$k$-TH ROOTS OF $p$-HYPONORMAL OPERATORS

BHAGWATI P. DUGGAL, IN HO JEON(*) AND EUNGIL KO

ABSTRACT. In this paper we prove that if $T$ is a $k$-th root of a $p$-
hyponormal operator when $T$ is compact or $T^n$ is normal for some
integer $n > k$, then $T$ is (generalized) scalar, and that if $T$ is a $k$-th
root of a semi-hyponormal operator and have the property $\sigma(T)$
is contained in an angle $< \frac{2\pi}{k}$ with vertex in the origin, then $T$ is
subscalar.

1. Introduction

Let $H$ and $K$ be complex Hilbert spaces and let $\mathcal{L}(H, K)$ denote the
space of all bounded linear operators from $H$ to $K$. If $H = K$, we write
$\mathcal{L}(H)$ in place of $\mathcal{L}(H, K)$.

A bounded linear operator $S$ on $H$ is called scalar of order $m$ if it
has a spectral distribution of order $m$, i.e., if there is a continuous unital
morphism of topological algebras

$$
\Phi : C_0^m(\mathbb{C}) \rightarrow \mathcal{L}(H)
$$

such that $\Phi(z) = S$, where as usual $z$ stands for the identity function on
$\mathbb{C}$ and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on
$\mathbb{C}$, continuously differentiable of order $m$, $0 \leq m \leq \infty$. An operator is
subscalar if it is similar to the restriction of a scalar operator to a closed
invariant subspace.

Let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let $D$
be a bounded open disc in $\mathbb{C}$. We shall denote by $L^2(D, H)$ the Hilbert

Received May 3, 2004.
2000 Mathematics Subject Classification: 47B20, 47A15.
Key words and phrases: $k$-th roots of $p$-hyponormal operator, subscalar operator.
(*) This work was supported by Korea Research Foundation Grant(KRF-2001-050-D0001).
space of measurable functions \( f : D \rightarrow H \), such that
\[
\Vert f \Vert_{2,D} = \left( \int_D \Vert f(z) \Vert^2 \, d\mu(z) \right)^{\frac{1}{2}} < \infty.
\]
The space of functions \( f \in L^2(D, H) \) which are analytic functions in \( D \) (i.e., \( \bar{\partial} f = 0 \)) is defined by
\[
A^2(D, H) = L^2(D, H) \cap \mathcal{O}(D, H),
\]
where \( \mathcal{O}(D, H) \) denotes the Fréchet space of \( H \)-valued analytic functions on \( D \) with respect to uniform topology. \( A^2(D, H) \) is called the Bergman space for \( D \). Let us define a Sobolev type space, denoted \( W^2(D, H) \). \( W^2(D, H) \) will be the space of those functions \( f \in L^2(D, H) \) whose derivatives \( \bar{\partial} f, \bar{\partial}^2 f \) in the sense of distributions still belong to \( L^2(D, H) \). Endowed with the norm \( \Vert f \Vert^2_{W^2} = \sum_{i=0}^{2} \Vert \bar{\partial}^i f \Vert^2_{L^2(D, H)} \), \( W^2(D, H) \) becomes a Hilbert space contained continuously in \( L^2(D, H) \). Now, for \( f \in C_0^2(\mathbb{C}) \), let \( M_f \) denote the operator on \( W^2(D, H) \) given by multiplication by \( f \). This has a spectral distribution of order 2, defined by the functional calculus
\[
\Phi_M : C_0^2(\mathbb{C}) \rightarrow \mathcal{L}(W^2(D, H)), \quad \Phi_M(f) = M_f.
\]
Therefore \( M_f \) is a scalar operator of order 2. Consider a bounded open disk \( D \) which contains \( \sigma(T) \) and the quotient space
\[
(1.1) \quad H(D) = W^2(D, H)/(T - z)W^2(D, H)
\]
endowed with the Hilbert space norm. We denote the class of a vector \( f \) or an operator \( A \) on \( H(D) \) by \( \widehat{f} \), respectively \( \widehat{A} \). Let \( M_z \) be the operator of multiplication by \( z \) on \( W^2(D, H) \). As noted above, \( M_z \) is a scalar of order 2 and has a spectral distribution \( \Phi \). Let \( S \equiv \widehat{M_z} \). Since \( (T - z)W^2(D, H) \) is invariant under every operator \( M_f, f \in C_0^2(\mathbb{C}) \), we infer that \( S \) is a scalar operator of order 2 with spectral distribution \( \Phi \). Consider the natural map \( V : H \rightarrow H(D) \) defined by \( V h = 1 \otimes \hat{h} \), for \( h \in H \), where \( 1 \otimes h \) denotes the constant function identically equal to \( h \). In [11], Putinar showed that if \( T \in \mathcal{L}(H) \) is a hyponormal operator then \( V \) is one-to-one and has closed range such that \( VT = SV \), and so \( T \) is subscalar of order 2.

An operator \( T \in \mathcal{L}(H) \) is said to be \( p \)-hyponormal, \( 0 < p \leq 1 \), if \( (T^*T)^p \geq (TT^*)^p \) where \( T^* \) is the adjoint of \( T \). If \( p = 1 \), \( T \) is hyponormal and if \( p = \frac{1}{2} \), \( T \) is called \emph{semi-hyponormal}. Semi-hyponormal operators were introduced by Xia (see [12]) and there are many works on general \( p \)-hyponormal operators ([1], [3], [5], [6], [9]).
Löwner-Heinz’s inequality. Let $A, B \in \mathcal{L}(H)$ be $A \geq B \geq 0$ and $p \in (0, 1]$. Then

$$A^p \geq B^p.$$ 

This inequality gives the following implications:

- hyponormality $\Rightarrow$ $p$-hyponormality ($\frac{1}{2} < p < 1$)
- $\Rightarrow$ semi-hyponormality
- $\Rightarrow$ $p$-hyponormality ($0 < p < \frac{1}{2}$).

It is well known that all the above implications are strict (see [6] and [12]).

In this paper we prove that if $T$ is a $k$-th root of a $p$-hyponormal operator when $T$ is compact or $T^n$ is normal for some integer $n > k$, then $T$ is (generalized) scalar, and that if $T$ is a $k$-th root of a semi-hyponormal operator and has the property $\sigma(T)$ is contained in an angle $< 2\pi/k$ with vertex in the origin, then $T$ is subscalar. These results extend [8, Theorem 4.3].

2. Results

Theorem 2.1. Let $T$ be a $k$-th root of a $p$-hyponormal operator. If $T$ is compact or $T^n$ is normal for some integer $n > k$, then $T$ is a (generalized) scalar operator.

Proof. First, we claim that $T^k$ is normal. If $T$ is compact, then that is straightforward, since $T^k$ is compact and a compact $p$-hyponormal operator is normal ([5, Theorem 2]). If $T^n$ is normal for some integer $n > k$, then there exists an $n$-nilpotent operator $T_0$ and an operator $T_1$ which is quasi-similar to a normal operator $N$ with $\sigma(T_1) = \sigma(N)$ such that $T = T_0 \oplus T_1$ [7, Theorem 3.1]. Consider $T^k = T_0^k \oplus T_1^k$. Clearly, $T_0^k$ is nilpotent. Since the only quasi-nilpotent $p$-hyponormal operator is the zero operator, $T_0^k = 0$. Let $X$ be a quasi-affinity such that $T_1^k X = X N^k$. Applying the Putnam-Fuglede theorem for $p$-hyponormal operators ([3, Theorem 7]), it follows that $T_1^k$ is normal. Hence $T^k$ is normal. Now it follows from [2] and [7, Remark, p.141] that $T$ is a (generalized) scalar operator. $\square$
COROLLARY 2.2. Let $T$ be a $k$-th root of a $p$-hyponormal operator. If $T$ is compact or $T^n$ is normal for some integer $n > k$, then $T$ has hyperinvariant subspaces.

Proof. Since $T$ is a (generalized) scalar operator by Theorem 2.1, $T$ is decomposable. Hence $T$ has hyperinvariant subspaces. \qed

THEOREM 2.3. Let $T$ be a $k$-th root of a semi-hyponormal operator and have the property $\sigma(T)$ is contained in an angle $< 2\pi/k$ with vertex in the origin. Then $T$ is subscalar of order 2.

We need the following lemmas to prove Theorem 2.3.

LEMMA 2.4. ([11, Proposition 2.1]) For every bounded disk $D$ in $\mathbb{C}$ there is a constant $C_D$, such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D, H)$ we have

$$\|(I - P)f\|_{2,D} \leq C_D \left( \|(T - z)^* \partial f\|_{2,D} + |(T - z)^* \partial^2 f\|_{2,D} \right),$$

where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

LEMMA 2.5. ([9, Lemma 4]) Let $T$ be a semi-hyponormal. Then for a $z \in \mathbb{C}$ and a sequence $f_n \in L^2(D, H)$

$$\lim_{n \to \infty} \|(T - z)f_n\|_{2,D} = 0 \implies \lim_{n \to \infty} \|(T - z)^* f_n\|_{2,D} = 0.$$

Proof of Theorem 2.3. Consider a bounded disk $D$ which contains $\sigma(T)$ and $H(D)$ as in (1.1). Then we define the map $V : H \to H(D)$ by

$$V h = \hat{1} \otimes h \left( \equiv 1 \otimes h + \overline{(T - z)W^2(D, H)} \right),$$

where $1 \otimes h$ denotes the constant function sending any $z \in D$ to $h$. As mentioned in section 1, to prove Theorem 2.3 it suffices to show that $V$ is one-to-one and has closed range.

Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

$$\lim_{n \to \infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0. \quad (2.1)$$

Then equation (2.1) implies

$$\lim_{n \to \infty} \|(T - z)\partial^i f_n\|_{2,D} = 0 \quad \text{for} \quad i = 1, 2. \quad (2.2)$$
From (2.2), we get
\[ \lim_{n \to \infty} \| (T^k - z^k) \bar{\partial} f_n \|_{2,D} = 0 \quad \text{for} \quad i = 1, 2. \]

Since \( T^k \) is semi-hyponormal, by Lemma 2.5 we have
\[ (2.3) \quad \lim_{n \to \infty} \| (T^{*k} - z^k) \bar{\partial} f_n \|_{2,D} = 0. \]

Now we claim that
\[ (2.4) \quad \lim_{n \to \infty} \| (T - z)^* \bar{\partial} f_n \|_{2,D} = 0. \]

Indeed, since \( T - z \) is invertible for \( z \in D \setminus \sigma(T) \), the equation (2.2) implies that
\[ \lim_{n \to \infty} \| \bar{\partial} f_n \|_{2,D \setminus \sigma(T)} = 0. \]

Therefore,
\[ \lim_{n \to \infty} \| (T - z)^* \bar{\partial} f_n \|_{2,D \setminus \sigma(T)} = 0. \]

Also, since \( \sigma(T) \) is contained in an angle \( < \frac{2\pi}{k} \) with vertex in the origin, it is clear from the equation (2.3) that
\[ \lim_{n \to \infty} \| (T - z)^* \bar{\partial} f_n \|_{2,D} = 0. \]

Thus Lemma 2.4 and equation (2.4) imply
\[ \lim_{n \to \infty} \| (I - P)f_n \|_{2,D} = 0, \]

where \( P \) denotes the orthogonal projection of \( L^2(D,H) \) onto \( A^2(D,H) \).

Then by (2.1)
\[ \lim_{n \to \infty} \| (T - z)Pf_n + 1 \otimes h_n \|_{2,D} = 0. \]

Let \( \Gamma \) be a curve in \( D \) surrounding \( \sigma(T) \). Then for \( z \in \Gamma \)
\[ \lim_{n \to \infty} \| Pf_n(z) + (T - z)^{-1}(1 \otimes h_n) \| = 0, \quad \text{uniformly}. \]

Hence
\[ \lim_{n \to \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z) dz + h_n \right\| = 0. \]

But by Cauchy’s theorem,
\[ \int_{\Gamma} Pf_n(z) dz = 0. \]

Thus \( \lim_{n \to \infty} h_n = 0 \). Hence \( V \) is one-to-one and has closed range. This completes the proof. \( \square \)
Corollary 2.6. Let $T$ be a $k$-th root of a semi-hyponormal operator and have the property that $\sigma(T)$ is contained in an angle $< 2\pi/k$ with vertex in the origin. If $\sigma(T)$ has interior in the plane, then $T$ has a non-trivial invariant subspace.

Proof. The corollary follows from Theorem 2.3 and [4]. \hfill \Box

We say that an operator $T - z$ on the space $\mathcal{O}(D, H)$ has Bishop's property ($\beta$) if $T - z$ is one-to-one and has closed range for every disc $D$. Since every subscalar operator has Bishop's property ($\beta$) ([10]), from Theorem 2.3 we have the following.

Corollary 2.7. Let $T$ be as in Corollary 2.6. Then $T$ has Bishop's property ($\beta$).

Does Theorem 2.3 hold for $k$-th roots of arbitrary $p$-hyponormal operators? A partial answer is given by the following corollary.

Corollary 2.8. Let $T$ be the $k$-th root of a $p$-hyponormal operator $A$, $0 < p < \frac{1}{2}$, such that $0 \notin \sigma(|A|^{\frac{1}{2}})$. If $\sigma(T)$ is contained in angle $< 2\pi/k$ with vertex in the origin, $T$ is subscalar of order 2.

Proof. Letting $A$ have the polar decomposition $A = U|A|$, it is seen that the operator $S = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ is a semi-hyponormal operator such that $S = |A|^{\frac{1}{2}}A|A|^{-\frac{1}{2}}$. Since $S = |A|^{\frac{1}{2}}T^k|A|^{-\frac{1}{2}} = (|A|^{\frac{1}{2}}T|A|^{-\frac{1}{2}})^k$, $S$ has a $k$-th root $T_0 = |A|^{\frac{1}{2}}T|A|^{-\frac{1}{2}}$ with spectrum contained in an angle $< 2\pi/k$ with vertex in the origin. Hence $T_0$, and so also $T$, is subscalar of order 2 by Theorem 2.3. \hfill \Box

References


BHAGWATI P. DUGGAL, 8 REDWOOD GROVE, NORTHFIELDS AVENUE EALING, LONDON W5 4SZ, UNITED KINGDOM  
E-mail: bpduggal@yahoo.co.uk

IN HO JEON, DEPARTMENT OF MATHEMATICS, EWHAA WOMEN’S UNIVERSITY, SEOUL 120–750, KOREA  
RECENT ADDRESS: DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151–747, KOREA  
E-mail: jih@math.ewha.ac.kr

EUNGIL KO, DEPARTMENT OF MATHEMATICS, EWHAA WOMEN’S UNIVERSITY, SEOUL 120–750, KOREA  
E-mail: eiko@ewha.ac.kr