NOTES ON GRADING MONOIDS

JE YOON LEE* AND CHUL HWAN PARK

Abstract. Throughout this paper, a semigroup $S$ will denote a torsion free grading monoid, and it is a non-zero semigroup with 0. The operation is written additively. The aim of this paper is to study semigroup version of an integral domain ([1],[3],[4] and [5]).

1. Introduction

Let $S$ be an additive commutative semigroup with identity (denoted by 0), that is a monoid. A monoid $S$ is said to be cancellative if $x+y = x+z$ with $x, y$ and $z \in S$ implies $y = z$ and $S$ is said to be torsion-free if $nx = ny$ with $x, y \in S$ and $n \in N$ implies $x = y$ where $N$ denotes the set of all positive integers. A cancellative monoid is called a grading monoid [10,p.112]. In this paper, a semigroup $S$ will denote a torsion free grading monoid, and it is a non-zero semigroup with 0. The operation is written additively.

A nonempty subset $B$ of a semigroup $S$ is called an additive system if it satisfies the following condition $b_1, b_2 \in B \Rightarrow b_1 + b_2 \in B$. For an additive system $B$, the quotient semigroup $S_B$ is defined as follows: $\{s - b \mid s \in S, b \in B\}$. Especially, if $B = S$, then the quotient semigroup $S_S = \{s_1 - s_2 \mid s_1, s_2 \in S\}$ is called the quotient group of $S$, and is denoted by $q(s) = G$. $T$ is called an oversemigroup of $S$ if $T$ is a subsemigroup of $G$ containing $S$.

An ideal of $S$ is a nonempty subset $I$ of $S$ such that $s + I = \{s + i \mid i \in I\} \subseteq I$ for each $s \in S$. For an ideal $I$, $J$ of $S$, set $I^{-1} = \{x \in G \mid x + I \subseteq I\}$.
Let $J$ be an ideal of $S$. Set $\text{rad}(J) = \{ s \in S | ns \in J \text{ for some } n \in \mathbb{Z}_0 \}$. $I$ is called a radical ideal of $S$ if $I = \text{rad}(I)$. For each $x \in S$, set $(x) = x + S$. An ideal of $S$ is principal if $I = (x)$. An ideal $P$ of $S$ is prime if $x + y \in P$ implies $x \in P$ or $y \in P$ for $x, y \in S$. An element $s$ of $S$ is called a unit if $s + u = 0$ for some $u \in S$. Also set $M = \{ m \in S | m \text{ is a non-unit element of } S \}$. Then $M$ is the unique maximal ideal of $S$. A semigroup $S$ is called valuation semigroup if either $\alpha \in S$ or $-\alpha \in S$ for each $\alpha \in G$. Throughout this paper, we may refer to [6],[7],[8] and [9].

2. Results

A semigroup $S$ is called seminormal semigroup if for each $x \in G$ such that there is a positive integer $n$ with $nx \in S$ for all $m \geq n$ then $x \in S$, or equivalently, if $2\alpha, 3\alpha \in S$ for $\alpha \in G$ then $\alpha \in G$(cf.[3],[6],[8] and [9]).

**Theorem 2.1.** Let $S$ be a seminormal semigroup with quotient group $G$ and $I$ be an ideal of $S$. Then $(\text{rad}(I) : \text{rad}(I))$ is a seminormal and $(\text{rad}(I) : \text{rad}(I)) = \{ x \in G | nx \in (S : I) \text{ for all } n \geq 1 \}$.

**Proof.** Clearly $(\text{rad}(I) : \text{rad}(I))$ is a subsemigroup of $(S : I)$. We first show that $(\text{rad}(I) : \text{rad}(I)) = \{ x \in G | nx \in (S : I) \text{ for all } n \geq 1 \}$. Let $x \in G$ such that $nx \in (S : I)$ for all $m \geq 1$ and $a \in \text{rad}(I)$. Then $na \in I$ for some positive integer $n$. Hence, for all $m \geq n$, $m(x + a) \in S$. By seminormality of $S$ we have $x + a \in S$. Also $(n+1)(x + a) = (n+1)x + na + a \in (S : I) + I + \text{rad}(I) \subseteq \text{rad}(I)$. Thus $x + a \in \text{rad}(I)$. Therefore $x \in (\text{rad}(I) : \text{rad}(I))$. The converse inclusion is clear. Finally we will show that $(\text{rad}(I) : \text{rad}(I))$ is seminormal. Let $nx \in (\text{rad}(I) : \text{rad}(I))$ for all $n \geq 1$. Since $nx \in (R : I)$ for all $n \geq 1$, we have $x \in (\text{rad}(I) : \text{rad}(I))$. Therefore $(\text{rad}(I) : \text{rad}(I))$ is seminormal semigroup.

**Corollary 2.2.** If $S$ is a seminormal semigroup and $I$ is a radical ideal, then the semigroup $(I : I)$ is seminormal semigroup.

**Theorem 2.3.** Let $S$ be a seminormal semigroup with quotient group $G$ and $I$ be a prime ideal of $S$. Then $P^{-1}$ is a subsemigroup of $G$ if and only if $P^{-1} = (P : P)$. 
Proof. Since one direction is trivial, we assume $P^{-1}$ is a subsemigroup of $G$ and $P^{-1} \neq (P : P)$. Let $J = (S : P^{-1})$. We claim $J = P$. Since $P + P^{-1} \subseteq S$, we have $P \subseteq J$. Let $a \in J$, then $a + P^{-1} \subseteq S \subseteq (P : P)$ and so $(a + P^{-1}) + P = a + (P^{-1} + P) \subseteq P$ since $P$ is prime and $P + P^{-1} \not\subseteq P$, $a \in P$. Whence, $J = P$. This is contradictsthe fact that $P^{-1} \neq (P : P)$.

**Theorem 2.4.** Let $S$ be a seminormal semigroup with quotient group $G$ and $I$ be an ideal of $S$ for which $I^{-1}$ is a semigroup. Then

1. $\text{rad}(I)^{-1} = (\text{rad}(I) : \text{rad}(I))$;
2. $I^{-1} = (\text{rad}(I) : I) = (J : I)$ for each prime $I \subseteq J$.

Proof. (1) since $\text{rad}(I) \subseteq S$, $(\text{rad}(I) : \text{rad}(I)) \subseteq \text{rad}(I)^{-1}$. To prove $\text{rad}(I)^{-1} \subseteq (\text{rad}(I) : \text{rad}(I))$, let $x \in (\text{rad}(I))^{-1}$ and $a \in \text{rad}(I)$. Then $na \in I$ for some positive integer $n$. Since $(\text{rad}(I))^{-1} \subseteq I^{-1}$ and $I^{-1}$ is a semigroup, we have $2nx \in I^{-1}$. Hence $2nx + na \in I^{-1} + I \subseteq S$, whence $2n(x + a) = (2nx + na) + na \in S + I$. Since $x + a \in S$, this implies that $x + a \in \text{rad}(I)$. Therefore $\text{rad}(I)^{-1} = (\text{rad}(I) : \text{rad}(I))$.

(3) It is enough to establish the inclusion $I^{-1} = (J : I)$, for each prime $I \subseteq J$. Let $x \in I^{-1}$. Since $I^{-1}$ is a semigroup, we have $2x \in I^{-1}$, it follows that $2x + I \subseteq S$ and $2(x + I) = (2x + I) + I \subseteq I \subseteq J$. Since $x + I \subseteq S$, we have $x + I \subseteq J$. Thus $I + I^{-1} \subseteq J$, $I^{-1} \subseteq (J : I) \subseteq (S : I) = I^{-1}$, we have $I^{-1} = (J : I)$. Since this is true for each $J$, we have $I + I^{-1} \subseteq \text{rad}(I)$. Therefore $I^{-1} = (\text{rad}(I) : \text{rad}(I))$.

A prime ideal $P$ of $S$ is called strongly prime if $x, y \in G$ and $x + y \in P$ implies that $x \in P$ or $y \in P$. $S$ is called pseudo-valuation semigroup if every prime ideal of $S$ is strongly prime[3].

The following Lemma is useful restatement of definition of strongly prime ideal in semigroup $S$.

**Lemma 2.5.** Let $P$ be a prime ideal of a semigroup with quotient group $G$. Then $P$ is strongly prime ideal if and only if $-x + P \subseteq P$ for each $x \in G \setminus S$.

Proof. Suppose that $I$ is strongly prime. Let $x \in G \setminus S$ and $p \in P$. Since $p = (p - x) + x \in P$ and $P$ is strongly prime ideal, we have $(p - x) \in P$ or $x \in P$. Since $x \not\in S$ we must have $p - x \in P$. Thus
$-x + P \subseteq P$. To prove opposite implication, assume $-x + P \subseteq P$
whenever $x \in G \setminus S$, and let $a + b \in P$. If $a, b \in S$ there is nothing to
prove. Hence we may assume $a \notin S$ so that $-a + P \subseteq P$ and $b = -a + a + b \in P$. T

**Theorem 2.6.** Let $P$ be an ideal in semigroup $S$ with quotient
group $G$. Then following statements are equivalent.

1. $P$ is strongly prime;
2. $G \setminus P$ additive system;
3. $P$ is prime and is comparable to each (principal) fractional ideal
   of $S$;
4. $P : P$ is valuation semigroup with maximal ideal $P$;
5. $P$ is a prime ideal in some valuation oversemigroup of $S$.

**Proof.** Clearly (1) and (2) are equivalent. (1) $\Rightarrow$ (3): Suppose that $P$
is strongly prime ideal. Let $x \in G \setminus P$. Then $x + (-x + P) \subseteq P$. Since $P$
is strongly prime, $-x + P \subseteq P$ and hence $P \subseteq x + P \subseteq x + S$.
(3) $\Rightarrow$ (2): Let $x, y \in S$. Suppose that $x + y \in P$. Now $x \in S$ implies
$P \subseteq X + S$, so $-x + P \subseteq S$. Then $y = -x + (x + y) \in -x + P \subseteq S$.
Similarly, $x \in S$. But then we get contradiction that either $x \in P$ or $y \in P$
for which $P$ is prime. (1) $\Rightarrow$ (4). Suppose that $x \in G \setminus P$. From the
proof of (1) $\Rightarrow$ (3), we see that $-x + P \subseteq P$, and hence $-x \notin (P : P)$.
From this it easily follows that $P : P$ is a valuation semigroup with $P$
as its maximal ideal. Finally, the implications $4 \Rightarrow 5$ and $5 \Rightarrow$
(1) are obvious.

In the following Theorem we characterize pseudo-valuation semigroup with the maximal ideals

**Theorem 2.7.** Let $(S, M)$ be a semigroup. The following statement are equivalent:

1. $S$ is pseudo-valuation semigroup
2. For each pair $I, J$ of ideals of $S$, either $I \subseteq J$ or $M + J \subseteq M + I$;
3. For each pair $I, J$ of ideals of $S$, either $I \subseteq J$ or $M + J \subseteq I$;
4. $M$ is strongly prime.

**Proof.** (1) $\Rightarrow$ (2): Assume $I \nsubseteq J$. Let $a \in I \setminus J$. For each $b \in J$ we
have $a - b \notin S$, so that $-(a - b) + M \subseteq M$ and $M + b \subseteq M + a \subseteq M + I$.
It follows that $M + J \subseteq M + I$. 
(2) $\Rightarrow$ (3) straightforward.

(3) $\Rightarrow$ (4) Let $a, b \in S$ with $a - b \notin S$. By Lemma 2.5, this is enough to show that $-(a - b) + M \subseteq M$. Since $a - b \notin S$ we have $(a) \nsubseteq (b)$ whence $M + b \subseteq (a)$ and $-(a - b) + M \subseteq S$. If $-(a - b) + M = S$ then $M = S + (a - b)$ and $a - b \notin S$, this is a contradiction. Hence $-(a - b) + M \subseteq M$. Therefore $M$ is strongly prime ideal.

(4) $\Rightarrow$ (1) Let $x \in G$, $x \notin S$, and let $P$ be a prime ideal. Again, by Lemma 2.5, it is enough to show that $-x + P \subseteq P$. Let $p \in P$. Since $P \subseteq M$, we have $-x + p \in M$. Hence $-x + p - x \in M$, whence $2(-x+p)=(-x+p)+(-x+p)\in P$. Since $P$ is prime and $-x+p \in S$, we therefore have $-x+p \in P$.

REFERENCES


Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea
E-mail: jylee@ulsan.ac.kr
Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea
E-mail: chpark@ulsan.ac.kr