SECTIONAL ANALYTICITY IN SEQUENCE SPACES

T. BALASUBRAMANIAN\textsuperscript{a}, A. PANDIARANI\textsuperscript{b} AND T. TAMIZH CHELVAM\textsuperscript{c}

Abstract. The object of the present paper is to introduce Λ-dual and the concept of sectional analyticity (Abschinitta-analystique or AA property) of an FK-space. The motivation for AA-property is that a sequence space having AK-property possess AA-property.

1. Introduction

A sequence whose \( k \)-th term is \( x_k \) is denoted by \( (x_k) \) or \( x \). Let \( \omega \) denote the set of all sequences. A sequence \( x \) is said to be an entire sequence if \( |x_k|^{1/k} \to 0 \) as \( k \to \infty \). The set \( \Gamma \) of all entire sequences is an FK space \([3]\) with seminorms \( q_i = \sup \left\{ \sum_{k=1}^{\infty} x_k z_k : |z| = i \right\} \) for \( i = 1, 2, \ldots \). A sequence \( x \) is said to be an analytic sequence if \((|x_k|^{1/k})\) is bounded. Let \( \Lambda \) denote the set of all analytic sequences.

For each positive integer \( k \), let \( \delta^k \) stands for the sequence \((0, 0, \ldots, 0, 1, 0, \ldots)\) with 1 in the \( k \)-th place and zeros elsewhere. A sequence space \( X \) is said to be an AK space if \( x^{[n]} \to x \) for each \( x \in X \) where \( x^{[n]} = (x_1, \ldots, x_n, 0, 0, \ldots) \). For a sequence space \( X \) its conjugate space is denoted by \( X' \).

Let \( X \) be any sequence space. Then \( X^\alpha \) is the Kothe-Toeplitz dual of \( X \) introduced in \([7]\), \( X^\beta \) is the space called the “g-dual” of \( X \) by Chillingworth in \([1]\) and the \( \beta \)-dual of \( X \) by Kothe and others \([8, p. 427]\). For arbitrary sequences \( X \) and \( Y \), \( X^\gamma \) is the space called \( X \to Y \) by Goes \([5, p. 137]\) and elsewhere. For \( Y = bs \) and arbitrary \( X \), \( X^\gamma \) corresponds to the \( \gamma \)-dual of \( X \) of Garling \([4]\) and others. Let \( X \) be an FK space containing \( \phi \). Then the \( f \)-dual denoted by \( X^f \) is defined by \([10]\) \[ X^f = \{ [f(\delta^n)] : f \in X' \} \]. An FK space \( X \) is called an integral space \([2]\) if and only if \( \Gamma \subset X \). The work presented in this paper is motivated by the following questions. “Are all integral spaces \( A \)-perfect?” and “Are all AA-space having AK-property?”.
In the sequel the following sequence spaces are required.

\( \ell = \) the BK space of all sequences \((x_k)\) such that \(\sum_{k=1}^{\infty} |x_k|\) converges.

\( cs = \) the BK space of all sequences \((x_k)\) such that \(\sum_{k=1}^{\infty} x_k\) converges.

\( bs = \) the BK space of all sequences \((x_k)\) such that \(\sup_n \sum_{k=1}^{\infty} x_k < \infty\).

The rest of the paper is organized as follows:

In Section 2, we introduce the concepts of \(\Lambda\)-dual and \(\Lambda\)-perfect. We have also tried to find the \(\Lambda\)-dual and \(\Lambda\)-perfect space of \(X\) with \(\Gamma \subseteq X \subseteq \Lambda\).

In Section 3, we introduce the concept of sectional analyticity and try to find the relation between \(f\)-dual and \(\Lambda\)-dual.

2. ANALYTICAL DUAL OF A SEQUENCE SPACE \(X\)

**Definition 2.1.** Let \(X\) be an FK space. The \(\Lambda\)-dual of \(X\) (denoted by \(X^{\Lambda}\)) and may be called analytic dual of \(X\) is defined as \(X^{\Lambda} = \{x \in \omega : xu \in \Lambda \text{ for every } u \in X\} \).

**Definition 2.2.** An FK-space \(X\) is called a perfect space or a \(\Lambda\)-perfect space if \(X^{\Lambda\Lambda} = X\).

**Remark 2.3.** The definitions also hold when \(X\) is a singleton or a sequence space instead of an FK space.

**Lemma 2.4.** The \(\Lambda\)-dual of a sequence space has the following properties.

1. \(X^{\Lambda}\) is linear subspace of \(\omega\) for \(X \subset \omega\).
2. \(X \subset Y\) implies \(X^{\Lambda} \supset Y^{\Lambda}\) for every \(X, Y \subset \omega\).
3. \(X^{\Lambda\Lambda} = (X^{\Lambda})^{\Lambda} \supset X\) for every \(X \subset \omega\).

**Lemma 2.5.**

1. \(1^{\Lambda} = \Lambda\) where \(1 = (1,1,1,\ldots)\).
2. \(\emptyset^{\Lambda} = \omega\).
3. The \(\Lambda\)-dual of \(\chi\) = \(\{u \in \omega : [n!|u_n|]^{1/n} \to 0 \text{ as } n \to \infty\}\) is \(S_{\infty} = \{u \in \omega : (|u_n|/n!)^{1/n} \text{ is bounded}\}\) suppose if \(x \in S_{\infty}\) then \(|x_n u_n|^{1/n} \to 0\) as \(n \to \infty\) for all \(u \in X\). So the sequence \((|x_n u_n|^{1/n})\) is bounded and hence \(S_{\infty} \subset \chi^{\Lambda}\) on the other hand suppose \(x \notin S_{\infty}\). Then there exists an increasing sequence \(n_1 < n_2 < \cdots\) such that \(\left[\frac{|x_{n_k}|}{n_k!}\right]^{1/n_k} > k\).
Define \( u = (u_n) \) by
\[
u_n = \begin{cases}
\frac{1}{n!}, & \text{for } n = n_k \\
0, & \text{otherwise}
\end{cases}
\]
Then \( |n!|u_n|^{1/n} = \left[ \frac{1}{(n!)^{n-1}} \right]^{1/n} = \frac{1}{(n-1)!} \left( \frac{n}{n_k} \right)^{1/n} \to 0 \) as \( n \to \infty \) thus \( u \) is an element of \( \chi \).

But \( |x_n u_k|^{1/n} = \left[ \frac{|x_n|}{n_k} \right]^{1/n_k} > k \).

This contradicts the fact that \( x \in \chi^\Lambda \) and hence the \( \Lambda \)-dual of \( \chi \) is \( S_\infty \).

(iv) The \( \Lambda \)-dual of \( R = \{ x : (n!|x_n|) \text{ is bounded} \} \) is \( S_\infty \).

Now \( |x_k u_k| = \left[ \frac{|x_k|}{k!} \right] k!|u_k| \leq ||u|| \left[ \frac{|x_k|}{k!} \right], \ x = (x_k) \in S_\infty \). (9)

Therefore \( (|x_k u_k|^{1/k}) \) is bounded and \( x \) is an element of \( R^\Lambda \). On the other hand if \( x \in R^\Lambda \) then \( (|x_k u_k|^{1/k}) \) is bounded for all \( x \in R \).

Therefore \( (||x_k|/k!|^{1/k}) \) is bounded for a particular \( (1/k!) \in R \). Hence the \( \Lambda \)-dual of \( R \) is \( S_\infty \).

**Theorem 2.6.** Suppose \( \Gamma \subseteq X \subseteq \Lambda \). Then \( X^\Lambda = \Lambda \).

**Proof.** Step (i): We first claim that \( \Gamma^\Lambda = \Lambda \). If \( x \in \Lambda \) then \( (|x_k|^{1/k}) \) is bounded. For any \( u \in \Gamma \) and \( x \in \Lambda \), \( u \cdot x \in \Lambda \) therefore \( x \in \Gamma^\Lambda \).

On the other hand suppose \( x \notin \Lambda \) then there would exist an increasing sequence of positive integers \( n_1 < n_2 < \ldots < n_k < \ldots \) such that \( |x_{n_k}|^{1/n_k} > p^{n_k} \) where \( p > 1 \) is an integer. Construct a sequence \( u = (u_n) \) as follows.
\[
u_n = \begin{cases}
k^n, & \text{if } n = n_k \ (k = 1, 2, \ldots) \\
0, & \text{otherwise}
\end{cases}
\]

Obviously \( u \in \Gamma \).

But \( |x_{n_k} u_{n_k}|^{1/n_k} > k \), so that \( (|x_n u_n|^{1/n}) \) is unbounded which is a contradiction to the fact that \( x \in \Gamma^\Lambda \).

Thus \( \Gamma^\Lambda = \Lambda \).

Step (ii): We show that \( \Lambda^\Lambda = \Lambda \).

\( N \subseteq \Lambda \) implies \( \Lambda^\Lambda \subseteq \Gamma^\Lambda = \Lambda \) (by step (i)). That is \( \Lambda^\Lambda \subseteq \Lambda \). Also we have \( \Lambda \subseteq \Lambda^\Lambda \). Hence \( \Lambda \subseteq \Lambda^\Lambda \).

Step (iii): We show that \( X^\Lambda = \Lambda \).

\( N \subseteq X \subseteq \Lambda \) implies \( X^\Lambda \subseteq \Gamma^\Lambda \). Then by step (i) we have \( X^\Lambda \subseteq \Lambda \). Also \( X \subseteq \Lambda \) implies \( \Lambda^\Lambda \subseteq X^\Lambda \). Then by step (ii) we have \( \Lambda \subseteq X^\Lambda \). Thus \( X^\Lambda = \Lambda \).
Corollary 2.7. The only $\Lambda$ perfect space $X$ with $\Gamma \subseteq X \subseteq \Lambda$ is $\Lambda$.

Proof. Let $X$ be such that $X^{\wedge\wedge} = X$. Since $\Gamma \subseteq X$ we have $X^\wedge \subseteq \Gamma^\wedge = \Lambda$ (by step (i) of 2.6). By applying step (ii) of 2.6, $\Lambda = \Lambda^\wedge \subseteq X^{\wedge\wedge} = X$. Also by hypothesis $X \subseteq \Lambda$. □

3. Sectional Analyticity

Definition 3.1. Let $X$ and $Y$ be FK spaces containing $\phi$. Then $A^+$ is defined as $A^+(X) = \{ z \in \omega : (z f(\delta^k)) \in \Lambda \text{ for all } f \in X' \}$ and we put $A = A^+ \cap X$.

Lemma 3.2. Let $X$ and $Y$ be a be FK spaces containing $\phi$. Then $A^+(X) \subset A^+(Y)$ wherever $X \subset Y$.

Proof. Let $Z \in A^+(X)$. Then $(Z_n f(\delta^n)) \in \Lambda$ for all $f \in X'$. Accordingly $(z_n g(\delta^n)) \in \Lambda$ for all $g \in Y'$ since $g|X \in X'$. This shows that $z \in A^+(Y)$. Hence $A^+(x) \subset A^+(Y)$. □

Definition 3.3. Let $X$ be an FK space containing $\phi$. Then $X$ is said to have AA. Property (Abschnitts analytique) or sectional analyticity if and only if $X = A^+$.

Example 3.4. The space $c_0$ has both AK and AA properties. The space $c_0$ has AK [10]. It is enough to prove that $c_0$ has AA-property. For that we have to show that $c_0 \subset A^+$, $f \in c_0'$. Then $f(z) = \sum_{k=1}^{\infty} a_k z_k$ where $(a_k) \in l$. Therefore $f(\delta^k) = a_k$ for all $k$. But $l \subset \Lambda = c^\wedge$. Hence $(z f(\delta^k)) \in \Lambda$ and so $z \in A^+$. Hence $c_0 \subset A^+$. Therefore $A = A^+ \cap c_0 = c_0$.

Lemma 3.5. Let $X$ be an FK space containing $\phi$. Let $z \in \omega$. Then $z \in X^+ \wedge$ if and only if $z^{-1}X \supset \Gamma$.

Proof. Let $f \in (z^{-1}X)'$. Then by Theorem 4.4.10 of [10] $f(x) = \alpha x + g(zx)$ where $\alpha \in \phi$, $g \in X'$ and $\alpha x = \sum_{k=1}^{\infty} \alpha_k x_k$. Consequently $f(\delta^k) = \alpha_k + g(z\delta^k)$. That is $f(\delta^k) = \alpha_k + z_k g(\delta^k)$. Hence if $z \in A^+$, then $(z f(\delta^k)) \in A$ and so $(f(\delta^k)) \in \Lambda$ for all $f \in (z^{-1}X)'$. That is $(z^{-1}X)' \subset \Lambda$. But $\Lambda = \Gamma^f$. Since $\Gamma$ has AD by Theorem 8.6.1 of [10], $\Gamma \subset z^{-1}X$. The reverse implication follows similarly. □

Theorem 3.6. Let $X$ be an FK space containing $\phi$. Then $z \in X^{f^\wedge}$ if and only if $z^{-1}X \supset \Gamma$. 
Proof. First we note that by definition \( z \in A^+ \) if and only if \( z u \in A \) for every \( u \in X^f \). Hence \( A^+ = X^{f^\Lambda} \). By the Lemma 3.5, \( z \in A^+ \) if and only if \( z^{-1}X \supset \Gamma \). Hence \( z \in X^{f^\Lambda} \) if and only if \( z^{-1}X \supset \Gamma \). \( \square \)

**Theorem 3.7.** Let \( X \) be an FK space containing \( \phi \). If \( X \) has AA, then \( X^f \subset X^{\Lambda} \).

Proof. Suppose that \( X \) has AA. Then \( X = A = A^+ \cap X \). So that \( X \subset A^+ = X^{f^\Lambda} \). Hence \( X^\Lambda \supset X^{f^\Lambda} \). Therefore \( X^\Lambda \supset X^f \). \( \square \)

**Theorem 3.8.** Let \( X \) be an FK space \( \supset \phi \). If \( X \) has AK then \( X \) has AA.

Proof. Suppose \( X \) has AK. Then we have \( X^\beta = X^f \). This implies \( X \subset X^{\beta\beta} = X^{f\beta} \). Also we have \( X \subset X^{f\beta} \subset X^{f^\Lambda} \). That is \( X \subset X^{f^\Lambda} \). This means that \( X \subset A^+ \) consequently \( A = X \). Hence \( X \) has AA property. \( \square \)

**Remark 3.9.** The converse of Theorem 3.8 need not be true. Consider the space \( c, A^+(c) = c^{f^\Lambda} = f^\Lambda = \Lambda \). Now \( A = A^+ \cap c = \Lambda \cap c = c \). Therefore \( c \) has AA. But \( c \) does not posses AK-Property.

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**References**


*a* Department of Mathematics, Kamaraj College, Tuticorin, Tamil Nadu, India
Email address: satbalu@yahoo.com

*b* Department of Mathematics, G. Vengadaswamy Naidu College, Kovilpatti 628502, Tamil Nadu, India
Email address: raniseelan_92@yahoo.co.in

*c* Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627012, Tamil Nadu, India
Email address: tamche_59@yahoo.co.in