THE GENERALIZED OPEN SETS ON SUPRATOPOLOGY

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Abstract. We introduce the notion of \(s_\gamma\)-sets, and we investigate some properties of \(s_\gamma\)-sets. In particular, we characterize the \(s_\gamma\)-closure by terms of supra-convergence of filters.

1. Introduction

Let \(X\) be a nonempty set. A subclass \(\tau \subset P(X)\) is called a supratopology on \(X\) [3] if \(X \in \tau\) and \(\tau\) is closed under arbitrary union. \((X, \tau)\) is called a supratopological space. The members of \(\tau\) are called supraopen sets. The complement of supraopen sets are called supraclosed sets. Let \((X, \tau)\) be a supratopological space and \(S \subset X\). The supra-closure of \(S\), denoted by \(scl(S)\), is the intersection of supraclosed sets including \(S\). And the interior of \(S\), denoted by \(sint(S)\), the union of supraopen sets included in \(S\). Let \((X, \tau)\) be a supratopological space. Then \(\tau^*\) is called a supratopology associated with \(\tau\) if \(\tau \subset \tau^*\). Let \((X, \tau)\) and \((X, \mu)\) be topological spaces and let \(\tau^*\) and \(\mu^*\) be associated supratopologies with \(\tau\) and \(\mu\), respectively. Let \((X, \tau)\) be a topological space and \(S \subset X\). The closure (resp. interior) of \(S\) will be denoted by \(cl(S)\) (resp. \(int(S)\)). A subset \(S\) of \(X\) is called a semi-open set [2] if \(S \subset cl(int(S))\). The complement of a semi-open set is called a semi-closed set. The family of all semi-open sets in \(X\) will be denoted by \(SO(X)\).

A subset \(M(x)\) of a space \(X\) is called a supra(resp. semi)-neighborhood of a point \(x\) in \(X\) if there exists a supra(resp. semi)open set \(S\) such that...
$x \in S \subset M(x)$. In [1], R. M. Latif introduced the notion of semi-convergences of filters. And he investigated some characterizations related to semi-continuous functions. Now we recall the concept of semi-convergences of filters. Let $SO(x) = \{ A \in SO(X) : x \in A \}$ and let $SO_x = \{ A \subset X : \text{there exists } \mu \subset SO(x) \text{ such that } \mu \text{ is finite and } \cap \mu \subset A \}$. Then $SO_x$ is called the semi-neighborhood filter at $x$. For any filter $F$ on $X$, we say that $F$ semi-converges to $x$ if and only if $F$ is finer than the semi-neighborhood filter $SO_x$ at $x$. In this paper we introduce the concept of supra-convergences of filters, $s\gamma$-sets, and $s\gamma$ (resp. $s\gamma^*$)-continuity. And we investigate some properties, in particular, a function $f : X \to Y$ is $s\gamma$ (resp. $s\gamma^*$)-continuous if and only if whenever a filter $F$ supra-converges to $x$, then $f(F)$ converges (resp. supra-converges) to $f(x)$.

2. $s\gamma$-sets

**Definition 2.1.** Let $(X, \tau)$ be a supratopological space. A subset $U$ of $X$ is called an $s\gamma$-set in $X$ if whenever a filter $F$ on $X$ supra-converges to $x$ and $x \in U$, then $U \in F$.

The class of all $s\gamma$-sets in $X$ will be denoted by $s\gamma(X)$. In particular, The class of all $s\gamma$-sets induced by the supratopology $\tau$ will be denoted by $s\gamma_\tau$.

**Remark.** From the definition of supra-neighborhood filters and $s\gamma$-sets, easily we can show every supraopen set is an $s\gamma$-set, but the converse is always not true.

**Example 2.2.** Let $X$ be the real number set and let $S = \{(a, b] : a, b \in \mathbb{R}\} \cup \{(c, d) : c, d \in \mathbb{R}\}$ be a suprabase for the supratopology $\tau$. For each $x \in X$, since both $(a, x]$ and $[x, b)$ are supraopen sets containing $x$, $\{x\}$ is an element of $S_x$. For any filter $F$ on $X$, if $F$ supra-converges to $x$, then $F$ includes $S_x$ and so $\{x\}$ is an $s\gamma$-set. But it is not supra-open.

We recall that; Let $(X, \tau)$ be a topological space and $\tau^*$ be a supratopology associated with $\tau$. A subset $A$ of $X$ is called an $m$-set with $\tau^*$ if $A \cap T \in \tau^*$ for all $T \in \tau^*$. The class of all $m$-sets with $\tau^*$ will be denoted by $\tau_m[4]$.

Obviously, we get the following;
Theorem 2.3. If \((X, \tau)\) be a topological space and \(\tau^*\) be a supratopology associated with \(\tau\), then \(\tau \subset \tau_m \subset \tau^* \subset s_{\gamma} \tau\).

Theorem 2.4. Let \((X, \tau)\) be a supratopological space. The intersection of finitely many supra-open subsets in \(X\) is an \(s_{\gamma}\)-set.

Proof. Let \(U_1\) and \(U_2\) be supra-open sets in \(X\). For each \(x \in U_1 \cap U_2\), it is clearly \(U_1 \cap U_2 \in S_x\), and from the notion of the supra-convergence of filters, whenever every filter \(F\) supra-converges to \(x\), \(U_1 \cap U_2 \in F\). ☐

Definition 2.5. Let \((X, \tau)\) be a supratopological space. The \(s_{\gamma}\)-interior of a set \(A\) in \(X\), denoted by \(sI_{\gamma}(A)\), is the union of all \(s_{\gamma}\)-sets contained in \(A\).

Theorem 2.6. Let \((X, \tau)\) be a supratopological space and \(A \subset X\).

(a) \(sI_{\gamma}(A) = \{x \in A : A \in S_x\}\);
(b) \(A\) is \(s_{\gamma}\)-set if and only if \(A = sI_{\gamma}(A)\).

Proof. (a). For each \(x \in sI_{\gamma}(A)\), there exists an \(s_{\gamma}\)-set \(U\) such that \(x \in U\) and \(U \subset A\). From the notion of \(s_{\gamma}\)-sets, the subset \(U\) is in the supra-neighborhood filter \(S_x\). And since \(S_x\) is a filter, \(A \in S_x\). Conversely, let \(A \in S_x\), then there exist \(U_1, \ldots, U_n \in S(x)\) such that \(U = U_1 \cap \ldots \cap U_n \subset A\). By Theorem 2.4. \(U\) is an \(s_{\gamma}\)-set containing \(x\) and since \(S_x\) is a filter, \(A \in S_x\). Thus \(x \in sI_{\gamma}(A)\).

(b). Obvious. ☐

Theorem 2.7. Let \((X, \tau)\) be a supratopological space. Then the class \(s_{\gamma}(X)\) of all \(s_{\gamma}\)-subsets in \(X\) is a topology on \(X\).

Proof. Since \(\emptyset\) and \(X\) are supraopen sets, they are also \(s_{\gamma}\)-sets in \(X\). Let \(A\) and \(B\) be non-disjoint \(s_{\gamma}\)-subsets. For \(x \in A \cap B\), if a filter \(F\) on \(X\) supra-converges to \(x\), then both \(A\) and \(B\) are elements of \(F\) and since \(F\) is a filter, the intersection of \(A\) and \(B\) also an element of \(F\). Thus \(A \cap B\) is an \(s_{\gamma}\)-set. For each \(\alpha \in I\), let \(A_\alpha \in s_{\gamma}(X)\) and \(U = \cup A_\alpha\). For each \(x \in U\) and for a filter \(F\) supra-converging to \(x\), by the notion of \(s_{\gamma}\)-sets, there exists a subset \(A_\alpha\) of \(U\) such that \(x \in A_\alpha\) and \(A_\alpha \in F\), and since \(F\) is a filter, \(U\) is an element of the filter \(F\). Thus \(U = \cup A_\alpha\) is an \(s_{\gamma}\)-set. ☐

Let \((X, \tau)\) be a supratopological space. For a subset \(B\) of \(X\), we call \(B\) an \(s_{\gamma}\)-closed set if the complement of \(B\) is an \(s_{\gamma}\)-set. From Theorem 2.7, the intersection of any family of \(s_{\gamma}\)-closed sets is an \(s_{\gamma}\)-closed set and the union of finitely many \(s_{\gamma}\)-closed sets is an \(s_{\gamma}\)-closed set. Obviously we obtain the following, by the definition of \(s_{\gamma}\)-set.
Theorem 2.8. Let \((X, \tau)\) be a supratopological space. A set \(G\) is \(s\gamma\)-closed if and only if whenever \(F\) supra-converges to \(x\) and \(G \in F\), \(x \in G\).

Definition 2.9. Let \((X, \tau)\) be a supratopological space and \(A \subset X\),
\[scl_\gamma(A) = \{x \in X : A \cap U \neq \emptyset \text{ for all } U \in S_x\}.\]

We call \(scl_\gamma(A)\) the \(s\gamma\)-closure of \(A\).

Now we can get the following.

Theorem 2.10. Let \((X, \tau)\) be a supratopological space. For \(A \subset X\),
(1) \(A \subset scl_\gamma(A)\);
(2) \(A\) is \(s\gamma\)-closed if and only if \(A = scl_\gamma A\);
(3) \(sI_\gamma(A) = X - scl_\gamma(X - A)\);
(4) \(scl_\gamma(A) = X - sI_\gamma(X - A)\).

Theorem 2.11. Let \((X, \tau)\) be a supratopological space. \(x \in scl_\gamma(A)\) if and only if there exists a filter \(F\) on \(X\) such that \(A \in F\) and \(F\) supra-converges to \(x\).

Proof. Let \(x \in scl(A)\), then by the notion of the \(s\gamma\)-closure, the collection \(B = \{U \cap A : U \in S_x\}\) is a filter base. The filter \(F\) generated by filter base \(B\) supra-converges to \(x\) and \(A \in F\). Suppose that there is a filter \(F\) supra-converging to \(x\) such that \(A \in F\). Since \(F\) contains \(S_x\) and \(F\) is a filter, for all \(U \in S_x, U \cap A \neq \emptyset\). Thus \(x \in scl_\gamma(A)\). 

References

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