DOMINATION IN GRAPHS WITH MINIMUM DEGREE SIX

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ABSTRACT. A set $D$ of vertices of a graph $G = (V(G), E(G))$ is called a dominating set if every vertex of $V(G) - D$ is adjacent to at least one element of $D$. The domination number of $G$, denoted by $\gamma(G)$, is the size of its smallest dominating set. Haynes et al.[5] present a conjecture: For any graph $G$ with $\delta(G) \geq k$, $\gamma(G) \leq \frac{k - 1}{3k - 1}n$. When $k \neq 6$, the conjecture was proved in [7], [8], [10], [12] and [13] respectively. In this paper we prove that every graph $G$ on $n$ vertices with $\delta(G) \geq 6$ has a dominating set of order at most $\frac{6}{17}n$. Thus the conjecture was completely proved.

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1. Introduction

The graphs considered here are finite, undirected, and simple. The set of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. The minimum degree of graph $G$ is denoted by $\delta(G)$. A set $D$ of vertices of a graph $G$ is called a dominating set if every vertex of $V(G) - D$ is adjacent to at least one element of $D$. The domination number of $G$, denoted by $\gamma(G)$, is the size of its smallest dominating set. It has been proved [4] that the decision problem corresponding to the domination number for arbitrary graphs is NP-complete. Thus, the exploration of lower and upper bounds for the domination number as sharp as possible is of great significance. In fact, many results on upper bounds on the domination number in terms of some basic parameters such as the numbers of vertices and edges, the minimum and maximum degree and so on, have been obtained. For a survey, we refer the reader to [5]. Haynes et al.[5] present a conjecture:

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Conjecture 1. For any graph $G$ with $\delta(G) \geq k$, $\gamma(G) \leq \frac{k}{3k-1} n$.

For $\delta(G) \geq 1, 3, 4, 5$, Conjecture 1 was partially proved in [8], [10], [12] and [13] respectively. For $\delta(G) \geq 2$, McCraig and Shepherd [7] proved that $\gamma(G) \leq \frac{2n}{5}$ except for seven graphs. For $\delta(G) \geq 7$, Caro and Roditty (see [1], [2]) gave the following better bound. For any graph $G$,

$$\gamma(G) \leq n \left[ 1 - \delta \left( \frac{1}{\delta + 1} \right)^{1+\frac{1}{\delta}} \right].$$

Thus, the question remains open only for graphs $G$ with $\delta(G) \geq 6$. In this paper one shall prove the question for $\delta(G) \geq 6$. The following theorem and above results will complete the proof of Conjecture 1.

**Theorem 1.** Let $G$ be a graph of order $n$ with $\delta(G) \geq 6$. Then

$$\gamma(G) \leq \frac{6}{17} n.$$

The proof of Theorem 1 will be completed by choosing a dominating set $D$ of $G$ based on the so-called vertex disjoint paths cover, which was introduced by Reed in [10]. In this paper, for $x, y \in V(G)$, $xy$ denotes the edge with ends $x$ and $y$. If $xy \in E(G)$, we say that $y$ is a neighbor of $x$ or $y$ is adjacent to $x$, and the set of neighbors of $x$ is denote by $N(x)$, $d(x) = |N(x)|$ is called the degree of $x$. A subgraph $H$ is said to be induced by $U$ if $V(H) = U$ and $xy \in E(H)$ if and only if $xy \in E(G)$, $x, y \in U$. The number of vertices of the graph $G$ is denoted by $|V(G)|$.

A vertex disjoint paths cover of $G$, or simply called a vdp – cover, is a set of vertex disjoint paths $P_1, \ldots, P_k$ such that $V(G) = V(P_1) \cup \cdots \cup V(P_k)$. A path $P$ is called a 0–1– or 2–path if $|P|$ is congruent to 0, 1 or 2 mod 3, respectively. For a vdp – cover $S$ of $G$, let $S_i$ ($i = 0, 1, 2$) be the set of i-paths in $S$. If $P = P'xP''$, where $P'$ is an $i$-path and $P''$ is a $j$-path ($x$ is on neither $P'$ nor $P''$), then we say $x$ is an $(i,j)$ – vertex of $P$. Let $P \in S$ and $x$ be an endvertex of $P$. We say that $x$ is an out-endvertex if it has a neighbor which is not on $P$. If $P$ is a 2-path, we say that $x$ is a $(2,2)$-endvertex if it is not an out-endvertex and is adjacent to some $(2,2)$-vertex of $P$.

2. Choose a dominating set

We assume that $G$ is a graph with order $n$ and $\delta(G) \geq 6$. For convenience, we assume that $G$ is connected. We first choose a vdp – cover $S$ of $G$ such that

1. $2|S_1| + |S_2|$ is minimized.
2. Subject to (1), $|S_2|$ is minimized.
(3) Subject to (2), \[ \sum_{P_i \in S_0} |P_i| \] is minimized.

(4) Subject to (3), \[ \sum_{P_i \in S_1} |P_i| \] is minimized.

Let \( x \) be an out-endvertex of \( P_i \in S_1 \cup S_2 \) and \( y \) be a neighbor of \( x \) on some path \( P_j \) distinct from \( P_i \). Let \( P_j = P'_j y P''_j \), then we have the following assertion (for the proof, see [10], Observation 1-3).

**Assertion 1.** \( P_j \) is not a 1-path. If \( P_j \) is a 0-path, then both \( P'_j \) and \( P''_j \) are 1-paths; if \( P_j \) is a 2-path, then both \( P'_j \) and \( P''_j \) are 2-paths.

Having chosen the minimal \( vdp - cover \) \( S = \{P_1, \ldots, P_k\} \), we rearrange the paths of \( S \) to obtain a new \( vdp - cover \) \( S' = \{P'_1, \ldots, P'_k\} \) such that:

(i) \( P'_i (1 \leq i \leq k) \) is a Hamiltonian path in \( G[P_i] \);

(ii) subject to (i), the number of out-endvertices in \( S' \) is maximized;

(iii) subject to (ii), the number of \((2,2)\)-vertices in \( S' \) is maximized.

Clearly, \( S' \) is still minimal with respect to the above conditions. For convenience, we still denote the new \( vdp - cover \) of \( G \) by \( S \).

If a 1-path \( P \) in \( S \) has at least one out-endvertex, then we choose an out-endvertex \( x \) of \( P \) and a vertex \( y \notin P \) which is adjacent to \( x \), we say that \( y \) is the acceptor for \( P \). If a 2-path \( P \) in \( S \) has two out-endvertices, then for each of the two out-endvertices, we choose a vertex of \( V(G) - V(P) \) which is adjacent to it and designate it as the acceptor corresponding to the out-endvertex. If a 2-path \( P \) in \( S \) which has precisely one out-endvertex \( x \) and \( |P| \leq 14 \), we choose a vertex \( y \notin P \) which is adjacent to \( x \) and designate \( y \) as the acceptor for \( P \).

We call a path in \( S \) accepting if it contains an acceptor. Now we specify a set \( A \subseteq S \) of 2-paths. Initially, let \( A \) be the set of accepting 2-paths. While there is any out-endvertex \( x \) of a path in \( A \) for which we have not chosen an acceptor, we choose a neighbor of this endvertex in \( V(G) - V(P) \) and designate it as an acceptor for \( x \). If this new acceptor is on a previously non-accepting 2-path \( P' \), then we add \( P' \) to \( A \). We continue this process until there is an acceptor for every out-endvertex of the paths in \( A \). In addition, for any \((2,2)\)-endvertex \( x \) of any path \( P \) in \( A \), we choose a \((2,2)\)-vertex \( y \) of \( P \) which is adjacent to \( x \) and designate \( y \) as an inacceptor for \( x \).

For any accepting 2-path \( P \), we partition \( P = P_1 P_2 P_3 \) such that \( P_1 \) and \( P_3 \) are maximal 1-paths which contain neither acceptors nor inacceptors. We say that \( P_1 \) and \( P_3 \) are tips of \( P \) and \( P_2 \) is its central path. By the maximality of \( P_1, P_3 \), and by Assertion 1, if \( x \in P_2 \) is adjacent to an endvertex of \( P_2 \), then it is an acceptor or an inacceptor. Before the description of choosing the dominating set, one present the following assertion.

**Assertion 2.** Let \( P \in S \) be a 2-path with at most one out-endvertex. If \( |P| \leq 14 \), then all vertices of \( V(P) \) except for the possible out-endvertex can be dominated by \( \left\lfloor \frac{|P|}{3} \right\rfloor \) vertices.
One will prove Assertion 2 in section 3. Now one choose a dominating set \( D \) of \( G \) in the following manner:

**Step 1:** For each 0-path \( P \), we put every \((1, 1)\)-vertex of \( P \) in \( D \).

**Step 2:** For each accepting 2-path \( P \), we put into \( D \) every \((2, 2)\)-vertex of \( P \) which is in the central path of \( P \).

**Step 3:** For each 1-path \( P \) with at least one out-endvertex, we choose \( \left\lfloor \frac{|P|}{3} \right\rfloor \) vertices of \( P \) which dominate all of the vertices of \( P \) except for the endvertex of \( P \) which is adjacent to the acceptor of \( P \). We put these vertices in \( D \). For each non-accepting 2-path \( P \) with two out-endvertices, we choose \( \left\lfloor \frac{|P|}{3} \right\rfloor \) vertices of \( P \) to dominate its interior vertices. We put these vertices in \( D \). For each non-accepting 2-path \( P \) which has precisely one out-endvertex \( x \) and \( |P| \leq 14 \), by assertion 2, we can choose \( \left\lfloor \frac{|P|}{3} \right\rfloor \) vertices of \( P \) which dominate all of the vertices of \( P \) except for the endvertex \( x \) of \( P \) which is adjacent to the acceptor of \( P \). We put these vertices in \( D \).

**Step 4:** For each 1-path \( P \) with no out-endvertex, we choose a subset of \( V(P) \) which dominate \( V(P) \) and put it in \( D \). If possible, we choose a set of \( \left\lfloor \frac{|P|}{3} \right\rfloor \) vertices; otherwise we choose a set of \( \left\lfloor \frac{|P|}{3} \right\rfloor + 1 \) vertices. For each non-accepting 2-path \( P \) with at most one out-endvertex and \( |P| \geq 11 \), we choose a subset of \( V(P) \) which dominate \( V(P) \) and put it in \( D \). If possible, we choose a set of \( \left\lfloor \frac{|P|}{3} \right\rfloor \) vertices, otherwise we choose a set of \( \left\lfloor \frac{|P|}{3} \right\rfloor \) vertices.

**Step 5:** For each tip \( P_1 \) of an accepting 2-path \( P \), if the common endvertex \( x \) of \( P_1 \) and \( P \) is adjacent to a vertex chosen in step 1 or 2, we choose \( \left\lfloor \frac{|P_1|}{3} \right\rfloor \) of vertices of \( P_1 \) which dominate the remaining vertices of \( P_1 \) and put them in \( D \). If \( x \) is not adjacent to a vertex chosen in step 1 or 2, we choose a set which dominates \( P_1 \) and put it in \( D \). If possible, we choose \( \left\lfloor \frac{|P_1|}{3} \right\rfloor \) vertices, otherwise we choose \( \left\lfloor \frac{|P_1|}{3} \right\rfloor + 1 \) vertices.

It is easy to see that \( D \) is a dominating set of \( G \) (see [10], Observation 5-8). To calculate the size of \( D \), we define the following sets.

(i) \( O_1 \): the set of 1-paths \( P \) which either have at least one out-endvertex or contain a dominating set of size \( \left\lfloor \frac{|P|}{3} \right\rfloor \).

(ii) \( O_2 \): the set of non-accepting 2-paths \( P \) which have two out-endvertices or contain a dominating set with size \( \left\lfloor \frac{|P|}{3} \right\rfloor \) that dominates all the vertices of \( P \), and all non-accepting 2-paths which have precisely one out-endvertex and \( |P| \leq 14 \).

(iii) \( I_1 \): the set of 1-paths not in \( O_1 \).
(iv) \( I_2 \): the set of non-accepting 2-paths not in \( O_2 \).
(v) \( E \): the tip \( T \) of an accepting 2-path \( P \) is in \( E \) if and only if the corresponding endvertex of \( P \) is neither an out-endvertex nor a (2, 2)-endvertex and we cannot dominate \( T \) by using \( \left\lfloor \frac{|T|}{3} \right\rfloor \) vertices.
(vi) \( W \): the set of (2, 2)-endvertices of accepting 2-paths for which we have chosen an inacceptor.

Then, the size of \( D \) can be calculated easily as

\[
|D| = \sum_{P \in O_1} \frac{|P| - 1}{3} + \sum_{P \in O_2} \frac{|P| - 2}{3} + \sum_{P \in I_1} \frac{|P| + 2}{3} + \sum_{P \in I_2} \frac{|P| + 1}{3} + \sum_{P \in S_0} \frac{|P|}{3} + \sum_{P \in A} \frac{|P| - 2}{3} + |E|.
\]

Equivalently,

\[
|D| = \frac{n}{3} - \frac{1}{3}|O_1| - \frac{2}{3}|O_2| + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| - \frac{2}{3}|A| + |E|.
\]

Note that each accepting 2-path corresponds to an endvertex of some path in \( O_1 \cup O_2 \) or to an endvertex of an accepting 2-path of \( A \) which is not in \( E \cup W \). Thus, we have \( |A| \leq |O_1| + 2|O_2| + 2|A| - |E| - |W| \), so \( |E| \leq |O_1| + 2|O_2| + |A| - |W| \). Also, \( |E| \leq 2|A| - |W| \). Thus,

\[
|D| \leq \frac{n}{3} + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{|E|}{2} - \frac{|W|}{2}.
\]

To any element \( T \) of \( E \) there corresponds an accepting 2-path \( P_T \) such that \( T \) is a tip of \( P_T \). Now we define a set \( E' \), \( E' \subseteq E \) by saying that each \( T \in E \) is in \( E' \) if the endvertex of \( P_T \) not in \( T \) is not an element of \( W \).

Clearly, \( |E'| \geq |E| - |W| \), and so

\[
|D| \leq \frac{n}{3} + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{1}{2}|E'| \quad (\ast)
\]

3. Proof of Theorem 1

The proof will be completed by a sequence of two lemmas and four assertions. The following three observations are straightforward to verify.

**Observation 1.** Let \( P = x_1 x_2 \cdots x_{3k+1} \) \((k \geq 1)\) be a path. If \( x_1 \) is adjacent to a vertex \( x_{3i} \) for some \( 1 \leq i \leq k \), then \( V(P) \) can be dominated by \( k \) vertices.

**Observation 2.** Let \( C \) be a circle with \( 3k+1 \) \((k \geq 1)\) vertices and let \( L = x_1 x_2 x_3 \) be a path such that \( V(C) \cap V(L) = \emptyset \). If \( x_2 \) has a neighbor in \( C \), then \( V(C) \cup V(L) \) can be dominated by \( k + 1 \) vertices.
Observation 3. Let $P = x_1x_2 \cdots x_{3k-1}$ $(k \geq 1)$ be a path and let $x \notin P$. If $x$ is adjacent to one vertex of $\bigcup_{i=1}^{k} \{x_{3i-2}, x_{3i-1}\}$, then $V(P) \cup \{x\}$ can be dominated by $k$ vertices.

From the above three observations, one shall prove Lemma 1 and Lemma 2.

Lemma 1. Let $C = x_1x_2 \cdots x_{3k+1}x_1$ $(2 \leq k \leq 7)$ be a circle of $G$, $H$ be a subgraph of $G$ induced by $V(C)$. For $v \in V(C)$, if in $H$ there is a Hamiltonian path between $v$ and $x_{3k+1}$, we have $N(v) \subseteq V(C)$, then $H$ can be dominated by $k$ vertices.

Proof. If $k = 2$, the conclusion is obvious. Here we only prove the case for $k = 7$, the cases for $3 \leq k \leq 6$ can be proved by similar reasoning and omitted.

When $k = 7$, then $C = x_1x_2 \cdots x_{22}x_1$. Let $C^+ = x_1x_2 \cdots x_{22}$, for $1 \leq i < j \leq 22$, let $x_iC^+x_j$ (or $x_jC^-x_i$) denotes the path between $x_i$ and $x_j$ of $C^+$ (both $x_i$ and $x_j$ are contained). We prove by contradiction, assume $H$ can not be dominated by $k$ vertices. We first check the neighbors of $x_1$, then deduce a contradiction.

Obviously, there are Hamiltonian paths between $x_1$ and $x_{22}$, $x_{21}$ and $x_{22}$ in $H$, so we have $N(x_1) \subseteq V(C)$, $N(x_{21}) \subseteq V(C)$, by Observation 1, $x_1$ is not adjacent to $x_{3i} (1 \leq i \leq 7)$, by symmetry, $x_{21}$ is not adjacent to $x_{3i+1} (0 \leq i \leq 6)$. In the following, we check the neighbors of $x_1$.

Case 1. $x_1$ is not adjacent to $x_{10}$.

We prove by contradiction. Assume $x_1$ is adjacent to $x_{10}$, now we check the neighbors of $x_{21}$, then deduce a contradiction.

For $x_{20}x_{21}x_{22}$ and the circle $x_1C^+x_{10}x_1$, by Observation 2, $x_{21}$ is not adjacent to the circle. So,

$$N(x_{21}) - \{x_{20}, x_{22}\} \subseteq \{x_{11}, x_{12}, x_{14}, x_{15}, x_{17}, x_{18}\}.$$

Since $d(x_{21}) \geq 6$, then $x_{21}$ must be adjacent to both $x_{11}$ and $x_{12}$, or both $x_{14}$ and $x_{15}$, or both $x_{17}$ and $x_{18}$.

Firstly, if $x_{21}$ is adjacent to both $x_{11}$ and $x_{12}$, then there is a Hamiltonian path $x_{13}C^+x_{21}x_{12}C^-x_1x_{22}$, so $N(x_{13}) \subseteq V(C)$, for $x_{12}x_{13}x_{14}$ and the circle $x_1C^+x_{11}x_{21}x_{22}x_1$, by Observation 2, $x_{13}$ is not adjacent to the circle. By Observation 1, $x_{13}$ is not adjacent to $x_{15}$ or $x_{18}$. So $N(x_{13}) \subseteq \{x_{15}, x_{17}, x_{19}, x_{20}\}$. Since $d(x_{13}) \geq 6$, then $x_{13}$ must adjacent to all vertices of $\{x_{16}, x_{17}, x_{19}, x_{20}\}$. Now there is a Hamiltonian path

$$x_{15}x_{14}x_{13}x_{16}C^+x_{21}x_{12}C^-x_1x_{22},$$

so, $N(x_{15}) \subseteq V(C)$. For $x_{14}x_{15}x_{16}$ and the circle $x_1C^+x_{13}x_{17}C^+x_{22}x_1$, by Observation 2, $x_{15}$ is not adjacent to the circle, this means $d(x_{15}) \leq 5$, a contradiction. So $x_{21}$ is at most adjacent to one of $\{x_{11}, x_{12}\}$.
Secondly, by similar reasoning, $x_{21}$ is at most adjacent to one of $\{x_{14}, x_{15}\}$, or at most one of $\{x_{17}, x_{18}\}$. This means $d(x_{21}) \leq 5$, a contradiction. So $x_1$ is not adjacent to $x_{10}$. This proves Case 1.

By similar reasoning, $x_1$ is not adjacent to the vertex of $\{x_{13}, x_{16}, x_{19}\}$.

**Case 2.** $x_1$ is not adjacent to $x_{11}$.

We prove by contradiction. Assume $x_1$ is adjacent to $x_{11}$, now $x_1$ dominates $x_2, x_{22}$ and $x_{11}$, for $x_{21}$ and the path $x_3C^+x_{10}$, by Observation 3, we have

$$N(x_{21}) - \{x_{20}, x_{22}\} \subseteq \{x_2, x_5, x_8, x_{11}, x_{12}, x_{14}, x_{15}, x_{17}, x_{18}\}.$$

**Case 2.1.** $x_{21}$ is not adjacent to $x_2$.

Assume $x_{21}$ is adjacent to $x_2$, now $x_{21}$ dominates $x_{20}, x_{22}$ and $x_2$, for $x_1$ and the path $x_3C^+x_{10}$, by Observation 3, we have $N(x_1) \subseteq \{x_5, x_8, x_{11}, x_{14}, x_{17}, x_{20}\}$.

Firstly, if $x_1$ is adjacent to $x_5$, there is a Hamiltonian path $x_3C^+x_{21}x_2x_{21}x_{22}$, so $N(x_3) \subseteq V(C)$. For $x_2x_3x_4$ and the circle $x_5C^+x_{22}x_1x_5$, by Observation 2, $x_3$ has no neighbor in $x_5C^+x_{22}x_1x_5$, a contradiction to $d(x_3) \geq 6$. So $x_1$ is not adjacent to $x_5$, by similar reasoning, $x_1$ is not adjacent to $x_8$. Then $x_1$ must be adjacent to all the vertices of $\{x_{11}, x_{17}, x_{20}\}$.

Secondly, $x_1$ is adjacent to all the vertices of $\{x_{11}, x_{17}, x_{20}\}$.

Now there is a Hamiltonian path $x_19C^-x_{12}x_{21}x_{22}$, so $N(x_{19}) \subseteq V(C)$. For $x_{18}x_{19}x_{20}$ and the circle $x_3C^+x_{21}x_1x_2x_{21}x_3$, by Observation 2, $x_{19}$ has no neighbor in the circle, a contradiction to $d(x_{19}) \geq 6$, so $x_1$ is not adjacent to all the vertices of $\{x_{11}, x_{17}, x_{20}\}$. This means $d(x_1) \leq 6$, a contradiction. This proves Case 2.1. by similar reasoning, $x_{21}$ is not adjacent to $x_5$ or $x_8$. So

$$N(x_{21}) - \{x_{20}, x_{22}\} \subseteq \{x_{11}, x_{12}, x_{14}, x_{15}, x_{17}, x_{18}\}.$$

**Case 2.2.** $x_{21}$ is at most adjacent to one of $\{x_{11}, x_{12}\}$.

Assume $x_{21}$ is adjacent to both $x_{11}$ and $x_{12}$, there is a Hamiltonian path
\[
(x_{17}C^+x_{21})
\]
\[
x_{18}C^-x_{12}x_{22},
\]
so $N(x_{13}) \subseteq V(C)$. For $x_{12}x_{13}x_{14}$ and the circle $x_1C^+x_{17}x_{21}x_{22}x_1$, by Observation 2, $x_{13}$ has no neighbor in the circle, and by Observation 1, $x_{13}$ must be adjacent to all the vertices of $\{x_{16}, x_{17}, x_{19}, x_{20}\}$, then there is a Hamiltonian path
\[
x_{15}x_{14}x_{13}x_{16}C^+x_{21}x_{12}C^-x_{1}x_{22},
\]
so $N(x_{15}) \subseteq V(C)$. For $x_{14}x_{15}x_{16}$ and the circle $x_1C^+x_{17}C^+x_{21}x_{12}x_1$, by Observation 2, $x_{15}$ has no neighbor in the circle, a contradiction to $d(x_{15}) \geq 6$. This proves Case 2.2. By similar reasoning, $x_{21}$ is at most adjacent to one of $\{x_{14}, x_{16}\}$ or at most one of $\{x_{17}, x_{18}\}$.

From the Case 2.1 and Case 2.2, we have $d(x_{21}) \leq 5$, a contradiction. So $x_1$ is not adjacent to $x_{11}$. This proves Case 2. By similar reasoning as Case 2, $x_1$ is not adjacent to the vertex of $\{x_{14}, x_{17}, x_{20}\}$.
Case 3. \( x_1 \) is at most adjacent to one of \( \{x_4, x_5\} \).

Assume \( x_1 \) is adjacent to both \( x_4 \) and \( x_5 \), then there is a Hamiltonian path \( x_3x_2x_1x_4C^+x_22 \), so \( N(x_3) \subseteq V(C) \). For \( x_2x_3x_4 \) and the circle \( x_5C^+x_2x_1x_5 \), by Observation 2, \( x_3 \) has no neighbor in the circle, a contradiction to \( d(x_3) \geq 6 \). This proves Case 3. By similar reasoning, \( x_1 \) is at most adjacent to one of \( \{x_7, x_8\} \).

From the above three cases we have \( d(x_1) \leq 5 \), a contradiction. This proves Lemma 1.

\[
\text{Lemma 2. Let } C = x_1x_2\cdots x_{3k+2}x_1 \text{ (} 2 \leq k \leq 7 \text{) be a circle of } G, \text{ } H \text{ be a subgraph of } G \text{ induced by } V(C). \text{ For } v \in V(C), \text{ if in } H \text{ there is a Hamiltonian path between } v \text{ and } x_{3k+2}, \text{ we have } N(v) \subseteq V(C), \text{ then } V(C) - \{x_{3k+2}\} \text{ can be dominated by } k \text{ vertices.}
\]

**Proof.** When \( k = 2 \), the conclusion is obvious. We only prove the conclusion for \( k = 7 \). The proof of other cases is similar and thus omitted.

When \( k = 7 \), \( C = x_1x_2\cdots x_{23}x_1 \). Similarly, let \( C^+ = x_1x_2\cdots x_{23}x_1 \). For \( 1 \leq i < j \leq 23 \), \( x_iC^+x_j \) or \( x_jC^-x_i \) denotes the path between \( x_i \) and \( x_j \) of \( C^+(x_i \text{ and } x_j \text{ are contained}). \)

We prove by contradiction. Assume \( V(C) - \{x_{3k+2}\} \) can not be dominated by \( k \) vertices. First we check the neighbors of \( x_1 \), then deduce a contradiction.

Since there are Hamiltonian paths between \( x_1 \) and \( x_{23}, x_{22} \) and \( x_{23} \), we have \( N(x_1) \subseteq V(C) \) and \( N(x_{22}) \subseteq V(C) \). Noted that \( x_1 \) and \( x_{22} \) are symmetrical about \( x_{23} \) on \( C \), so the properties about the neighbors of \( x_1 \) are the same as the neighbors of \( x_{22} \). By Observation 1, \( x_1 \) is not adjacent to \( x_{3k} (1 \leq k \leq 7) \), and symmetrically, \( x_{22} \) is not adjacent to \( x_{3k-1} (1 \leq k \leq 7) \).

Case 1. \( x_1 \) is not adjacent to \( x_{22} \).

Assume \( x_1 \) is adjacent to \( x_{22} \), now in \( H \) there are Hamiltonian paths between \( x_2 \) and \( x_{23}, x_{21} \) and \( x_{23} \), this is similar to the Lemma 1, so by the similar proof of Case 1-2 of Lemma 1 we get \( N(x_1) - \{x_2, x_{23}, x_{22}\} \subseteq \{x_4, x_5, x_7, x_8\} \), similar to the proof of Case 3 of Lemma 1, we get \( x_1 \) is at most adjacent to one of \( \{x_4, x_5\} \), or at most adjacent to one of \( \{x_7, x_8\} \). This means \( d(x_1) \leq 5 \), a contradiction.

By similar reasoning as the proof of Lemma 1 we have \( x_1 \) is not adjacent to the vertex of \( \{x_{13}, x_{16}, x_{19}\} \). By symmetry, \( x_{22} \) is not adjacent to the vertex of \( \{x_1, x_4, x_7, x_{10}\} \).

Case 2. \( x_1 \) is not adjacent to \( x_{20} \).
Assume $x_1$ is adjacent to $x_{20}$, then $x_{20}$ dominates $\{x_1, x_{19}, x_{21}\}$, for $x_{22}$ and the path $x_2C^+x_{18}$, by Observation 3,

$$N(x_{22}) - \{x_{21}, x_{23}\} \subseteq \{x_{13}, x_{16}, x_{19}\}. $$

This is a contradiction to $d_{20} \geq 6$. This proves Case 2.

By the same reason as Case 2, $x_1$ is not adjacent to $x_{14}$ or $x_{17}$. By symmetry,

$$N(x_{22}) - \{x_{21}, x_{23}\} \subseteq \{x_{12}, x_{13}, x_{15}, x_{16}, x_{18}, x_{19}\}. $$

**Case 3.** $x_1$ is at most adjacent to one of $\{x_{10}, x_{11}\}$.

Assume $x_1$ is adjacent to both $x_{10}$ and $x_{11}$, now we check the neighbors of $x_{22}$.

**Case 3.1.** $x_{22}$ is at most adjacent to one of $\{x_{12}, x_{13}\}$.

If $x_{22}$ is adjacent to both $x_{12}$ and $x_{13}$, then there is a Hamiltonian path $x_{21}C^-x_{13}x_{22}x_{12}C^-x_{23}$, so $N(x_{21}) \subseteq V(C)$. For $x_{20}x_{21}x_{22}$ and the circle $x_1C^+x_{10}x_1$, by Observation 2, $x_{21}$ is not adjacent to the circle. And by Observation 1, we have

$$N(x_{21}) - \{x_{20}, x_{22}\} \subseteq \{x_{12}, x_{14}, x_{15}, x_{17}, x_{18}, x_{23}\}. $$

Now we check the neighbors of $x_{21}$.

**Case 3.1.1.** $x_{21}$ is at most adjacent to one vertex of $\{x_{12}, x_{23}\}$.

Assume $x_{21}$ is adjacent to both $x_{12}$ and $x_{23}$, there is a Hamiltonian path $x_{14}C^+x_{22}

x_{13}C^-x_{12}x_{23}$, for $x_{13}x_{14}x_{15}$ and the circle $x_1C^+x_{10}x_1$, by Observation 2, $x_{14}$ is not adjacent to the circle. By Observation 1,

$$N(x_{14}) - \{x_{13}, x_{15}\} \subseteq \{x_{12}, x_{17}, x_{18}, x_{20}, x_{21}, x_{23}\}. $$

Firstly, if $x_{14}$ is adjacent to $x_{12}$, then there are 7 vertices $\{x_2, x_5, x_8, x_{11}, x_{12}, x_{16}, x_{19}\}$ dominate $V(C) - \{x_{23}\}$, this is contrary to the supposition that $V(C) - \{x_{23}\}$ can not be dominated by 7 vertices. So $x_{14}$ is not adjacent to $x_{12}$.

Secondly, if $x_{14}$ is adjacent to $x_{20}$, for $x_{13}x_{22}x_{21}$ and the circle $x_{14}C^+x_{20}x_{14}$, by Observation 2, $x_{22}$ is not adjacent to the circle, then

$$N(x_{22}) - \{x_{21}, x_{23}\} \subseteq \{x_{12}, x_{13}\}, $$

a contradiction, so $x_{14}$ is not adjacent to $x_{20}$.

Similarly, $x_{14}$ is not adjacent to $x_{17}$.

This means $d(x_{14}) \leq 5$, a contradiction. So $x_{21}$ is at most adjacent to one vertex of $\{x_{12}, x_{23}\}$. This proves Case 3.1.1.

**Case 3.1.2.** $x_{21}$ is at most adjacent to one vertex of $\{x_{17}, x_{18}\}$.

Assume $x_{21}$ is adjacent to both $x_{17}$ and $x_{18}$, then there is a Hamiltonian path $x_{19}x_{20}x_{21}x_{18}C^-x_{13}x_{22}x_{12}C^-x_{1}x_{23}$,
for $x_{18}x_{19}x_{20}$ and the circle $x_1C^+x_{10}x_1$ or the circle $x_{13}C^+x_{17}x_{21}x_{22}x_{13}$, by Observation 2, $x_{19}$ is not adjacent to the circles, this means $d(x_{19}) \leq 5$, a contradiction to $d(x_{19}) \geq 6$. So $x_{21}$ is at most adjacent to one vertex of $\{x_{17}, x_{18}\}$.

Similarly, $x_{21}$ is at most adjacent to one vertex of $\{x_{14}, x_{15}\}$.

From the two cases 3.1.1-3.1.2, we get $d(x_{21}) \leq 5$, a contradiction. So $x_{22}$ is at most adjacent to one of $\{x_{12}, x_{13}\}$. This proves Case 3.

By similar reasoning, we have $x_{20}$ is at most adjacent to one vertex of $\{x_{15}, x_{16}\}$, or one of $\{x_{18}, x_{19}\}$. This means $d(x_{22}) \leq 5$, a contradiction. So $x_{1}$ is at most adjacent to one of $\{x_{10}, x_{11}\}$. This proves Case 3.

By similar reasoning as Case 3, $x_{1}$ is at most adjacent to one of $\{x_{4}, x_{5}\}$ or one of $\{x_{7}, x_{8}\}$.

From Case 1-3, we get $d(x_{1}) \leq 5$, a contradiction. This proves Lemma 2. □

A lasso $L$ is defined as a graph formed by identifying any vertex in a circle $C$ with an endvertex of a path $P$. The other endvertex of the path $P$ is called the end of $L$ and the common vertex of $C$ and $P$ is called the connecting vertex of $L$. Especially, a cycle can be regarded as a lasso.

**Assertion 3.** Let $P \in S$ be a 2-path with at most one out-endvertex. If $|P| \leq 14$, then all vertices of $V(P)$ except for the possible out-endvertex can be dominated by $\left\lceil \frac{|P|}{3} \right\rceil$ vertices.

**Proof.** If $|V(G)| = 14$, then the conclusion is immediate. We may assume that $|V(G)| > 14$. Let $P = x_1x_2 \cdots x_{3m+2}$ ($2 \leq m \leq 4$) be a 2-path in $S$ with at most one out-endvertex. Let $H$ be a subgraph of $G$ induced by $V(P)$. Since $\delta \geq 6$, when $|P| = 8, 11$, the conclusion is obvious, so in the following we prove only $|P| = 14$, i.e., $P = x_1x_2 \cdots x_{14}$.

**Case 1.** $P$ has no out-endvertex.

As $G$ is connected, there is at least one edge between $V(P)$ and $V(G) - V(P)$. If there is a Hamiltonian circle of $H$, then each vertex of $H$ is an out-endvertex of some Hamiltonian path, a contradiction. So there has no Hamiltonian circle in $H$.

Now we choose a lasso $L$ in $H$ such that the number of vertices on the circle of the lasso is maximum. For convenience, we label the vertices of $L$ along a Hamiltonian path on $L$ from the end of $L$ as $x_{14}, x_{13}, \cdots, x_2, x_1$. Since $x_1$ and $x_{14}$ are not out-endvertices, so $x_1$ and $x_{14}$ are only adjacent to the vertices of $P$. As there has no Hamiltonian circle in $H$, $x_1$ is not adjacent to $x_{14}$, and $d(x_1) \geq 6$, let $u$ be the connecting vertex of the lasso, we have

$$u \in \{x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},$$

by the labeling, $x_1$ is adjacent to $u$.

In the following, we prove only $u = x_{13}$ because the proof of the other can be done in a similar way.
Case 1.1. When \( u = x_{13} \).

We prove by contradiction, assume \( V(P) \) can not be dominated by 4 vertices. We check the neighbors of \( x_1 \).

\[ \text{Case 1.1.1. } x_1 \text{ is not adjacent to } x_6. \]

Firstly, if \( x_1 \) is adjacent to \( x_6 \), we check the neighbors of \( x_{14} \). Now \( x_5 \) is an endvertex of a Hamiltonian path \( x_5C^-x_1x_5C^+x_{13}x_{14} \), so by the choice of the circle of \( L \), \( x_{14} \) is not adjacent to \( x_5 \).

Secondly, as \( x_1 \) dominates \( \{x_2, x_8, x_{13}\} \), if \( x_{14} \) is adjacent to one of \( \{x_4, x_8, x_{11}\} \), then \( \{x_1, x_4, x_8, x_{11}\} \) dominate \( V(P) \), this is contrary to the supposition that \( V(P) \) can not be dominated by 4 vertices.

Finally, since \( x_6 \) dominate \( \{x_1, x_5, x_7\} \), if \( x_{14} \) is adjacent to one vertex of \( \{x_3, x_6, x_9, x_{12}\} \), then \( \{x_3, x_6, x_9, x_{12}\} \) dominate \( V(P) \), a contradiction.

So \( N(x_{14}) \subseteq \{x_2, x_7, x_{10}, x_{13}\} \), this means \( d(x_{14}) \leq 4 \), a contradiction to \( d(x_{14}) \geq 6 \). So \( x_1 \) is not adjacent to \( x_6 \). By similar reasoning, \( x_1 \) is not adjacent to the vertex of \( \{x_3, x_9, x_{12}\} \). So \( x_1 \) is adjacent to at least four vertices of \( \{x_4, x_5, x_7, x_8, x_{10}, x_{11}\} \).

\[ \text{Case 1.1.2. } x_1 \text{ is at most adjacent to one of } \{x_4, x_5\}. \]

Now there is a Hamiltonian path \( x_2C^+x_4x_1x_5C^+x_{13}x_{14} \), so \( N(x_2) \subseteq V(P) \), and by the choice of the circle of \( L \), we have \( x_2 \) is not adjacent to \( x_{14} \). Since \( x_4 \) dominates \( \{x_1, x_3, x_5\} \), for \( x_2 \) and the path \( x_5C^+x_{13} \), by Observation 3, we have

\[ N(x_2) - \{x_1, x_3\} \subseteq \{x_5, x_8, x_{11}\}, \]

this means \( d(x_2) \leq 5 \), a contradiction.

By similar reasoning, \( x_1 \) is at most adjacent to one of \( \{x_7, x_8\} \), or one of \( \{x_{10}, x_{11}\} \).

From Cases 1.1.1-1.1.2, we get \( d(x_1) \leq 5 \), a contradiction. This completes Case 1.

Case 2. \( P \) has precisely one out-endvertex.

When \( P \) has precisely one out-endvertex, assume \( x_{14} \) is an out-endvertex of \( P \), as \( x_1 \) is not an out-endvertex, \( x_1 \) is only adjacent to the vertices of \( P \).

Similarly as Case 1, we choose a lasso \( L \) in \( H \) such that the vertices on the circle of the lasso is maximum, let \( C \) be the circle of the lasso, and \( x_{14} \) be an out-endvertex of the lasso. For convenience, we label the vertices of \( L \) along a Hamiltonian path on \( L \) from the end of \( L \) as \( x_{14}, \ldots, x_1 \). Let \( u \) be the connecting vertex. By the labeling, \( x_1 \) is adjacent to \( u \).

We prove by contradiction. By proposition 1, \( x_1 \) is not adjacent to \( x_{3k}(1 \leq i \leq 4) \), so \( u = x_{3k+1} \) or \( x_{3k+2} \) \( (2 \leq k \leq 4) \). Assume \( x_i \in V(C) \), Noted that if there is a path from \( x_1 \) to \( u \) that all the vertices of \( C \) are on the path, then there is a Hamiltonian path from \( x_i \) to \( x_{14} \), thus \( N(x_i) \subseteq V(C) \), i.e., the vertex of \( C \) satisfy the conditions of Lemma 1 or Lemma 2, so by Lemma 1 or Lemma 2, we have the conclusion. \( \square \)
**Assertion 4.** Let \( T \in E' \) be a tip of a 2-path \( P \) in \( A \). If \(|T| \leq 22\), then \( T \) can be dominated by \( \left\lfloor \frac{|T|}{3} \right\rfloor \) vertices.

**Proof.** We prove by contradiction, assume \( T \) can not be dominated by \( \left\lfloor \frac{|T|}{3} \right\rfloor \) vertices, then deduce a contradiction.

Let \( T = a_0 \cdots a_k \in E' \) be a tip of 2-path \( P \), \( C = c_0 \cdots c_l \) be a central path of \( P \). Assume \( c_0 \) is adjacent to \( a_k \) on the path \( P \), by definition, \( c_1 \) is an acceptor or inacceptor. As \( T \in E' \), there is not \((2, 2)\)-endvertex in \( P \), so \( c_1 \) is an acceptor.

We first present a Claim proved by Reed (for the proof, see [10] p285, Fact 11).

**Claim 1.** \( a_0 \) is only adjacent to the vertex of \( V(T) \cup \{c_0\} \).

If \( a'_0 \cdots a'_k \) is a Hamiltonian path on \( V(T) \) such that \( a'_k \) is adjacent to \( c_0 \), then by the choice of \( S \), \( a'_k \) also is only adjacent to the vertex of \( T \cup \{c_0\} \).

As \( T \) is 1-path, then \(|T| = 3m + 1(0 \leq m \leq 6)\). Let \( H \) be a subgraph of \( G \) induced by \( V(T) \cup \{c_0\} \). As \( c_0 \) is only adjacent to the vertex of \( V(T) \cup \{c_0\} \), there is a lasso in \( H \) with one endvertex \( c_0 \). Now we choose a lasso \( L \) in \( H \) such that the number of vertex on the circle of the lasso is maximum, and \( c_0 \) is an endvertex of the lasso (perhaps there is a Hamiltonian circle of \( H \)). We label the vertices of \( L \) along a Hamiltonian path on \( L \) from the end of \( L \) as \( c_0 x_3x_{3m+1} \cdots x_1 \), \( 1 \leq m \leq 6 \). Let \( u \) be the connecting vertex, by the labeling, \( x_1 \) is adjacent to \( u \), by Observation 1, \( x_1 \) is not adjacent to \( x_{3k} \), \( 1 \leq k \leq m \leq 6 \). Since \( d(x_1) \geq 6 \) and \( x_1 \) is not an out-endvertex, then \( u = x_{3k+1} \) or \( u = x_{3k+2} \), where \( 2 \leq k \leq m \leq 6 \). Designate the circle of \( L \) as \( C \), similarly we can deduce that the vertices of \( C \) satisfy the conditions of Lemma 1 or Lemma 2, so by Lemma 1 or Lemma 2, we have the conclusion that \( V(T) \) can be dominated by \( \left\lfloor \frac{|T|}{3} \right\rfloor \) vertices, a contradiction.

**Assertion 5.** Let \( P \in S \) be a 1-path with no out-endvertex. If \(|P| \leq 31\), then \( P \) can be dominated by \( \left\lfloor \frac{|P|}{3} \right\rfloor \) vertices.

**Proof.** Here we only prove \(|P| = 31\), the other cases can be proved by similar reasoning and omitted. Let \( H \) be a subgraph of \( G \) induced by \( V(P) \).

**Case 1.** When \(|V(G)| = 31\) and there has a Hamiltonian circle in \( G \).

If \(|V(G)| = 31\) and there has a Hamiltonian circle in \( G \), \( G = H \), by Lemma 1, \( G \) can be dominated by 10 vertices, as \( 10 \leq \frac{6n}{17} = \frac{6 \times 31}{17} \), This satisfies Theorem 1.

**Case 2.** When \(|V(G)| \geq 31\) and there has no Hamiltonian circle in \( H \).

If \(|V(G)| > 31\) and there has a Hamiltonian circle in \( H \), since \( P \) has no out endvertex, this is contrary to that \( G \) is connected. So in the following, we always
assume $|V(G)| \geq 31$ and there has no Hamiltonian circle in $H$. We prove by contradiction, assume $P$ can not be dominated by $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices.

Now we choose a lasso $L$ in $H$ such that the number of vertices on the circle of the lasso is maximum, $x_{31}$ is an endvertex of the lasso. For convenience, we label the vertices of $L$ along a Hamiltonian path on $L$ from the end of $L$ as $x_{31}x_{30}\cdots x_1$. Let $u$ be the connecting vertex, by the labeling, $x_1$ is adjacent to $u$.

We prove by contradiction, assume $P$ can not be dominated by $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices.

By Observation 1, $x_1$ is not adjacent to $x_{31}, (1 \leq i \leq 10)$, by the choice of the circle of $L$, $x_1$ is not adjacent to $x_{31}$, since $d(x_1) \geq 6$ and $x_1$ is not an out-endvertex, then $u = x_{3k+1}$ or $x_{3k+2}, (2 \leq k < 10)$. Let $C$ be the circle of the lasso. When $k \leq 7$, the vertices of $C$ satisfy the conditions of Lemma 1 or Lemma 2, so by the Lemma 1 or Lemma 2, we have $\left\lfloor \frac{|P|}{3} \right\rfloor$ vertices dominate $V(P)$, a contradiction. Thus, assume $k \geq 8$, for convenience, we denote $C^+ = x_1x_2\cdots x_{31}$. For $1 \leq i < j \leq 31$, let $x_ix_j$ (or $x_jC^+x_i$) denote the path between $x_i$ and $x_j$ of $C^+$ (both $x_i$ and $x_j$ are contained). Here we only prove $k = 8$, i.e., $u = x_{25}$, the other cases can be similarly proved and omitted.

When $u = x_{3k+1} = x_{25}$, by the choice of the lasso, $x_1$ is not adjacent to the vertex of $\{x_{26}, x_{27}, \cdots, x_{31}\}$ (otherwise there is a longer circle). So

$$N(x_1) - \{x_2, x_{25}\} \subseteq \{x_4, x_5, x_7, x_8, x_{10}, x_{11}, x_{13}, x_{14}, x_{16}, x_{17}, x_{19}, x_{20}, x_{22}, x_{23}\}.$$ 

Now we check the neighbors of $x_{31}$. Since $x_{31}$ is not an out-endvertex, $x_{31}$ is only adjacent to the vertices of $P$. By the choice of the lasso, $x_{31}$ is not adjacent to the vertex of $\{x_1, x_2, \cdots, x_6\}$, by symmetry, $x_{31}$ is not adjacent to the vertex of $\{x_{19}, x_{20}, \cdots, x_{24}\}$, by Observation 1, $x_{31}$ is not adjacent to the vertex of $\{x_8, x_{11}, x_{14}, x_{17}, x_{26}, x_{29}\}$. Thus,

$$N(x_{31}) - \{x_{30}\} \subseteq \{x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{18}\} \cup \{x_{25}, x_{27}, x_{28}\}.$$ 

Case 2.1. $x_{31}$ is not adjacent to $x_7$.

If $x_{31}$ is adjacent to $x_7$, now we check the neighbors of $x_1$. By the choice of the lasso, $x_1$ is not adjacent to the vertex of $\{x_8, \cdots, x_{13}\}$.

Case 2.1.1. $x_1$ is not adjacent to $x_{22}$.

If $x_1$ is adjacent to $x_{22}$, since $x_{24}$ is an endvertex of a Hamiltonian path of $H$, so $N(x_{24}) \subseteq V(P)$, for $x_{23}x_{24}x_{25}$ and the circle $x_1C^+x_{22}x_1$, by Observation 2, $x_{24}$ is not adjacent to the circle, this means $d(x_{24}) \leq 5$, a contradiction. So $x_1$ is not adjacent to $x_{22}$, similarly, $x_1$ is not adjacent to $x_{16}$ or $x_{19}$.

Case 2.1.2. $x_1$ is not adjacent to $x_{23}$.

Now we check the neighbors of $x_{24}$.
Firstly, by Observation 1, \(x_{24}\) is not adjacent to \(x_{27}, x_{30}\) and \(x_{3i+1}, (0 \leq i \leq 7)\).

Secondly, by the choice of the circle of \(L\), \(x_{24}\) is not adjacent to the vertex of \(x_1C^+x_6\) and \\{\(x_{25}, \ldots, x_{31}\)\}.

Finally, since \(x_1\) dominates \{\(x_2, x_{23}, x_{25}\)\}, for \(x_{24}\) and the path \(x_3C^+x_{22}\), by Observation 3, we have \(x_{24}\) is not adjacent to \(x_{3i}\) and \(x_{3i+1}, 0 \leq i \leq 7\).

So \(N(x_{24}) \subseteq \{x_8, x_{11}, x_{14}, x_{17}, x_{20}\}\). If \(x_{24}\) is adjacent to \(x_{20}\), then there is a Hamiltonian path \(x_{22}C^-x_1x_{23}x_{24}x_{25}x_{31}\), so \(N(x_{24}) \subseteq V(H)\), for \(x_{22}x_{23}x_{25}\) and the circle \(x_1C^+x_{20}x_{24}x_{25}x_1\), by Observation 2, \(x_{22}\) is not adjacent to the circle, this means \(d(x_{22}) \leq 5\), a contradiction. So \(x_{24}\) is not adjacent to \(x_{20}\), similarly, \(x_{24}\) is not adjacent to \(x_{17}\), this means \(d(x_{24}) \leq 5\), a contradiction. So \(x_1\) is not adjacent to \(x_{23}\), similarly, \(x_1\) is not adjacent to \(x_{20}\) or \(x_{17}\).

Combining with Case 2.1.1, since \(d(x_1) \geq 6\), then \(x_1\) must be adjacent to the four vertices of \(\{x_4, x_5, x_7, x_{14}\}\).

**Case 2.1.3.** \(x_1\) is at most adjacent to three vertices of \(\{x_4, x_5, x_7, x_{14}\}\).

If \(x_1\) is adjacent to all vertices of \(\{x_4, x_5, x_7, x_{14}\}\). Now there is a Hamiltonian path \(x_3x_2x_1x_4C^+x_{25}x_{31}\), so \(N(x_3) \subseteq V(H)\). For \(x_2x_3x_4\) and the circle \(x_5C^+x_{25}x_1x_5\), by Observation 2, \(x_3\) is not adjacent to the circle, this means \(d(x_3) \leq 5\), a contradiction.

So \(x_{31}\) is not adjacent to \(x_7\). This proves Case 2.1.

Similarly, \(x_{31}\) is not adjacent to \(x_{10}\) or \(x_{13}\). By symmetry, \(x_{31}\) is not adjacent to the vertex of \(\{x_{18}, x_{15}, x_{12}\}\).

So

\[N(x_{31}) - \{x_{30}\} \subseteq \{x_9, x_{16}, x_{25}, x_{27}, x_{28}\}\]

Since \(d(x_{31}) \geq 6\), \(x_{31}\) must be adjacent to all vertices of \(\{x_9, x_{16}, x_{25}, x_{27}, x_{28}\}\).

**Case 2.2.** \(x_{31}\) is at most adjacent to one vertex of \(\{x_{27}, x_{28}\}\).

Assume \(x_{31}\) is adjacent to both \(x_{27}\) and \(x_{28}\). Now there is a Hamiltonian path \(x_{29}x_{30}x_{31}x_{28} \cdots x_{25}C^-x_1\), so \(N(x_{29}) \subseteq V(H)\). Since \(x_{27}\) dominates \(x_{26}\) and \(x_{31}\), for \(x_{28}x_{29}x_{30}\) and the circle \(C\), by Observation 2, \(x_{29}\) is not adjacent to \(C\), this means \(d(x_{29}) \leq 5\), a contradiction.

From Cases 2.1-2.1, we get \(d(x_{31}) \leq 5\), a contradiction. This proves Case 2. It completes Assertion 4.

Now by using the three assertions we deduce Theorem 1. By Assertion 2 and Assertion 4, if \(P \in I_2\), then \(|P| \geq 17\). If \(P \in I_1\), then \(|P| \geq 34\). Hence

\[\sum_{P \in I_1} |P| \geq 34|I_1|\]

\[\sum_{P \in I_2} |P| \geq 17|I_2|\]
By Assertion 3,
\[ \sum_{P \in A} |P| \geq 26|E'|. \]
So we have
\[ n \geq \sum_{P \in I_1} |P| + \sum_{P \in I_2} |P| + \sum_{P \in A} |P| \geq 34|I_1| + 17|I_2| + 26|E'|, \]
i.e.,
\[ \frac{n}{51} \geq \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{26}{51}|E'|. \]
Combining with (\(\ast\)), we have \(|D| \leq \frac{6}{17} n\). This completes Theorem 1.

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