αm-OPEN SETS AND αM-CONTINUOUS FUNCTIONS

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Abstract. In this paper, we introduce the notions of αm-open sets and αM-continuous functions and investigate some properties of such concepts.

1. Introduction

In [4], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of M-continuous functions as functions defined between minimal structures. They showed that the M-continuous functions on minimal structures have properties similar to those of continuous functions between topological spaces.

In this paper, we introduce the notions of αm-open sets, α-interior and α-closure operators in minimal structures. We investigate some basic properties of such notions. Also we introduce the notion of αM-continuous functions and study characterizations of αM-continuous functions by using the α-interior and α-closure operators.

2. Preliminaries

Definition 2.1 ([1, 4]). A subfamily \( m_X \) of the power set \( P(X) \) of a nonempty set \( X \) is called a minimal structure on \( X \) if \( \emptyset \in m_X \) and \( X \in m_X \). By \( (X, m_X) \), we denote a nonempty set \( X \) with a minimal structure \( m_X \) on \( X \). Simply we call \( (X, m_X) \) a minimal structure on \( X \). Set \( M(x) = \{U \in m_X : x \in U\} \).

Definition 2.2 ([1, 4]). Let \( (X, m_X) \) be a minimal structure. For a subset \( A \) of \( X \), the closure of \( A \) and the interior of \( A \) are defined as the following:

1. \( mInt(A) = \bigcup\{U : U \subseteq A, U \in m_X\} \).
2. \( mCl(A) = \bigcap\{F : A \subseteq F, X - F \in m_X\} \).

Theorem 2.3 ([1, 4]). Let \( (X, m_X) \) be a minimal structure and \( A \subseteq X \).

1. \( X = mInt(X) \) and \( \emptyset = mCl(\emptyset) \).

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(2) \( m \text{Int}(A) \subseteq A \) and \( A \subseteq m \text{Cl}(A) \).
(3) If \( A \in m_X \), then \( m \text{Int}(A) = A \) and if \( X = F \in m_X \), then \( m \text{Cl}(F) = F \).
(4) If \( A \subseteq B \), then \( m \text{Int}(A) \subseteq m \text{Int}(B) \) and \( m \text{Cl}(A) \subseteq m \text{Cl}(B) \).
(5) \( m \text{Int}(m \text{Int}(A)) = m \text{Int}(A) \) and \( m \text{Cl}(m \text{Cl}(A)) = m \text{Cl}(A) \).
(6) \( m \text{Cl}(X - A) = X - m \text{Int}(A) \) and \( m \text{Int}(X - A) = X - m \text{Cl}(A) \).

**Definition 3.1.** Let \((X, m_X)\) be a minimal structure. The intersection of any two \(m\)-open sets is called an \(m\)-open set if \(X \subseteq m_X\). Then \( U \subseteq M(x) \) such that \( f(U) \subseteq V \).

3. \(am\)-open sets and \(\alpha M\)-continuity

**Definition 3.1.** Let \((X, m_X)\) be a minimal structure and \( A \subseteq X \). A subset \( A \) of \( X \) is called an \(am\)-open set if \( A \subseteq m \text{Int}(m \text{Int}(A)) \). The complement of an \(am\)-open set is called an \(am\)-closed set. The family of all \(am\)-open sets in \( X \) will be denoted by \(\alpha M(X)\).

**Remark 3.2.** Let \((X, \tau)\) be a topological space and \( A \subseteq X \). \( A \) is called an \(\alpha\)-open set \([3]\) if \( A \subseteq \text{int}(\text{int}(A)) \). The minimal structure \( m_X \) is a topology, clearly an \(am\)-open set is \(\alpha\)-open.

From Definition 3.1, obviously the following statements are obtained:

**Lemma 3.3.** Let \((X, m_X)\) be a minimal structure. Then

1. Every \(m\)-open set is \(am\)-open.
2. \( A \) is an \(am\)-closed set if and only if \( m \text{Cl}(m \text{Cl}(A)) \subseteq A \).

**Theorem 3.4.** Let \((X, m_X)\) be a minimal structure. Any union of \(am\)-open sets is \(am\)-open.

**Proof.** Let \( A_i \) be an \(am\)-open set for \( i \in J \). From Definition 3.1 and Theorem 2.3(4), it follows

\[
A_i \subseteq m \text{Int}(m \text{Cl}(m \text{Int}(A_i))) \subseteq m \text{Int}(m \text{Cl}(m \text{Int}(\cup A_i))).
\]

This implies \( \cup A_i \subseteq m \text{Int}(m \text{Cl}(m \text{Int}(\cup A_i))) \). Hence \( \cup A_i \) is an \(am\)-open set.

**Remark 3.5.** Let \((X, m_X)\) be a minimal structure. The intersection of any two \(am\)-open sets may not be \(am\)-open set as shown in the next example.

**Example 3.6.** Let \( X = \{a, b, c\} \) and \( m_X = \{\emptyset, \{a, b\}, \{a, c\}, X\} \) a minimal structure in \( X \). Then obviously \( \{a, b\} \) and \( \{a, c\} \) are \(\alpha\-m\) open sets. But \( \{a\} \) is not \(am\)-open because of \( m \text{Int}(m \text{Cl}(m \text{Int}(\{a\}))) = \emptyset \). Thus the intersection of two \(am\)-open sets \( \{a, b\} \) and \( \{a, c\} \) is not \(am\)-open.

**Definition 3.7.** Let \((X, m_X)\) be a minimal structure. For a subset \( A \) of \( X \), the \(\alpha\)-closure of \( A \) and the \(\alpha\)-interior of \( A \), denoted by \(am \text{Cl}(A)\) and \(am \text{Int}(A)\), respectively, are defined as the following:

\[
am \text{Cl}(A) = \cap \{ F : A \subseteq F, F \text{ is } am \text{-closed in } X \},
\]

\[
am \text{Int}(A) = \cup \{ U : U \subseteq A, U \text{ is } am \text{-open in } X \}.
\]
Theorem 3.8. Let \((X, m_X)\) be a minimal structure and \(A \subseteq X\). Then

1. \(\alpha m\text{-Int}(A) \subseteq A\).
2. If \(A \subseteq B\), then \(\alpha m\text{-Int}(A) \subseteq \alpha m\text{-Int}(B)\).
3. \(A\) is \(\alpha m\)-open if and only if \(\alpha m\text{-Int}(A) = A\).
4. \(\alpha m\text{-Int}(\alpha m\text{-Int}(A)) = \alpha m\text{-Int}(A)\).
5. \(\alpha m\text{-Cl}(X - A) = X - \alpha m\text{-Int}(A)\) and \(\alpha m\text{-Int}(X - A) = X - \alpha m\text{-Cl}(A)\).

Proof. (1), (2) Obvious.

(3) It follows from Theorem 3.4.

(4) It follows from (3).

(5) For \(A \subseteq X\),
\[
X - \alpha m\text{-Int}(A) = X - \cup\{U : U \subseteq A, U \text{ is } \alpha m\text{-open}\}
= \cap\{X - U : U \subseteq A, U \text{ is } \alpha m\text{-open}\}
= \cap\{X - U : X - A \subseteq X - U, U \text{ is } \alpha m\text{-open}\}
= \alpha m\text{-Cl}(X - A).
\]
Similarly, we have \(\alpha m\text{-Int}(X - A) = X - \alpha m\text{-Cl}(A)\).

Theorem 3.9. Let \((X, m_X)\) be a minimal structure and \(A \subseteq X\). Then

1. \(A \subseteq \alpha m\text{-Cl}(A)\).
2. If \(A \subseteq B\), then \(\alpha m\text{-Cl}(A) \subseteq \alpha m\text{-Cl}(B)\).
3. \(F\) is \(\alpha m\)-closed if and only if \(\alpha m\text{-Cl}(F) = F\).
4. \(\alpha m\text{-Cl}(\alpha m\text{-Cl}(A)) = \alpha m\text{-Cl}(A)\).

Proof. It is similar to the proof of Theorem 3.8.

Theorem 3.10. Let \((X, m_X)\) be a minimal structure and \(A \subseteq X\). Then

1. \(x \in \alpha m\text{-Cl}(A)\) if and only if \(A \cap V \neq \emptyset\) for every \(\alpha m\)-open set \(V\) containing \(x\).
2. \(x \in \alpha m\text{-Int}(A)\) if and only if there exists an \(\alpha m\)-open set \(U\) such that \(U \subseteq A\).

Proof. (1) Suppose there is an \(\alpha m\)-open set \(V\) containing \(x\) such that \(A \cap V = \emptyset\). Then \(X - V\) is an \(\alpha m\)-closed set such that \(A \subseteq X - V, x \notin X - V\). This implies \(x \notin \alpha m\text{-Cl}(A)\).

The reverse relation is obvious.

(2) Obvious.

Definition 3.11. Let \(f : X \to Y\) be a function between minimal structures \((X, m_X)\) and \((Y, m_Y)\). Then \(f\) is said to be \(\alpha M\)-continuous if for each \(x\) and each \(m\)-open set \(V\) containing \(f(x)\), there exists an \(\alpha m\)-open set \(U\) containing \(x\) such that \(f(U) \subseteq V\).

Every \(M\)-continuous function is \(\alpha M\)-continuous but the converse may not be true.
Example 3.12. Let $X = \{a, b, c\}$. Consider two minimal structures defined as follows: $m_1 = \{\emptyset, \{a\}, X\}$, $m_2 = \{\emptyset, \{a, b\}, \{a, c\}, X\}$.

Let $f : (X, m_1) \to (X, m_2)$ be the identity function. Then $f$ is $\alpha M$-continuous but not $M$-continuous.

Remark 3.13. Let $f : X \to Y$ be an $\alpha M$-continuous function between minimal structures $(X, m_X)$ and $(Y, m_Y)$. If the minimal structures $(X, m_X)$ and $(Y, m_Y)$ are topologies on $X$ and $Y$, respectively, then $f$ is $\alpha$-continuous [2].

Theorem 3.14. Let $f : X \to Y$ be a function on two minimal structures $(X, m_X)$ and $(Y, m_Y)$. Then the following statements are equivalent:

1. $f$ is $\alpha M$-continuous.
2. $f^{-1}(V)$ is an $\alpha M$-open set for each $m$-open set $V$ in $Y$.
3. $f^{-1}(B)$ is an $\alpha M$-closed set for each $m$-closed set $B$ in $Y$.
4. $f(\alpha M(Cl(A))) \subseteq m Cl(f(A))$ for $A \subseteq X$.
5. $m Cl(f^{-1}(B)) \subseteq f^{-1}(m Cl(B))$ for $B \subseteq Y$.
6. $f^{-1}(m Int(B)) \subseteq \alpha M Int(f^{-1}(B))$ for $B \subseteq Y$.

Proof. (1) $\Rightarrow$ (2) Let $V$ be an $m$-open set in $Y$ and $x \in f^{-1}(V)$. By hypothesis, there exists an $\alpha M$-open set $U_x$ containing $x$ such that $f(U_x) \subseteq V$. This implies $x \in U_x \subseteq f^{-1}(V)$ for all $x \in f^{-1}(V)$. Hence by Theorem 3.4, $f^{-1}(V)$ is $\alpha M$-open.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) For $A \subseteq X$,

$$f^{-1}(m Cl(f(A))) = f^{-1}(\cap \{F \subseteq Y : f(A) \subseteq F \text{ and } F \text{ is } m \text{-closed}\})$$
$$= \cap \{f^{-1}(F) \subseteq X : A \subseteq f^{-1}(F) \text{ and } F \text{ is } \alpha M \text{-closed}\}$$
$$\supseteq \cap \{K \subseteq X : A \subseteq K \text{ and } K \text{ is } \alpha M \text{-closed}\}$$
$$= \alpha M(Cl(A)).$$

Hence $f(\alpha M(Cl(A))) \subseteq m Cl(f(A))$.

(4) $\Rightarrow$ (5) For $A \subseteq X$, from (4), it follows

$$f(\alpha M(Cl(f^{-1}(A)))) \subseteq m Cl(f(f^{-1}(A))) \subseteq m Cl(A).$$

Hence we get (5).

(5) $\Rightarrow$ (6) For $B \subseteq Y$, from $m Int(B) = Y - m Cl(Y - B)$ and (5), it follows:

$$f^{-1}(m Int(B)) = f^{-1}(Y - m Cl(Y - B))$$
$$= X - f^{-1}(m Cl(Y - B))$$
$$\subseteq X - \alpha M(Cl(f^{-1}(Y - B))$$
$$= \alpha M Int(f^{-1}(B)).$$

Hence (6) is obtained.

(6) $\Rightarrow$ (1) Let $x \in X$ and $V$ an $m$-open set containing $f(x)$. Then from (6) and Theorem 2.5, it follows $x \in f^{-1}(V) = f^{-1}(m Int(V)) \subseteq \alpha M Int(f^{-1}(V)).$
So from Theorem 3.10, we can say that there exists an \(\alpha\)-open set \(U\) containing \(x\) such that \(x \in U \subseteq f^{-1}(V)\). Hence \(f\) is \(\alpha\)-continuous. \(\square\)

**Lemma 3.15.** Let \((X, m_X)\) be a minimal structure and \(A \subseteq X\). Then

\[
(1)\quad mCl(mInt(mCl(A))) \subseteq mCl(mInt(mCl(\alpha mCl(A)))) \subseteq \alpha mCl(A).
\]

\[
(2)\quad \alpha mInt(A) \subseteq mInt(mCl(\alpha mInt(A))) \subseteq mInt(mCl(mInt(A))).
\]

**Proof.** (1) For \(A \subseteq X\), by Theorem 3.9, \(\alpha mCl(A)\) is an \(\alpha\)-closed set. Hence from Lemma 3.3, we have

\[
mCl(mInt(mCl(A))) \subseteq mCl(mInt(mCl(\alpha mCl(A)))) \subseteq \alpha mCl(A).
\]

(2) It is similar to the proof of (1). \(\square\)

**Theorem 3.16.** Let \(f : X \to Y\) be a function on two minimal structure \((X, m_X)\) and \((Y, m_Y)\). Then the following statements are equivalent:

\[
(1)\quad f\text{ is }\alpha M\text{-continuous}. \\
(2)\quad f^{-1}(V) \subseteq mInt(mCl(f^{-1}(V)))\text{ for each }m\text{-open set }V\text{ in }Y. \\
(3)\quad mCl(mInt(mCl(f^{-1}(F)))) \subseteq f^{-1}(F)\text{ for each }m\text{-closed set }F\text{ in }Y. \\
(4)\quad f(mCl(mInt(mCl(A)))) \subseteq mCl(f(A))\text{ for each }A \subseteq X. \\
(5)\quad mCl(mInt(mCl(f^{-1}(B)))) \subseteq f^{-1}(mCl(B))\text{ for }B \subseteq Y. \\
(6)\quad f^{-1}(mInt(B)) \subseteq mInt(mCl(mInt(f^{-1}(B))))\text{ for }B \subseteq Y. \\
\]

**Proof.** (1) \(\Leftrightarrow\) (2) It follows from Theorem 3.14 and definition of \(\alpha m\)-open sets.

(1) \(\Leftrightarrow\) (3) It follows from Theorem 3.14 and Lemma 3.3.

(3) \(\Rightarrow\) (4) Let \(A \subseteq X\). Then from Theorem 3.14 (4) and Lemma 3.15, it follows

\[
mCl(mInt(mCl(A))) \subseteq \alpha mCl(A) \subseteq f^{-1}(mCl(f(A))).
\]

Hence \(f(mCl(mInt(mCl(A)))) \subseteq mCl(f(A))\).

(4) \(\Rightarrow\) (5) Obvious.

(5) \(\Rightarrow\) (6) From (5) and Theorem 2.3, it follows:

\[
f^{-1}(mInt(B)) = f^{-1}(Y - mCl(Y - B)) \\
= X - f^{-1}(mCl(Y - B)) \\
\subseteq X - mCl(mInt(mCl(f^{-1}(Y - B)))) \\
= mInt(mCl(mInt(f^{-1}(B)))).
\]

Hence, (6) is obtained.

(6) \(\Rightarrow\) (1) Let \(V\) be an \(m\)-open set in \(Y\). Then by (6) and Theorem 2.3, we have

\[
f^{-1}(V) = f^{-1}(mInt(V)) \subseteq mInt(mCl(mInt(f^{-1}(V)))) \subseteq \alpha mCl(\alpha mCl(f^{-1}(V))).
\]

This implies \(f^{-1}(V)\) is an \(\alpha m\)-open set. Hence by (2), \(f\) is \(\alpha M\)-continuous. \(\square\)

**References**


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