j} \) in \( \mathbb{R}^d \), \( 1 \leq j \leq q + r \). The projector onto \( W_j \) in the direct sum decomposition

\[
\mathbb{R}^d = W_1 \oplus \cdots \oplus W_{q+r}
\]

is written as \( U_j \). We denote the kernel of \( (Q - \sigma_j)^{n_j} \) in \( C^d \), \( 1 \leq j \leq q + 2r \), by \( V_j \), that is, \( V_j \) is the eigenspace of \( Q \) in the wide sense associated with the eigenvalue \( \sigma_j \) for \( 1 \leq j \leq q + 2r \). Denote by \( T_j \) the projector onto \( V_j \) in the decomposition

\[
C^d = V_1 \oplus \cdots \oplus V_{q+2r}.
\]

We set \( J(\alpha) = \{ j : 1 \leq j \leq q + 2r, \alpha_j = \frac{\alpha}{2} \} \), \( K(\alpha) = \{ j : 1 \leq j \leq q + r, \alpha_j > \frac{\alpha}{2} \} \), \( W_{K(\alpha)} = \bigoplus_{j \in K(\alpha)} W_j \) and \( S_{K(\alpha)} = \{ \xi \in W_{K(\alpha)} : |\xi| = 1, \|u^{\alpha} \xi\| > 1 \text{ for all } u > 1 \} \). We write for \( x \neq 0 \) in \( \mathbb{R}^d \), \( \alpha(x) = \min\{\alpha_j : 1 \leq j \leq q + 2r, T_j x \neq 0\} \), and for \( j \) such that \( T_j x \neq 0 \), we set \( n(x, j) = \max\{n \geq 0 : (Q - \sigma_j)^n T_j x \neq 0\} \). For \( x \neq 0 \) in \( \mathbb{R}^d \), we denote \( n(x) = \max\{n(x, j) : 1 \leq j \leq q + 2r, U_j x \neq 0, \alpha_j = \alpha(x)\} \), and \( N = \max\{n_j : 1 \leq j \leq q + 2r\} \).

The following theorem characterizes the class of all \((Q, b, \alpha)\)-semi-stable purely non-Gaussian distributions without assuming fullness. The first necessary and sufficient condition for a purely non-Gaussian distribution on \( \mathbb{R}^d \) to be \((Q, b, \alpha)\)-semi-stable was obtained in [7,8]. But, from the results in [7,8], it is not easy to find the relations between the
Lévy measure of operator semi-stable distributions and that of operator stable distributions. With this consideration, we rewrite the Lévy measure of a \((Q, b, \alpha)\)-semi-stable distribution in a form similar to that of the Lévy measure of an operator stable distribution. Our description of the Lévy measure of a \((Q, b, \alpha)\)-semi-stable distribution in the case of \(Q = I\) is that of a semi-stable distribution in \([1]\). Let \(R_{+} = (0, \infty)\), the open half line. Denote the support of a measure \(\rho\) by \(\text{Spt} \ \rho\). The indicator function of \(E\) is denoted by \(I_{E}(x)\).

**Theorem 4.1.** Fix \(b \in (0, 1)\), \(\alpha > 0\) and \(Q \in M_{+}(R^{d})\). Let \(\mu\) be infinitely divisible on \(R^{d}\) with the Lévy representation \((\gamma, 0, \nu)\). Then a necessary and sufficient condition for \(\mu\) to be \((Q, b, \alpha)\)-semi-stable is that

\[
\nu(E) = \int_{S_{K(\alpha)}} \lambda(d\xi) \int_{0}^{\infty} I_{E}(u^{Q}\xi) d \left\{ \frac{-H_{\xi}(u)}{u^{\alpha}} \right\}
\]

for all Borel sets \(E \subset R^{d}\), where \(\lambda\) is a finite measure on \(S_{K(\alpha)}\), \(\frac{H_{\xi}(u)}{u^{\alpha}}\) is nonincreasing in \(u\), \(H_{\xi}(u)\) is right-continuous in \(u\) and measurable in \(\xi\), \(H_{\xi}(1) = 1\) and \(H_{\xi}(bu) = H_{\xi}(u)\) for any \(u\) and \(\xi\). If \(\mu\) is \((Q, b, \alpha)\)-semi-stable, then the measure \(\lambda\) is unique and the function \(H_{\xi}(u)\) is unique for \(\lambda\)-almost every \(\xi \in S_{K(\alpha)}\). For any finite measure \(\lambda\) on \(S_{K(\alpha)}\) and for any function \(H_{\xi}\) satisfying the conditions above, there exists a \((Q, b, \alpha)\)-semi-stable purely non-Gaussian distribution \(\mu\) with the Lévy measure \(\nu\) of (4.2).

Since \(W_{K(\alpha)}\) is \(Q\)-invariant, using Lemma 2.4, we see that any point \(x \neq 0\) in \(W_{K(\alpha)}\) has unique expression \(x = u^{Q}\xi\) with \(\xi \in S_{K(\alpha)}\) and \(u > 0\). From Lemma 4.1 in \([11]\) (see Lemma 5.1 in \([12]\) or Lemma 5.6 in \([10]\)), we see that there is \(C_{1}\) such that

\[
|u^{Q}\xi| \leq C_{1}u^{\alpha(\xi)}|\log u|^{N-1} \quad \text{for} \quad 0 < u \leq 1/e.
\]

Put \(h(u) = \frac{u^{2}}{1+u^{2}}\). Then, by (4.3), there is \(C_{2}\) such that

\[
h(|u^{Q}\xi|) \leq C_{2}u^{2\alpha(\xi)}|\log u|^{2N-2} \quad \text{for} \quad 0 < u \leq 1/e.
\]

Here \(C_{1}\) and \(C_{2}\) are constants independent of \(u\) and \(\xi\).
Lemma 4.2. If $\mu$ is $(Q, b, \alpha)$-semi-stable, purely non-Gaussian with Lévy measure $\nu$, then

$$Spt \, \nu \subset W_{K(\alpha)}.$$ 

Proof. Define a finite measure $\nu'$ by $\nu'(E) = \int_E h(|x|)\nu(dx)$ for $E \in B(R^d)$. Let $n$ be a positive integer such that $0 < b^n < \frac{1}{e}$. By Lemma 2.2, we obtain that

$$\nu'(b^nQ E) = b^{-n\alpha} \int h(|b^nQ x|) I_E(x) \nu(dx).$$

Using Lemma 4.1 in [11], we see that there is a positive function $b_0(x)$ for $x \neq 0$ in $R^d$ such that

$$b^{-n\alpha} \int h(|b^nQ x|) I_E(x) \nu(dx) \geq b^{-n\alpha} \int h(b_0(x)b^{n\alpha(x)}|x|) I_E(x) \nu(dx).$$

Let $x_0 \notin W_{K(\alpha)}$. Choose a bounded open neighborhood $E$ of $x_0$ such that $\alpha(x) \leq \frac{\alpha}{2}$ for $x \in E$. By Fatou's lemma we have

$$\liminf_{n \to \infty} \nu'(b^nQ E)$$

$$\geq \int \liminf_{n \to \infty} b^{-n\alpha} h(b_0(x)b^{n\alpha(x)}|x|) I_E(x) \nu(dx).$$

Let $E_1$ be the set of $x \in E$ such that $\alpha(x) < \frac{\alpha}{2}$, and $E_2$ be the set of $x \in E$ such that $\alpha(x) = \frac{\alpha}{2}$. Then,

$$\liminf_{n \to \infty} b^{-n\alpha} h(b_0(x)b^{n\alpha(x)}|x|) = \begin{cases} \infty & \text{for } x \in E_1 \\ b_0(x)^2|x|^2 & \text{for } x \in E_2. \end{cases}$$

Hence, we see that $\nu(E_1) = 0$. By (4.4), we have that

$$|b^nQ x| \leq C_1 b^n \frac{\log b^n}{b^n} |x|$$

for $x \in E_2$, if $b^n \leq 1/e$. This leads to

$$\liminf_{n \to \infty} \nu'(b^nQ E_2) = \nu'({\{0}\}) = 0.$$ 

Since

$$\liminf_{n \to \infty} \nu'(b^nQ E_2) \geq \int b_0(x)^2|x|^2 I_{E_2}(x) \nu(dx),$$

we get $\nu(E_2) = 0$. Hence $\nu(E) = 0$, which means that $x_0 \notin Spt \, \nu$. \( \square \)
Proof of Theorem 4.1. Suppose that \( \mu \) is \( (Q,b,\alpha) \)-semi-stable. For any \( B \in \mathcal{B}(S_K(\alpha)) \), define \( \lambda(B) = \nu(\{u^Q \xi : \xi \in B, u > 1\}) \) and \( N(s,B) = \nu(\{u^Q \xi : \xi \in B, u > s\}) \). Then for any positive real number \( r \), we can choose integer \( m \) such that \( r > b^m > 0 \), so

\[
0 \leq N(r,B) \leq N(b^m,B) = b^{-m\alpha} \lambda(B).
\]

Hence \( N(r,B) \) is absolutely continuous with respect to \( \lambda \). Thus for each positive real number \( r \), there is a nonnegative measurable function \( N_\xi(r) \) of \( \xi \) such that

\[
N(r,B) = \int_B N_\xi(r) \lambda(d\xi), \quad B \in \mathcal{B}(S_K(\alpha)).
\]

Here \( N_\xi(r) \) is uniquely defined for \( \lambda \)-almost every \( \xi \). We can take \( N_\xi(r) \) nonincreasing right-continuous in \( r \). For \( E = \{u^Q \xi : \xi \in B, u \in F\} \), with \( F \in \mathcal{B}(\mathbb{R}_+) \), we obtain

\[
\nu(E) = - \int_B \lambda(d\xi) \int_F dN_\xi(u).
\]

From the fact that \( \nu(b^{-Q}\{u^Q \xi : \xi \in B, u \in (s,\infty)\}) = \nu(\{u^Q \xi : \xi \in B, u \in (b^{-1}s,\infty)\}) = b^{\alpha} \nu(\{u^Q \xi : \xi \in B, u \in (s,\infty)\}) \), we see that \( N_\xi(bu) = b^{-\alpha} N_\xi(u) \). Putting \( N_\xi(u) = H_\xi(u) u^{-\alpha} \), we see that \( \frac{H_\xi(u)}{u^\alpha} \) is nonincreasing in \( u \), \( H_\xi(u) \) is right-continuous in \( u \) and measurable in \( \xi \), \( H_\xi(1) = 1 \) and \( H_\xi(bu) = H_\xi(u) \) for any \( u \) and \( \xi \). Since \( \mathcal{B}(W_K(\alpha)) \) is generated by sets \( E \) of the above form, we get (4.2) for all \( E \in \mathcal{B}(W_K(\alpha)) \), which shows (4.2) by Lemma 4.2.

Conversely, assume that \( \lambda \) is a finite measure on \( S_K(\alpha) \) and define a measure \( \nu \) on \( \mathbb{R}^d \) by (4.2). Let \( \alpha^+ = \min\{\alpha_j : 1 \leq j \leq q + 2r, \alpha_j > \frac{\alpha}{2}\} \) and let \( \alpha^{++} = \max\{\alpha_j : 1 \leq j \leq q + 2r, \alpha_j > \frac{\alpha}{2}\} \). Then \( \alpha^+ \leq \alpha(\xi) \leq \alpha^{++} \) for \( \xi \in S_K(\alpha) \). Let \( M \) be the positive integer satisfying \( 1 \leq e^{-1}b^{-\alpha^+} < b^{-M} \). For any \( \xi \in S_K(\alpha) \), we have the following. By (4.4), we see that

\[
\int_0^{e^{-1}} h(|u^Q \xi|) d\left\{ \frac{-H_\xi(u)}{u^\alpha} \right\} \\
\leq C_2 \sum_{n=0}^{\infty} \int_{e^{-1}b^{n+1}}^{e^{-1}b^n} u^{2\alpha(\xi)} |\log u|^{2N-2} d\left\{ \frac{-H_\xi(u)}{u^\alpha} \right\}.
\]
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For \( n = 0, 1, \cdots \), we have that

\[
\int_{-b^{n+1}} e^{-b^{n}} u^{2\alpha(\xi)} |\log u|^{2N-2} d \left\{ \frac{-H_\xi(u)}{u^\alpha} \right\} \\
\leq \left| (n+1) \log b - 1 \right|^{2N-2} \int_{-b^{n+1}} e^{-b^{n}} u^{2\alpha(\xi)} d \left\{ \frac{-H_\xi(u)}{u^\alpha} \right\},
\]

because \( |\log u| \leq |\log(e^{-b^{n+1}})| = |(n+1) \log b - 1| \) for \( e^{-b^{n+1}} < e^{-b^n} \). Letting \( u = b^{n+1+M} v \), we obtain that

\[
\int_{-b^{n+1}} e^{-b^{n}} u^{2\alpha(\xi)} d \left\{ \frac{-H_\xi(u)}{u^\alpha} \right\} \\
= b^{2(n+1+M)\alpha(\xi)-(\frac{\alpha}{2})} \int_{-b^{n-M}} e^{-b^{n-M-1}} v^{2\alpha(\xi)} d \left\{ \frac{-H_\xi(v)}{v^\alpha} \right\}.
\]

Since \( \int_{1}^{\infty} d \left\{ \frac{-H_\xi(v)}{v^\alpha} \right\} = H_\xi(1) = 1 \), we have that

\[
b^{2(n+1+M)\alpha(\xi)-(\frac{\alpha}{2})} \int_{-b^{n-M}} e^{-b^{n-M-1}} v^{2\alpha(\xi)} d \left\{ \frac{-H_\xi(v)}{v^\alpha} \right\} \\
\leq b^{2(n+1+M)\alpha(\xi)-(\frac{\alpha}{2})} \int_{1}^{b^{n-M-1}} v^{2\alpha(\xi)} d \left\{ \frac{-H_\xi(v)}{v^\alpha} \right\} \\
\leq b^{2(n+1+M)\alpha(\xi)-(\frac{\alpha}{2})} b^{-2(M+1)\alpha} \int_{1}^{\infty} d \left\{ \frac{-H_\xi(v)}{v^\alpha} \right\} \\
\leq b^{2(n+1+M)(\alpha+\frac{\alpha}{2})-2(M+1)\alpha}.\]

Hence

\[
\int_{S_{K(\alpha)}} \lambda(d\xi) \int_{0}^{e^{-1}} h(|uQ\xi|) d \left\{ \frac{-H_\xi(u)}{u^\alpha} \right\} \\
\leq C_2 \lambda(S_{K(\alpha)}) \sum_{n=0}^{\infty} \left| (n+1) \log b - 1 \right|^{2N-2} b^{2(n+1+M)(\alpha+\frac{\alpha}{2})-2(M+1)\alpha} \\
< \infty,
\]

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since $\alpha^+ - \frac{\alpha}{2} > 0$. Since $h(\cdot) \leq 1$, we see that

$$\int_{e^{-1}}^{\infty} h(|u^Q\xi|) d \left\{ \frac{-H_\xi(u)}{u^\alpha} \right\} \leq \int_{e^{-1}}^{\infty} d \left\{ \frac{-H_\xi(u)}{u^\alpha} \right\}.$$  

Using the fact that

$$\sup_{u>0} H_\xi(u) = \sup_{1 \leq u < b^{-1}} H_\xi(u) = \sup_{1 \leq u < b^{-1}} u^\alpha H_\xi(u) = b^{-\alpha} H_\xi(1) = b^{-\alpha}$$

and

$$\inf_{u>0} H_\xi(u) = \inf_{1 \leq u < b^{-1}} H_\xi(u) = \inf_{1 \leq u < b^{-1}} u^\alpha H_\xi(u) \geq b^\alpha H_\xi(b^{-1}) = b^\alpha,$$

we obtain that $\lim_{u \to \infty} -\frac{H_\xi(u)}{u^\alpha} = 0$. It follows that

$$\int_{e^{-1}}^{\infty} d \left\{ \frac{-H_\xi(u)}{u^\alpha} \right\} = \frac{H_\xi(e^{-1})}{e^{-\alpha}} \leq e^\alpha b^{-\alpha}.$$  

Hence

$$\int_{S_K(\alpha)} \lambda(d\xi) \int_{e^{-1}}^{\infty} h(|u^Q\xi|) d \left\{ \frac{-H_\xi(u)}{u^\alpha} \right\} \leq e^\alpha b^{-\alpha} \lambda(S_K(\alpha)) < \infty.$$  

Therefore

$$\int_{\mathbb{R}^d} h(|x|) \nu(dx) < \infty.$$  

Hence $\nu$ is the Lévy measure of a purely non-Gaussian infinitely divisible distribution $\mu$. It is easy to see that $\nu$ satisfies (2.2). Thus, $\mu$ is $(Q, b, \alpha)$-semi-stable.

Let $\mu$ be $(Q, b, \alpha)$-semi-stable and centered purely non-Gaussian with Lévy measure $\nu$. Let $W_\mu, W_\nu$ be the smallest linear subspaces that contain $Spt \mu, Spt \nu$, respectively. From Lemma 4.2, we see that $W_\nu$ is a linear subspace of $W_K(\alpha)$. Using Lemma 5.2 and Theorem 5.2 in [12], we get the following remark.
Operator semi-stable distributions

Remark 4.3. Suppose that $\mu$ is $(Q, b, \alpha)$-semi-stable and centered purely non-Gaussian with Lévy measure $\nu$. Then $W_\mu = W_\nu$, $\mu$ is full in $W_\mu$ and $W_\mu$ is $bQ$-invariant.

Remark 4.4. Suppose that $\mu$ is a $(Q, b, \alpha)$-semi-stable distribution on $R^2$ with Lévy representation $(0, 0, \nu)$. If the subspace $W_\nu$ is contained in $R = \{x = (x_1)_{i=1,2} : x_2 = 0\}$, then $\mu$ is a semi-stable distribution with some exponent $\alpha$ on $R$ in the sense of [1].

5. Relations between $(Q, b, \alpha)$-semi-stable distributions and operator semi-stable distributions

R. Jajte in the Theorem of [4] described that, if $\mu$ is full, then a necessary and sufficient condition for $\mu$ to be an operator semi-stable distribution is that it is infinitely divisible and there exist a number $a \in (0, 1)$, a vector $c_0 \in R^d$, and $A \in Aut(R^d)$ such that

$$\mu^a = A\mu * \delta_{c_0}.$$  

(5.1)

In [3], V. Chorny pointed out that the relation (5.1) was equivalent to

$$\mu^b = bQ\mu * \delta_{c_1}$$

with some $b \in (0, 1)$, $Q \in M_+(R^d)$ and $c_1 \in R^d$. This distribution is a $(Q, b, 1)$-semi-stable distribution.

The following Remarks 5.1 and 5.2 for operator semi-stable distributions are given in R. Jajte [4].

Remark 5.1. Fix $b \in (0, 1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let $\mu$ be $(Q, b, \alpha)$-semi-stable on $R^d$. Then $\mu$ is an operator semi-stable distribution.

Remark 5.2. If $\mu$ is a full operator semi-stable distribution on $R^d$, then $\mu$ is $(Q, b, \alpha)$-semi-stable on $R^d$ with some $b \in (0, 1)$, $\alpha > 0$ and $Q \in M_+(R^d)$.

Proposition 5.3. If $\mu$ is an operator semi-stable distribution on $R^d$ and $T \in End(R^d)$, then $T\mu$ is an operator semi-stable distribution.
Proof. We choose $T_n \in \text{Aut}(\mathbb{R}^d)$ such that $T_n \to T$. By the definition of an operator semi-stable distribution, there are $A_n \in \text{Aut}(\mathbb{R}^d)$ and $a_n \in \mathbb{R}^d$ such that

$$
\lim_{n \to \infty} A_n \mu^{k_n} \ast \delta_{a_n} = \mu,
$$

where $k_n^{-1}k_{n+1} \to r$ for some $r \in [1, \infty)$. Hence, we have that

$$
\lim_{n \to \infty} T_n A_n \mu^{k_n} \ast \delta_{T_n a_n} = T \mu.
$$

This shows that $T \mu$ is an operator semi-stable distribution. \qed

In [10,14], there are examples of operator stable distributions that are not $(Q, \alpha)$-stable. Modifying it, we get the following example. These will show that the converse of Remark 5.1 is not true without the condition of fullness.

**Example 5.4.** Let $d = 2$, $Q = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}$ and $\xi_0 = 2^{-\frac{1}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $u^Q = \begin{pmatrix} u^{\frac{3}{2}} \\ 0 \\ u^2 \end{pmatrix}$ and $u^Q \xi_0 = 2^{-\frac{1}{2}} \begin{pmatrix} \frac{u^{3}}{u^2} \\ 0 \end{pmatrix}$. Fix $b \in (0,1)$ and $\alpha \in (0,2)$. We choose a positive number $C_0$ such that $C_0 = (\frac{2\pi}{\log b} + 1 < 1$. Let

$$
H_{\xi_0}(u) = C_0 \cos \left( \frac{2\pi}{\log b} \log u \right) + 1.
$$

Then the function $H_{\xi_0}(u)$ satisfies the conditions in Theorem 4.1. We consider the $(Q, b, \alpha)$-semi-stable distribution $\mu$ having the Lévy representation $(0,0,\nu)$ with

$$
\nu(E) = \int_0^\infty I_E(u^Q \xi_0) d\left\{ \frac{-H_{\xi_0}(u)}{u^\alpha} \right\}.
$$

This shows that $\mu$ is an operator semi-stable distribution by Remark 5.1. Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then, by Proposition 5.3, $T \mu$ is an operator semi-stable distribution. We have, for some positive real number $s$,

$$
\text{Spt} \ T \nu = \{ x = (x_1)_{i=1,2} : x_1 \leq s, x_2 = 0 \}.
$$
Operator semi-stable distributions

Suppose that $T\mu$ is a $(\hat{Q}, \hat{b}, \hat{\alpha})$-semi-stable distribution on $\mathbb{R}^2$ with some $\hat{b}, \hat{\alpha}$ and $\hat{Q} \in \mathcal{M}_+(\mathbb{R}^2)$. Then, by Remark 4.4, $T\mu$ is regarded as a semi-stable distribution with some exponent on $\mathbb{R}$ in the sense of [1]. But, if the support of the Lévy measure of a semi-stable distribution on $\mathbb{R}$ is not contained in $(-\infty, 0]$, then it must be unbounded to both directions. So we get a contradiction. Thus we conclude that there are no numbers $\hat{b}, \hat{\alpha}$ and $\hat{Q} \in \mathcal{M}_+(\mathbb{R}^2)$ such that $T\mu$ is a $(\hat{Q}, \hat{b}, \hat{\alpha})$-semi-stable distribution on $\mathbb{R}^2$.

EXAMPLE 5.5. Let $d = 2$. Let $Q, T, \xi_0$ be as in Example 5.4. Fix $\alpha \in (0, 2)$. Let $n$ be an integer. Consider a $(Q, b, \alpha)$-semi-stable distribution $\mu$ having Lévy representation $(0, 0, \nu)$ with

$$\nu(E) = \int_0^\infty I_E(u^Q\xi_0)d \left\{ \frac{-H_{\xi_0}(u)}{u^\alpha} \right\},$$

where $\frac{-H_{\xi_0}(u)}{u^\alpha} = \sum_{b^{-n} > u} b^{n\alpha}$. We see that

$$\text{Spt } T\mu = \{x = (x_i)_{i=1,2} : x_1 = 2^{-\frac{x}{2}}(b^n)^{\frac{x}{2}} - 2^{-\frac{x}{2}}(b^n)^2, x_2 = 0\}.$$

By a similar argument to the previous example, we can show that $T\mu$ is an operator semi-stable distribution on $\mathbb{R}^2$. But, we can not find numbers $\hat{b}, \hat{\alpha}$ and $\hat{Q} \in \mathcal{M}_+(\mathbb{R}^2)$ such that $T\mu$ is a $(\hat{Q}, \hat{b}, \hat{\alpha})$-semi-stable distribution on $\mathbb{R}^2$.

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