MULTIPLICITY RESULTS AND THE M-PAIRS OF TORUS-SHERE
VARIATIONAL LINKS OF THE STRONGLY INDEFINITE FUNCTIONAL

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ABSTRACT. Let \( I \in C^{1,1} \) be a strongly indefinite functional defined on a Hilbert space \( H \). We investigate the number of the critical points of \( I \) when \( I \) satisfies two pairs of Torus-Sphere variational linking inequalities and when \( I \) satisfies \( m \) \((m \geq 2)\) pairs of Torus-Sphere variational linking inequalities. We show that \( I \) has at least four critical points when \( I \) satisfies two pairs of Torus-Sphere variational linking inequality with \((P.S.)^*_c\) condition. Moreover we show that \( I \) has at least \( 2m \) critical points when \( I \) satisfies \( m \) \((m \geq 2)\) pairs of Torus-Sphere variational linking inequalities with \((P.S.)^*_c\) condition. We prove these results by use of Theorem 2.2 (Theorem 1.1 in [1]) and the critical point theory on the manifold with boundary.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let \( I \in C^{1,1} \) be a strongly indefinite functional defined on a Hilbert Space \( H \). In this paper, we investigate the number of the critical points of \( I \) when \( I \) satisfies \( m \) \((m \geq 2)\) pairs of Torus-Sphere variational linking inequalities and \((P.S.)^*_c\) condition, \( m \in \mathbb{N} \). We show that \( I \) has at least two critical points each when \( I \) satisfies each one pair of Torus-Sphere variational linking inequality and \((P.S.)^*_c\) condition. We prove these results by use of Theorem 2.2 and the critical point theory on the manifold with boundary. In the case that \( I \) is not strongly indefinite functional Marino, A., Micheletti, A.M., Pistoia, Schechter, M., Tintarev, K., and Rabinowitz, P., proved in Theorem (3.4) of [4], [7] and [8] a theorem of existence of two solutions when \( I \) satisfies one pair of Sphere-Torus variational linking inequality by the mountain pass theorem and degree theory. Marino, A., Micheletti, A.M. and Pistoia, A. proved in Theorem (8.4) of [5] a theorem of existence of three solutions when \( I \) satisfies two pairs of Sphere-Torus variational linking inequalities and \((P.S.)^*_c\) condition by the mountain pass theorem and degree theory. In this paper we obtain the following results for the strongly indefinite functional case:

Received by the editors November 10, 2008.

2000 Mathematics Subject Classification. 35A15.

Key words and phrases. Strongly indefinite functional, Torus-Sphere variational linking inequality, \((P.S.)^*_c\) condition, critical point theory, limit relative category.

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Theorem 1.1. (Two pairs of Torus-Sphere variational links) Let $H$ be a Hilbert space with a norm $\| \cdot \|$, which is topological direct sum of the four subspaces $X_0$, $X_1$, $X_2$ and $X_3$. Let $I \in C^{1,1}(H, R)$ be a strongly indefinite functional. Assume that

1. $\dim X_i < \infty$, $i = 1, 2$;
2. There exist a small number $\rho > 0$, $r^{(1)} > 0$ and $R^{(1)}$ such that
   $$r^{(1)} < R^{(1)} \text{ and } \sup_{\Sigma_{r^{(1)}}(S_1(\rho), X_0)} I < \inf_{S_{r^{(1)}}(X_1 \oplus X_2 \oplus X_3)} I,$$

   where $S_1(\rho) = \{ u \in X_1 | \| u \| = \rho \}$;
3. There exist a small number $\rho > 0$, $r^{(2)} > 0$ and $R^{(2)} > 0$ such that
   $$r^{(2)} < R^{(2)} \text{ and } \sup_{\Sigma_{r^{(2)}}(S_2(\rho), X_0 \oplus X_1)} I < \inf_{S_{r^{(2)}}(X_2 \oplus X_3)} I,$$

   where
   $$S_{r^{(2)}}(X_2 \oplus X_3) = \{ u \in X_2 \oplus X_3 | \| u \| = r^{(2)} \},$$
   $$\Sigma_{r^{(2)}}(S_2(\rho), X_0 \oplus X_1) = \{ u = u_0 + u_1 + u_2 | u_2 \in S_2(\rho), u_0 \in X_0, u_1 \in X_1, \| u_2 \| = \rho, 1 \leq \| u_0 + u_1 \| \leq R^{(2)} \};$$
4. $R^{(2)} < R^{(1)} \Rightarrow \Delta^{(2)}(S_2(\rho), X_0 \oplus X_1) \subset \Sigma_{r^{(1)}}(S_1(\rho), X_0)$;
5. $\beta^{(1)} = \sup_{\Delta_{r^{(1)}}(S_1(\rho), X_0)} I < +\infty$, where
   $$\Delta_{r^{(1)}}(S_1(\rho), X_0) = \{ u = u_0 + u_1 | u_1 \in S_1(\rho), u_0 \in X_0, \| u_1 \| = \rho, 1 \leq \| u_0 + u_1 \| \leq R^{(1)} \};$$
6. (P.S.) condition holds for any $c \in [\alpha^{(1)}, \beta^{(1)}]$, where
   $$\alpha^{(1)} = \inf_{S_{r^{(2)}}(X_2 \oplus X_3)} I;$$
7. There exists one critical point $e$ in $X_0 \oplus X_3$ with $I(e) < \alpha^{(1)}$. Then there exist at least four distinct critical points except $e$, $u_j^1$, $j = 1, 2$, in $X_1$, $u_j^2$, $j = 1, 2$ in $X_2$, of $I$ with
   $$\alpha^{(1)} = \inf_{S_{r^{(2)}}(X_2 \oplus X_3)} I \leq I(u_j^2) \leq \sup_{\Delta_{r^{(2)}}(S_2(\rho), X_0 \oplus X_1)} I \leq \sup_{\Sigma_{r^{(1)}}(S_1(\rho), X_0)} I < \inf_{S_{r^{(1)}}(X_1 \oplus X_2 \oplus X_3)} I \leq I(u_j^1) \leq \sup_{\Delta_{r^{(1)}}(S_1(\rho), X_0)} I = \beta^{(1)} < +\infty.$$
**Theorem 1.2.** (m pairs of Torus-Sphere variational links) Let $H$ be a Hilbert space with a norm $\| \cdot \|$, which is a topological direct sum of the $m + 2$ subspaces $X_0, X_1, \cdots, X_m$ and $X_{m+1}$. Let $I \in C^{1,1}(H, R)$ be a strongly indefinite functional. Assume that

1. $\dim(X_i) < \infty$, $i = 1, \cdots, m$;
2. There exist a small number $\rho > 0$, $r(k) > 0$ and $R(k) > 0$ such that
   $$r(k) < R(k)$$
   and
   $$\sup_{\Sigma_{R(k)}(S_k(\rho), X_0 \oplus \cdots \oplus X_{k-1})} I < \inf_{S_r(k)(X_k \oplus \cdots \oplus X_{m+1})} I,$$
3. $R(k) < R(k-1) \Rightarrow$
   $$\Delta_{R(k)}(S_k(\rho), X_0 \oplus \cdots \oplus X_{k-1}) \subset \Sigma_{R(k-1)}(S_{k-1}(\rho), X_0 \oplus \cdots \oplus X_{k-2}),$$
4. $\beta^{(m)} = \sup_{\Delta_{R(1)}(S_1(\rho), X_0)} I < +\infty$;
5. $(P.S.)^c_\alpha$ condition holds for any $c \in [\alpha^{(m)}, \beta^{(m)}]$, where
   $$\alpha^{(m)} = \inf_{S^{(m)}(X_{m+1})} I;$$
6. There exists one critical points $e$ in $X_0 \oplus X_{m+1}$ with $I(e) < \alpha^{(m)}$.

Then there exist at least $2m$ distinct critical points except $e, u^k_j, j = 1, 2, \text{ in } X_k, 1 \leq k \leq m$, of $I$ with

$$\alpha^{(m)} = \inf_{S^{(m)}(X_{m+1})} I \leq I(u^1_j) \leq \sup_{\Sigma_{R(m-1)}(S_{m-1}(\rho), X_0 \oplus \cdots \oplus X_{m-2})} I \leq \cdots \leq \sup_{\Sigma_{R(k)}(S_k(\rho), X_0 \oplus \cdots \oplus X_{k-1})} I \leq \inf_{S^{(k)}(X_k \oplus \cdots \oplus X_{m+1})} I \leq I(u^k_j) \leq \sup_{\Sigma_{R(k-1)}(S_{k-1}(\rho), X_0 \oplus \cdots \oplus X_{k-2})} I \leq \cdots \leq \inf_{S^{(1)}(X_1 \oplus \cdots \oplus X_{m+1})} I \leq I(u^1_j) \leq \sup_{\Delta_{R(1)}(S_1(\rho), X_0)} I = \beta^{(m)}.$$

For the proofs of the main results we use Theorem 2.2 and the critical point theory on the manifold with boundary. Since the functional $I$ is strongly indefinite functional, it is convenient to use the notion of the limit relative category instead of the relative category and the $(P.S.)^c_\alpha$ condition which is a suitable version of the Palais-Smale condition. We restrict the functional $I$ to the manifold $C_k$ with boundary, where $C_k$ is introduced in section 4. We study the geometry and topology of the sub-levels of $I$ and $I_k$ and investigate the limit relative category of the
sub-level sets of $\tilde{I}_k$ and $(P.S.)^*_c$ condition in $C_k$. By Theorem 2.2 and the critical point theory on the manifold with boundary, we obtain at least two distinct critical points of $\tilde{I}_k$, in each linked subspace $X_k, k = 1, \ldots, m$. So we obtain at least two distinct critical points of $I$, in each linked subspace $X_k, k = 1, \ldots, m$.

2. CRITICAL POINT THEORY ON THE MANIFOLD WITH BOUNDARY

Now, we consider the critical point theory on the manifold with boundary. Let $H$ be a Hilbert space and $M$ be the closure of an open subset of $H$ such that $M$ can be endowed with the structure of $C^2$ manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where $W$ is an open set containing $M$. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for $f$ on $M$. Since the functional $I(u)$ is strongly indefinite, the notion of the $(P.S.)^*_c$ condition and the limit relative category (see [2]) is a useful tool for the proof of the main theorems.

**Definition 2.1.** If $u \in M$, the lower gradient of $f$ on $M$ at $u$ is defined by

$$\text{grad}_M^L f(u) = \begin{cases} \nabla f(u) & \text{if } u \in \text{int}(M), \\ \nabla f(u) + \langle \nabla f(u), \nu(u) \rangle - \nu(u) & \text{if } u \in \partial M, \end{cases}$$ (2.1)

where we denote by $\nu(u)$ the unit normal vector to $\partial M$ at the point $u$, pointing outwards. We say that $u$ is a lower critical for $f$ on $M$, if $\text{grad}_M^L f(u) = 0$.

Let $(H_n)_n$ be a sequence of closed finite dimensional subspace of $H$ with $\dim H_n < +\infty$, $H_0 \subset H_{n+1}$, $\cup_{n \in N} H_n$ is dense in $H$.

Let $M_n = M \cap H_n$, for any $n$, be the closure of an open subset of $H_n$ and has the structure of a $C^2$ manifold with boundary in $H_n$. We assume that for any $n$ there exists a retraction $r_n : M \to M_n$. For given $B \subset H$, we will write $B_n = B \cap H_n$.

**Definition 2.2.** Let $c \in R$. We say that $f$ satisfies the $(P.S.)^*_c$ condition with respect to $(M_n)_n$, on the manifold with boundary $M$, if for any sequence $(k_n)_n$ in $N$ and any sequence $(u_n)_n$ in $M$ such that $k_n \to \infty, \forall n, f(u_n) \to c, \text{grad}_{M_{k_n}} f(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $\text{grad}_M^L f(u) = 0$.

Let $Y$ be a closed subspace of $M$.

**Definition 2.3.** Let $B$ be a closed subset of $M$ with $Y \subset B$. We define the relative category $\text{cat}_{M,Y}(B)$ of $B$ in $(M,Y)$, as the least integer $h$ such that there exist $h + 1$ closed subsets $U_0, U_1, \ldots, U_h$ with the following properties:

- $B \subset U_0 \cup U_1 \cup \ldots \cup U_h$,
- $U_0, \ldots, U_h$ are contractible in $M$;
- $Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0, 1] \to M$ such that
  
  $$F(x, 0) = x \quad \forall x \in U_0,$$
  $$F(x, t) \in Y \quad \forall x \in Y, \forall t \in [0, 1],$$
  $$F(x, 1) \in Y \quad \forall x \in U_0.$$

If such an $h$ does not exist, we say that $\text{cat}_{M,Y}(B) = +\infty$. 

**Definition 2.4.** Let \((X, Y)\) be a topological pair and \((X_n)_n\) be a sequence of subsets of \(X\). For any subset \(B\) of \(X\) we define the limit relative category of \(B\) in \((X, Y)\), with respect to \((X_n)_n\), by

\[
\text{cat}^\ast_{(X, Y)}(B) = \lim_{n \to \infty} \sup_{i} \text{cat}_{(X_n, Y_n)}(B_n).
\]

Let \(Y\) be a fixed subset of \(M\). We set

\[
B_i = \{B \subset M| \text{cat}^\ast_{(M, Y)}(B) \geq i\},
\]

\[
c_i = \inf_{B \in B_i} \sup_{x \in B} f(x).
\]

We have the following multiplicity theorem, which was proved in [6].

**Theorem 2.1.** Let \(i \in N\) and assume that

1. \(c_i < +\infty\),
2. \(\sup_{x \in Y} f(x) < c_i\),
3. the \((P.S.)_c^\ast\) condition with respect to \((M_n)_n\) holds.

Then there exists a lower critical point \(x\) such that \(f(x) = c_i\). If

\[
c_i = c_{i+1} = \ldots = c_{i+k-1} = c,
\]

then

\[
\text{cat}_M(\{x \in M| f(x) = c, \text{grad}_M f(x) = 0\}) \geq k.
\]

Jung and Choi [1] prove the following theorem which will be used to prove the main results:

**Theorem 2.2.** (One pair of Torus-Sphere variational link) Let \(H\) be a Hilbert space with a norm \(\| \cdot \|\), which is topological direct sum of the three subspaces \(X_0, X_1\) and \(X_2\). Let \(I \in C^{1,1}(H, R)\) be a strongly indefinite functional. Assume that

1. \(\dim X_1 < +\infty\);
2. There exist a small number \(\rho > 0, r > 0\) and \(R > 0\) such that \(r < R\) and

\[
\sup_{\Sigma_R(S_1(\rho), X_0)} I < \inf_{S_r(X_1 \oplus X_2)} I,
\]

where

\[
S_1(\rho) = \{u \in X_1| \|u\| = \rho\},
\]

\[
S_r(X_1 \oplus X_2) = \{u \in X_1 \oplus X_2| \|u\| = r\},
\]

\[
B_r(X_1 \oplus X_2) = \{u \in X_1 \oplus X_2| \|u\| \leq r\},
\]

\[
\Sigma_R(S_1(\rho), X_0) = \{u = u_1 + u_2| u_1 \in S_1(\rho), u_2 \in X_0, \|u_1\| = \rho, 1 \leq \|u_1 + u_2\| = R\} \cup \{u = u_1 + u_2| u_1 \in S_1(\rho), \|u_1\| = \rho, 1 \leq \|u_1\| \leq R\},
\]

\[
\Delta_R(S_1(\rho), X_0) = \{u = u_1 + u_2| u_1 \in S_1(\rho), u_2 \in X_0, \|u_1\| = \rho, 1 \leq \|u_1 + u_2\| \leq R\};
\]

3. \(\beta = \sup_{\Delta_R(S_1(\rho), X_0)} I < +\infty\).
(4) $(P.S.)_\alpha$ condition holds for any $c \in [\alpha, \beta]$ where
\[ \alpha = \inf_{S_r(X_1 \oplus X_2)} I; \]

(5) There exists one critical point $c$ in $X_0 \oplus X_2$ with $I(c) < \alpha$. Then there exist at least two distinct critical points except $c$, $u_i$, $i = 1, 2$, in $X_1$, of $I$ with
\[ \inf_{S_r(X_1 \oplus X_2)} I \leq I(u_i) \leq \sup_{\Delta_R(S_1(\rho), X_0)} I. \]

3. PROOF OF THEOREM 1.1

We will apply Theorem 2.2 to the case when $H$ is the topological direct sum of $X_0 \oplus X_1$, $X_2$ and $X_3$ and to the case when $H$ is the topological direct sum of $X_0$, $X_1$ and $X_2 \oplus X_3$. By the conditions (1), (2), (3), (4), we have that
\[ \alpha^{(1)} = \inf_{S_{r(2)}(X_2 \oplus \cdots \oplus X_m)} I \leq \sup_{\Delta_R(S_2(\rho), X_0 \oplus X_1)} I \leq \sup_{\Delta_R(S_{r(1)}(S_1(\rho), X_0))} I \]
\[ \alpha^{(2)} = \inf_{S_{r(1)}(X_1 \oplus \cdots \oplus X_m)} I \leq \sup_{\Delta_R(S_1(\rho), X_0)} I. \]

The condition (6) implies that $I$ satisfies $(P.S.)_\alpha$ condition for any $c$ with
\[ \inf_{S_{r(1)}(X_1 \oplus \cdots \oplus X_m)} I \leq c \leq \sup_{\Delta_R(S_{r(1)}(S_1(\rho), X_0))} I. \]

and $I$ also satisfies $(P.S.)_\gamma$ condition for any $\gamma$ with
\[ \inf_{S_{r(2)}(X_2 \oplus \cdots \oplus X_m)} I \leq \gamma \leq \sup_{\Delta_R(S_2(\rho), X_0 \oplus X_1)} I. \]

By the condition (5),
\[ \sup_{\Delta_R(S_1(\rho), X_0)} I = \beta < +\infty. \]

Now, we apply Theorem 2.2 to the case when $H$ is the topological direct sum of $X_0$, $X_1$ and $X_2 \oplus X_3$. In this case we set the smooth manifold $C^{(1)} = \{ u \in H | \|P_{X_1} u\| \geq 1 \}$.

$\psi^{(1)} : H \backslash (X_0 \oplus (X_2 \oplus X_3)) \to H$ by
\[ \psi^{(1)}(u) = u - \frac{P_{X_1} u}{\|P_{X_1} u\|} = P_{X_0 \oplus (X_2 \oplus X_3)} u + \left( 1 - \frac{1}{\|P_{X_1} u\|} \right) P_{X_1} u \]

and $I_1 = I \cdot \psi^{(1)} \in C^{1,1}_{loc}(C^{(1)}, H)$. Then by Theorem 2.2 with the conditions (1), (2), (4), (5), (7) and (3.2), I has at least two critical points $u_j^1$, $j = 1, 2$, in $X_1$, except $c$, with
\[ \inf_{S_{r(1)}(X_1 \oplus \cdots \oplus X_m)} I \leq I(u_j^1) \leq \sup_{\Delta_R(S_1(\rho), X_0)} I. \]
Next we apply Theorem 2.2 once more to the case when \( H \) is the topological direct sum of \( X_0 \oplus X_1, X_2 \) and \( X_3 \). In this case we set the smooth manifold
\[
C^{(2)} = \{ u \in H \mid \|P_{X_2}u\| \geq 1 \},
\]
\[\psi^{(2)} : H \setminus \{(X_0 \oplus X_1) \oplus X_3\} \to H \]
by
\[
\psi^{(2)}(u) = u - \frac{P_{X_2}u}{\|P_{X_2}u\|} = P_{(X_0 \oplus X_1) \oplus X_3}u + \left(1 - \frac{1}{\|P_{X_2}u\|}\right)P_{X_2}u
\]
and \( \tilde{I}_2 = I \cdot \psi^{(2)} \in C^{(2)}_{loc}(C^{(2)}, H) \). Then by Theorem 2.2 with the conditions (1), (3), (7), (3.3) and (3.4), I has at least two critical points, \( u_j^2 \), \( j = 1, 2 \), in \( X_2 \), except \( e \), with
\[
\inf_{S_j^{(2)}(X_2 \oplus \cdots \oplus X_{m+1})} I \leq I(u_j^2) \leq \sup_{\Delta_{R^{(2)}}(S_2(X_0 \oplus X_1))} I.
\]
Using the condition (4), we can combine (3.5) with (3.6). Then we have
\[
\alpha^{(1)} = \inf_{S_j^{(2)}(X_2 \oplus \cdots \oplus X_{m+1})} I \leq I(u_j^2) \leq \sup_{\Delta_{R^{(2)}}(S_2(X_0 \oplus X_1))} I \leq \sup_{\Sigma R^{(1)}(S_1(X_0))} I = \beta^{(1)}.
\]
Thus I has at least four nontrivial distinct critical points except \( e \). So we prove the theorem.

4. PROOF OF THEOREM 1.2

We will apply Theorem 2.2 \( m \) times to the case when \( H \) is the topological direct sum of \( X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1}, X_k, X_{k+1} \oplus \cdots \oplus X_{m+1} \), for each \( 1 \leq k \leq m \). The conditions (1), (2) and (3) implies that
\[
\alpha^{(m)} = \inf_{S_j^{(m)}(X_m \oplus X_{m+1})} I \leq \sup_{\Delta_{R^{(m)}}(S_m(X_0 \oplus \cdots \oplus X_{m-2}))} I \leq \sup_{\Sigma R^{(m-1)}(S_{m-1}(X_0 \oplus \cdots \oplus X_{m-2}))} I \leq \sup_{\Delta_R^{(k)}(S_k(X_0 \oplus \cdots \oplus X_{k-1}))} I \leq \sup_{\Sigma R^{(k-1)}(S_{k-1}(X_0 \oplus \cdots \oplus X_{k-2}))} I \leq \sup_{\Delta_R^{(1)}(S_1(X_0))} I \leq \sup_{\Sigma R^{(1)}(S_1(X_0))} I \leq \inf_{S_j^{(1)}(X_1 \oplus \cdots \oplus X_{m+1})} I \leq \sup_{\Delta_R^{(1)}(S_1(X_0))} I = \beta^{(m)}.
\]
The condition (5) implies that I satisfies $(P.S.)^{(k)}_{c^{(k)}}$ condition for any $c^{(k)}$ with
\[ \inf_{S_{c^{(k)}}(X_k \oplus \cdots \oplus X_{m+1})} I \leq c^{(k)} \leq \sup_{\Delta_{R(k)}(S_k(\rho), X_0 \oplus \cdots \oplus X_{k-1})} I, \quad k = 1, \ldots, m. \quad (4.2) \]
By the condition (4),
\[ \sup_{\Delta_{R(1)}(S_1(\rho), X_0)} I = \beta^{(m)} < +\infty, \quad (4.3) \]
We apply Theorem 2.2 to the case when $H$ is the topological direct sum of $X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1}$, $X_k$, $X_{k+1} \oplus \cdots \oplus X_{m+1}$, $k = 1, \ldots, m$. In this case we set
\[ C^{(k)} = \{ u \in H | \|P_{X_k}u\| \geq 1 \}, \quad k = 1, \ldots, m. \]
$\psi^{(k)} : H \setminus \{(X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1}) \oplus (X_{k+1} \oplus \cdots \oplus X_{m+1})\} \to H$ by
\[ \psi^{(k)}(u) = u - \frac{P_{X_k}u}{\|P_{X_k}u\|} = P_{(X_0 \oplus \cdots \oplus X_{k-1}) \oplus (X_{k+1} \oplus \cdots \oplus X_{m+1})}u + \left(1 - \frac{1}{\|P_{X_k}u\|}\right) P_{X_k}u, \]
$k = 1, \ldots, m$, and
\[ \tilde{I}_k = I \cdot \psi^{(k)} \in C_{loc}^{1,1}(C^{(k)}, H), \quad k = 1, \ldots, m. \]
Then by Theorem 2.2 with the conditions (1), (2), (3), (5), (6), (4.2) and (4.3), I has at least two critical points $u_j^k$, $j = 1, 2$, in $X_k$, except $e$ $k = 1, \ldots, m$ with
\[ \inf_{S_{c^{(k)}}(X_k \oplus \cdots \oplus X_{m+1})} I \leq I(u_j^k) \leq \sup_{\Delta_{R(k)}(S_k(\rho), X_0 \oplus \cdots \oplus X_{k-1})} I \leq \sup_{\Sigma_{R(k-1)}(S_{k-1}(\rho), X_0 \oplus \cdots \oplus X_{k-2})} \inf_{S_{c^{(k)}}(X_{k-1} \oplus \cdots \oplus X_{m+1})} I. \quad (4.4) \]
Using the condition (3), we can combine (4.4) for all $k = 1, \ldots, m$. So we have
\[ \alpha^{(m)} = \inf_{S_{c^{(m)}}(X_m \oplus X_{m+1})} I \leq I(u_j^m) \leq \sup_{\Delta_{R(m)}(S_m(\rho), X_0 \oplus \cdots \oplus X_{m-1})} I \leq \sup_{\Sigma_{R(m-1)}(S_{m-1}(\rho), X_0 \oplus \cdots \oplus X_{m-2})} \inf_{S_{c^{(m)}}(X_{m-1} \oplus \cdots \oplus X_{m+1})} I \leq \sup_{\Delta_{R(k)}(S_k(\rho), X_0 \oplus \cdots \oplus X_{k-1})} I \leq \sup_{\Sigma_{R(k-1)}(S_{k-1}(\rho), X_0 \oplus \cdots \oplus X_{k-2})} \inf_{S_{c^{(k)}}(X_{k-1} \oplus \cdots \oplus X_{m+1})} I \leq \sup_{\Sigma_{R(1)}(S_1(\rho), X_0)} I \leq \inf_{S_{c^{(1)}}(X_1 \oplus \cdots \oplus X_{m+1})} I \leq I(u_j^1) \leq \sup_{\Delta_{R(1)}(S_1(\rho), X_0)} I = \beta^{(m)}. \]
Thus I has at least $2m$ distinct critical points except $e$. Thus we prove the theorem.
References

[1] T. Jung and Q. H. Choi, The number of the critical points of the strongly indefinite functional with one pair of the Torus-Sphere variational linking sublevels, To be appeared in Korean J. Math..


