SOME APPLICATIONS OF THE UNION OF STAR-CONFIGURATIONS IN \( \mathbb{P}^n \)

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Abstract. It has been proved that if \( X(s,s) \) is the union of two linear star-configurations in \( \mathbb{P}^2 \) of type \( s \times s \), then \( (I_{X(s,s)})_s \neq \{0\} \) for \( s = 3, 4, 5 \), and \( (I_{X(s,s)})_s = \{0\} \) for \( s \geq 6 \). We extend \( \mathbb{P}^2 \) to \( \mathbb{P}^n \) and show that if \( X(s,s) \) is the union of two linear star-configurations in \( \mathbb{P}^n \), then \( (I_{X(s,s)})_s = \{0\} \) for \( n \geq 3 \) and \( s \geq 3 \). Using this generalization, we also prove that the secant variety \( \text{Sec}_1(\text{Split}_s(\mathbb{P}^n)) \) has the expected dimension \( 2ns + 1 \) for \( n \geq 3 \) and \( s \geq 3 \).

1. Introduction

We are interested in the secant variety to the variety \( X \subset \mathbb{P}^n \) and the tangent space ideal at a point in \( X \subset \mathbb{P}^n \), where \( X \) is a non-degenerate, reduced, and irreducible variety of dimension \( d \). We are also interested in the dimension of the secant variety to determine if the secant variety is not defective. Recent papers studied the secant varieties ([1, 3, 4, 5, 7, 8, 9, 10, 12, 13]).

In [13], the author showed that if \( X(t,s) \) is the union of two linear star-configurations in \( \mathbb{P}^2 \) of type \( t \times s \) with \( 3 \leq t \leq 9 \) and \( s \geq t \), then \( R/I_{X(t,s)} \) has generic Hilbert function, and \( (I_{X(s,s)})_s = \{0\} \) for \( s \geq 6 \). With these two results, the author also showed that when \( n = 2 \), the secant variety \( \text{Sec}_1(\text{Split}_s(\mathbb{P}^2)) \) has the expected dimension \( 4s + 1 \) for \( s \geq 6 \) and that the ideal \( I_{X(s,s)} \) has the following property:

\[
\dim_k(I_{X(s,s)})_s = \begin{cases} 
3,3,1 & \text{for } s = 3,4,5, \text{ respectively,} \\
0, & \text{for } s \geq 6.
\end{cases}
\]

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In this paper we attempt to generalize this result and find an answer to the following question.

Question 1.1. What is \( \dim_k(I_X)_s \) when \( X := X^{(s,s)} \) is the union of two linear star-configurations in \( \mathbb{P}^n \) of type \( s \times s \), \( n \geq 3 \) and \( s \geq 3 \)?

In [3] the author showed that when \( 3(s-1) \leq n \) and \( s > 2 \), the secant variety \( \text{Sec}_{r-1}(\text{Split}_s(\mathbb{P}^n)) \) has the expected dimension using Terracini’s Lemma, which will be introduced in the next section. We will however use the ideal of the union of two linear star-configurations \( X \) in \( \mathbb{P}^n \) instead of Terracini’s Lemma to find first the dimension, \( \dim_k(I_X)_s \), and then the secant variety \( \text{Sec}_{r-1}(\text{Split}_s(\mathbb{P}^n)) \).

Our goal is to find an answer to Question 1.1 and its applications. In Section 2, we briefly review some definitions, notations, and preliminary results of the secant varieties \( \text{Sec}_{r-1}(\text{Split}_s(\mathbb{P}^n)) \). In Section 3, we show that if \( X := X^{(s,s)} \) is the union of two linear star-configurations in \( \mathbb{P}^n \) with \( n \geq 3 \) and \( s \geq 3 \), then

\[
(I_X)_s = \{0\},
\]

which is the key element to the complete answer to Question 1.1. With this result, we introduce another method to prove that the secant variety

\[
\text{Sec}_1(\text{Split}_s(\mathbb{P}^n))
\]

has the expected dimension \( 2ns + 1 \) for \( n \geq 3 \) and \( s \geq 3 \).

2. Preliminary results and definitions

First, we recall definitions of Hilbert function, the secant varieties \( \text{Sec}_{r-1}(\text{Split}_s(\mathbb{P}^n)) \), and irreducible varieties respectively. Let \( R = k[x_0, x_1, \ldots, x_n] \) be an \((n+1)\)-variable polynomial ring over a field \( k \) of characteristic 0, \( R_d \) its homogeneous part of degree \( d \), and \( \mathbb{P}^n \) the projective \( n \)-space over a field \( k \). With these notations, \( \mathbb{P}(R_d) := \mathbb{P}^{(n+d)-1} \) is naturally identified with the set of hypersurfaces of degree \( d \) in \( \mathbb{P}^n \). Recall that if \( I \) is a homogeneous ideal in \( R \) or the ideal of a subscheme \( X \) in \( \mathbb{P}^n \), then \( R/I = \bigoplus_{t \geq 0} R_t/I_t \) is a graded ring. In this situation the \textit{Hilbert function} of \( X \) (or \( R/I \)) is the function of the subscheme \( X \) (or of the ring \( R/I \)) as follows:

\[
H_X(t) = H(R/I, t) := \dim_k R_t - \dim_k I_t.
\]

The first difference of the Hilbert function \( H \) is defined by \( \Delta H(0) = 1 \) and \( \Delta H(t) = H(t) - H(t-1) \) for \( t > 0 \).
Let $\lambda \vdash d$ denote a partition of the integer $d$, i.e.

$$\lambda = (\lambda_1, \ldots, \lambda_r)$$

where $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$ and $\sum_{i=1}^{r} \lambda_i = d$.

We associate a variety, denoted by $X_{\lambda,n}$, to $R = k[x_0, x_1, \ldots, x_n]$ and $\lambda$, which is defined by

$$X_{\lambda,n} := \{ F \in \mathbb{P}(R_d) \mid F = F_1 \cdots F_r, \ \deg F_i = \lambda_i \},$$

and we omit the $n$ if it is clear from the context. Such varieties are called varieties of reducible forms. If $\lambda$ is the $d$-tuple $(1, \ldots, 1)$, then the variety is often referred to as the variety of completely decomposable forms or split forms. In this case, $X_{\lambda,n}$ is denoted by $\text{Split}_d(P^n)$.

Let $X_1, \ldots, X_r$ all be non-degenerate, reduced and irreducible varieties in $P^n$ with $\dim X_i = d_i$.

**Definition 2.1** (Definition 2.1, [1]).

(a) Choose points $P_i \in X_i$ such that $\{P_1, \ldots, P_r\}$ are linearly independent (and so $r \leq n$). The join of $\{P_1, \ldots, P_r\}$ is the linear space spanned by the points, i.e.,

$$\Lambda(P_1, \ldots, P_r) := \langle P_1, \ldots, P_r \rangle \simeq \mathbb{P}^{r-1}.$$

(b) The join of $X_1, \ldots, X_r$ is

$$\Lambda(X_1, \ldots, X_r) = \bigcup \Lambda(P_1, \ldots, P_r)$$

for all $P_1, \ldots, P_r$ linearly independent with $P_i \in X_i$.

(c) If $X_1 = \cdots = X_r = \mathbb{X}$ with $\dim \mathbb{X} = d$, then we write

$$\Lambda(X_1, \ldots, X_r) = \text{Sec}_{r-1}(\mathbb{X})$$

and call it the $(r-1)$-st secant variety to $\mathbb{X}$.

The number of parameters shows that the upper bound of the dimension of the join is

$$\dim \Lambda(X_1, \ldots, X_r) \leq \min \left\{ n, \sum_{i=1}^{r} d_i + (r - 1) \right\},$$

and thus

$$\dim \text{Sec}_{r-1}(\mathbb{X}) \leq \min\{n, dr + (r - 1)\}.$$
i.e., the linear space of the tangent spaces at the given points.

**Definition 2.3.** Let $X \subset \mathbb{P}^n$ be a projective variety of dimension $d$. Then the *expected dimension* of the secant variety $\text{Sec}_{r-1}(X)$ to $X$ is defined by

$$\text{expdim}(\text{Sec}_{r-1}(X)) = \min\{n, dr + (r - 1)\}.$$ 

However, the expected dimension of $\text{Sec}_{r-1}(X)$ is not always the same as $\dim \text{Sec}_{r-1}(X)$. When $\delta_{r-1} = \text{expdim}(\text{Sec}_{r-1}(X)) - \dim \text{Sec}_{r-1}(X) > 0$, we say that the secant variety $\text{Sec}_{r-1}(X)$ to $X$ is *defective* and $\delta_{r-1}$ is called *defect*.

Since we are interested in secants to the varieties of reducible forms, we introduce another important result (in view of Terracini’s Lemma) in [7] to find a description of the tangent space at a generic point of those varieties.

**Proposition 2.4 ([7]).** Let $\lambda \vdash d$, $\lambda = (\lambda_1, \ldots, \lambda_r)$ and let $X_{\lambda,n} \subset \mathbb{P}^{(d+n)-1}$. Let $P = [F_1 \cdots F_r]$ be a generic point of $X_{\lambda,n}$ where $\deg F_i = \lambda_i$, $i = 1, \ldots, r$. Then

$$T_P X_{\lambda,n} = \mathbb{P}(V_P)$$

where $V_P$ is the subspace of $R_d = k[x_0, \ldots, x_n]$ defined by

$$V_P := \sum_{i=1}^{r} (F_1 \cdots \hat{F}_i \cdots F_r) R_{\lambda_i},$$

where $\hat{\ast}$ means that we omit $\ast$.

When we wish to find the dimension of the secant variety $\text{Sec}_{r-1}(X_{\lambda,n})$ to $X_{\lambda,n}$, Terracini’s Lemma clearly suggests that we choose first generic points, $P_1, \ldots, P_r$ on $X_{\lambda,n}$, and then find the dimension of the subspace

$$V_{P_1} + \cdots + V_{P_r} \subset k[x_0, \ldots, x_n].$$

We try to place this problem in a more general context.

**Definition 2.5.** The *tangent space ideal* of $X_{\lambda,n}$ at the point $P$ is the unique saturated ideal, $T_P$, in $R = k[x_0, \ldots, x_n]$, with the property that

$$(T_P)_d = V_P.$$ 

The following corollary shows the dimension of the secant variety, by which we can decide whether or not the dimension and the expected dimension of the secant variety are the same. In other words, we can determine if the given secant variety is not defective.
Corollary 2.6 ([7]). Let $\lambda \vdash d$, $\lambda = (\lambda_1, \ldots, \lambda_r)$ and let $X_{\lambda, n} \subset \mathbb{P}^{(d+n)-1}$. Let $P_1, \ldots, P_r$ be $r$ generic points on $X_{\lambda, n}$. Then

$$\dim \text{Sec}_{r-1}(X_{\lambda, n}) = \left[\binom{d+n}{n} - \mathbf{H}(A, d)\right] - 1 = \dim_k I_d - 1$$

where $A = R/I$ and $I = T_{P_1} + \cdots + T_{P_r}$.

Finally we introduce a star-configuration and a linear star-configuration in $\mathbb{P}^n$.

Proposition 2.7 ([1]). Let $F_1, F_2, \ldots, F_r$ be general forms in $R = k[x_0, x_1, \ldots, x_n]$ with $r \geq 3$. Then

$$\bigcap_{1 \leq i < j \leq r} (F_i, F_j) = \sum_{i=1}^{r} (F_1 \cdots \hat{F}_i \cdots F_r).$$

Definition 2.8. With notations in Proposition 2.7, the variety $X$ in $\mathbb{P}^n$ of the ideal $\bigcap_{1 \leq i < j \leq r} (F_i, F_j) = \sum_{i=1}^{r} (F_1 \cdots \hat{F}_i \cdots F_r)$ is called a star-configuration in $\mathbb{P}^n$ of type $r$. In particular, if $\deg F_i = 1$ for every $i = 1, \ldots, r$, we call $X$ a linear star-configuration in $\mathbb{P}^n$ of type $r$. Furthermore, if $X$ is the union of two star-configurations in $\mathbb{P}^n$ defined by $s$ general linear forms and $t$ general linear forms, respectively, then we call $X$ the union of two linear star-configurations $\mathbb{P}^n$ of type $s \times t$, denoted by $X := X(s, t)$.

Remark 2.9. (a) If $X$ is a star-configuration in $\mathbb{P}^n$, then $X$ is an arithmetically Cohen-Macaulay subscheme in $\mathbb{P}^n$ of codimension 2 (see Remark 2.2, [2]).

(b) Let $R = k[x_0, \ldots, x_n]$, $X$ be a star-configuration in $\mathbb{P}^n$ with $n \geq 3$, and $L$ be a general linear form in $R/I_X$. Since $X$ is an arithmetically Cohen-Macaulay subscheme in $\mathbb{P}^n$ of codimension 2, $L$ is a non-zero divisor of $R/I_X$. Thus $R/(I_X, L)$ is also a Cohen-Macaulay ring of codimension 2. In other words, $(I_X, L)/(L)$ is also the ideal of a star-configuration in $\mathbb{P}^{n-1}$.

3. Secant varieties $\text{Sec}_1(\text{Split}_s(\mathbb{P}^n))$

As mentioned in the introduction, in [3] they proved that the secant variety $\text{Sec}_{r-1}(\text{Split}_s(\mathbb{P}^n))$ has the expected dimension for $3(s-1) \leq n$ and $s > 2$ using Terracini’s Lemma. As these two conditions indicate, it has been unknown for $n = 2$. In [13], the author showed that the secant
variety $\text{Sec}_1(\text{Split}_s(\mathbb{P}^2))$, where $n = 2$, has the expected dimension for $s > 2$.

In this section, we shall find the dimension of the ideal of the union of two linear star-configurations in $\mathbb{P}^n$ of type $s \times s$ in degree $s$. With this result, we shall give another method to prove that $\text{Sec}_1(\text{Split}_s(\mathbb{P}^n))$ has the expected dimension for $n \geq 3$ and $s > 2$.

The following lemma is immediately from Proposition 2.7 and Remark 2.9 (a), (see also Corollary 2.5 in [2]).

**Lemma 3.1.** Let $L_i$ be general linear forms in $k[x_0, x_1, x_2, x_3]$ for $i = 1, \ldots, s$ with $s \geq 3$ and let

$$I_s := \sum_{i=1}^{s} (L_1 \cdots \hat{L}_i \cdots L_s)$$

be the ideal of a linear star-configuration in $\mathbb{P}^3$. Then the Hilbert function of $R/I_s$ is

$$H(R/I_s, t) = \begin{cases} \binom{3+t}{3}, & 0 \leq t \leq s-2, \\ \binom{3+(s-2)}{3} + (t - (s-2))\binom{2+(s-2)}{2}, & t \geq s-1. \end{cases}$$

*Proof.* Let $L$ be a general linear form in $R$. By Remark 2.9 (b), the Hilbert function of $R/(I_s, L)$ is the same as the Hilbert function of the linear star-configuration in $\mathbb{P}^2$ of type $s$. Thus the first difference of the Hilbert function of $R/I_s$ is

$$\Delta H(R/I_s, t) = \begin{cases} \binom{2+t}{2}, & 0 \leq t \leq s-2, \\ \binom{2+((s-2))}{2}, & t \geq s-1. \end{cases}$$

This implies that

$$H(R/I_s, t) = \begin{cases} \binom{3+t}{3}, & 0 \leq t \leq s-2, \\ \binom{3+(s-2)}{3} + (t - (s-2))\binom{2+(s-2)}{2}, & t \geq s-1, \end{cases}$$

as we wished. \qed

**Remark 3.2.** Let $n$ and $s$ be positive integers. By induction on $n$, we can easily obtain the following equation, and so we omit the proof.

$$\binom{s+n}{n} = \binom{(s-1)+n}{(s-1)} + \binom{(s-1)+(n-1)}{(s-1)} + \cdots + \binom{(s-1)+1}{(s-1)} + \binom{(s-1)+0}{(s-1)}.$$
Proposition 3.3. Let $R = k[x_0, x_1, \ldots, x_n]$ and $L_i$ be general linear forms in $R$ for $i = 1, \ldots, s$ with $s \geq 3$ and $n \geq 3$. Let

$$I^{[s]} := \sum_{i=1}^{s}(L_1 \cdots L_i)$$

be the ideal of a linear star-configuration in $\mathbb{P}^n$. Then

$$\dim_k I^{[s]}_j = ns + 1.$$  

Proof. We shall prove this by induction on $n$ with $n \geq 3$.

First, let $n = 3$. Then, by Lemma 3.1, the statement holds for this case.

Now assume $n > 3$. By Remark 2.9 (b), the first difference of the Hilbert function of $R/I^{[s]}$ is the Hilbert function of a linear star-configuration in $\mathbb{P}^{n-1}$ of type $s$. Hence, by induction on $n$, we have

$$\Delta \mathbf{H}(R/I^{[s]}, s) = \mathbf{H}(R/I^{[s]}, s) - \mathbf{H}(R/I^{[s]}, s - 1)$$

$$= \sum_{i=1}^{s}(L_1 \cdots L_i) = (s-2)\binom{n-3}{n-2} + (s-2)\binom{n-2}{3} + \cdots + (n-1-2)\binom{n-2}{2} + \cdots + (n-1-1)\binom{n-1-2}{2} + \cdots + (n-1-1)\binom{n-1-1}{1} + \cdots + (n-1-1)\binom{n-1-1}{1}.$$  

This implies that

(3.1)$$\mathbf{H}(R/I^{[s]}, s) = \mathbf{H}(R/I^{[s]}, s) + \sum_{i=1}^{s}(L_1 \cdots L_i)$$

$$= \left[\binom{(s-2)+(n-1)}{(n-1)} + 2 \cdot \binom{(s-2)+(n-2)}{(n-2)} + \cdots + (n-1-2)\binom{(s-2)+3}{3} + (n-1-1)\binom{(s-2)+2}{2} + \cdots + (n-1-1)\binom{(s-2)+1}{1}\right]$$

(by Remark 3.2)
which completes the proof.

From equation (3.2), we have

$$\dim_k I_s^{[n]} = \dim_k R_s - \text{H}(R/I^{[s]}, s) = ns + 1,$$

which completes the proof. □
We now find the dimension of the ideal of the union of two linear star-configurations in \( \mathbb{P}^3 \) of type \( s \times s \) in degree \( s \). This lemma is a bridge to the main theorem (see Theorem 3.7).

**Lemma 3.4.** Let \( R = k[x_0, x_1, x_2, x_3] \) and \( L_i, M_i \) be general linear forms in \( R \) for \( i = 1, 2, \ldots, s \) with \( s \geq 3 \) and let
\[
I^{[s]} := \sum_{i=1}^{s} (L_1 \cdots \hat{L}_i \cdots L_s),
\]
\[
J^{[s]} := \sum_{i=1}^{s} (M_1 \cdots \hat{M}_i \cdots M_s),
\]
i.e., the ideals of linear star-configurations in \( \mathbb{P}^3 \) of type \( s \) defined by linear forms \( L_1, \ldots, L_s \) and \( M_1, \ldots, M_s \), respectively. Then, for \( 3 \leq s \leq 5 \),
\[
\dim_k(I^{[s]} \cap J^{[s]}) = 0.
\]

**Proof.** We shall prove this lemma with 3 cases for \( s = 3, 4, \) and 5, respectively.

**Case 1.** Let \( s = 3 \).

Define the ideal \( I^{[2]} = (x_0, x_1) \). Without loss of generality, we may assume that
\[
M_1 = x_2, M_2 = x_3, M_3 = ax_0 + bx_1 + cx_2 + dx_3,
\]
where \( a, b, c, d \in k - \{0\} \). Consider the following exact sequence.
\[
(3.3) \quad 0 \to J^{[3]} \cap I^{[2]} \to J^{[3]} \to J^{[3]}/(J^{[3]} \cap I^{[2]}) \to 0.
\]
Since \( J^{[3]}/(J^{[3]} \cap I^{[2]}) \simeq (J^{[3]} + I^{[2]})/I^{[2]} \), we can rewrite equation (3.3) as
\[
(3.4) \quad 0 \to J^{[3]} \cap I^{[2]} \to J^{[3]} \to (J^{[3]} + I^{[2]})/I^{[2]} \to 0.
\]

Since the dimension of \( (J^{[3]} + I^{[2]})/I^{[2]} \) in degree 2 is represented by \( \dim_k(R/(x_0, x_1))_2 \):
\[
\dim_k((J^{[3]} + I^{[2]})/I^{[2]})_2 = \dim_k(((M_1 M_2, M_1 M_3, M_2 M_3) + (x_0, x_1))/(x_0, x_1))_2
\]
\[
= \dim_k((x_2 x_3, x_2(ax_0 + bx_1 + cx_2 + dx_3), x_3(ax_0 + bx_1 + cx_2 + dx_3)) + (x_0, x_1))/(x_0, x_1))_2
\]
\[
= \dim_k(((x_2 x_3, c x_3^2, dx_3^2) + (x_0, x_1))/(x_0, x_1))_2
\]
\[
= \dim_k((x_2^3, x_3 x_3^2, x_3^2) \bar{x}_3^2)
\]
\[
= \dim_k(R/(x_0, x_1))_2,
\]
we get that
\[
\dim_k((J^{[3]} + I^{[2]})/I^{[2]})_3 = \dim_k(R/(x_0, x_1))_3 = 4.
\]
Note that the Hilbert function of $R/J$ is $3t + 1$ for $t \geq 0$. It is from equation (3.4) that

$$\dim_k(J^{[3]} \cap I^{[2]}_3) = \dim_k J^{[3]} - \dim_k(J^{[3]} + I^{[2]}/I^{[2]}_3)$$

$$= 10 - 4 = 6.$$  

Now consider two ideals $I^{[3]}$ and $J^{[3]}$ and we assume

$$L_1 = x_0, L_2 = x_1, L_3 = x_2.$$  

Define

$$J^{[3],[1]} = (M_1, M_2) \cap (M_1, M_3) \cap (M_2, M_3) \cap (x_1, x_2), \quad \text{and} \quad J^{[3],[2]} = J^{[3],[1]} \cap (x_0, x_2) = J^{[3]} \cap (x_0, x_2) \cap (x_1, x_2).$$

Since $M_1, M_2,$ and $M_3$ are general linear forms,

$$\dim_k((J^{[3],[1]} + I^{[2]})/I^{[2]}_3) = \dim_k((M_1, M_2, M_1 M_3, M_2 M_3) \cap (x_2))_3 = \dim_k((x_2 M_1 M_2, x_2 M_1 M_3, x_2 M_2 M_3)_3 = 3.$$  

Moreover, it is from equation (3.5) that

$$\dim_k J^{[3],[1]}_3 = \dim_k((M_1, M_2) \cap (M_1, M_3) \cap (M_2, M_3) \cap (x_1, x_2))_3$$

$$= \dim_k((M_1, M_2) \cap (M_1, M_3) \cap (M_2, M_3) \cap (x_0, x_1))_3$$

$$= \dim_k (J^{[3]} \cap I^{[2]})_3 = 6.$$  

So, for every $t \geq 0,$

$$\dim_k J^{[3],[2]}_t = \dim_k (J^{[3]} \cap (x_1, x_2) \cap (x_0, x_2))_t$$

$$= \dim_k (J^{[3]} \cap (x_1, x_2) \cap (x_0, x_1))_t$$

$$= \dim_k (J^{[3],[1]} \cap I^{[2]})_t.$$  

Using equation (3.7) and the following exact sequence

$$0 \rightarrow J^{[3],[1]} \cap I^{[2]} \rightarrow J^{[3],[1]} \rightarrow (J^{[3],[1]} + I^{[2]})/I^{[2]} \rightarrow 0,$$

we obtain

$$\dim_k J^{[3],[2]}_3 = \dim_k (J^{[3],[1]} \cap I^{[2]})_3$$

$$= \dim_k J^{[3],[1]}_3 - \dim_k ((J^{[3],[1]} + I^{[2]})/I^{[2]})_3$$

$$= 3.$$  

Furthermore,

$$J^{[3],[2]} = J^{[3]} \cap (x_0, x_2)$$

$$= J^{[3]} \cap (x_1, x_2) \cap (x_0, x_2)$$

$$= J^{[3]} \cap (x_0 x_1, x_2).$$
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and so

\[
\dim_k ((J^{[3,2]} + I^{[2]})/I^{[2]}_3) \\
= \dim_k ((J^{[3,2]} + (x_0, x_1))/(x_0, x_1)_3) \\
= \dim_k ((J^{[3]} \cap (x_0 x_1, x_2) + (x_0, x_1))/(x_0, x_1)_3) \\
= \dim_k (x_2 M_1 M_2, x_2 M_1 M_3, x_2 M_2 M_3)_3 \\
= 3.
\]

Note that

\begin{align*}
J^{[3]} \cap I^{[3]} &= J^{[3]} \cap (x_0, x_1) \cap (x_1, x_2) \cap (x_0, x_2) \\
&= (J^{[3]} \cap (x_1, x_2) \cap (x_0, x_2)) \cap (x_0, x_1) \\
&= J^{[3,2]} \cap I^{[2]}.
\end{align*}

Using equation (3.9) and the following exact sequence

\[
0 \rightarrow J^{[3,2]} \cap I^{[2]} \rightarrow J^{[3,2]} \rightarrow (J^{[3,2]} + I^{[2]})/I^{[2]} \rightarrow 0,
\]

we have that

\begin{align*}
\dim_k (J^{[3]} \cap I^{[3]})_3 &= \dim_k J^{[3,2]} - \dim_k ((J^{[3,2]} + I^{[2]})/I^{[2]})_3 \\
&= 3 - 3 = 0.
\end{align*}

**Case 2.** Let \( s = 4 \).

Without loss of generality, assume that

\[
L_1 = x_0, L_2 = x_1, L_3 = x_2, L_4 = x_3.
\]

Since all the \( M_i \) are general linear forms, we have that

\[
\dim_k ((J^{[4]} + I^{[2]})/I^{[2]})_3 = \dim_k \left( \sum_{i=1}^{4} (M_1 \cdots M_i \cdots M_4) \right)_3 \\
= 4 = \dim_k (R/(x_0, x_1))_3.
\]

Thus

\[
\dim_k ((J^{[4]} + I^{[2]})/I^{[2]})_4 = 5.
\]

Using Proposition 3.3 and the following exact sequence

\[
0 \rightarrow J^{[4]} \cap I^{[2]} \rightarrow J^{[4]} \rightarrow (J^{[4]} + I^{[2]})/I^{[2]} \rightarrow 0,
\]

we obtain that

\begin{align*}
\dim_k (J^{[4]} \cap I^{[2]})_4 &= \dim_k J^{[4]} - \dim_k ((J^{[4]} + I^{[2]})/I^{[2]})_4 \\
&= 13 - 5 = 8.
\end{align*}

Define

\[
J^{[4,1]} := J^{[4]} \cap (x_1, x_2), \quad \text{and} \\
J^{[4,2]} := J^{[4]} \cap (x_1, x_2) \cap (x_2, x_3).
\]
By equation (3.12),
\[
\dim_k J_4^{[4,1]} = \dim_k [J^{[4]} \cap (x_1, x_2)]_4 = \dim_k [J^{[4]} \cap (x_0, x_1)]_4 \quad \text{(since } M_t \text{ are general)}
\]
\[
= \dim_k (J^{[4]} \cap I^{[2]})_4 = 8.
\]

Note that
\[
\dim_k ((J^{[4,1]} + I^{[2]})/I^{[2]})_4 = \dim_k (J^{[4]} \cap (x_1, x_2) + (x_0, x_1))/(x_0, x_1)_4
\]
\[
= \dim_k (J^{[4]} \cap (\bar{x}_2))_4 = \dim_k (J^{[4]})_3
\]
\[
(3.14)
\]
\[
\dim_k J_t^{[4,2]} = \dim_k (J^{[4]} \cap (x_1, x_2) \cap (x_2, x_3))_t = \dim_k (J^{[4]} \cap (x_1, x_2) \cap (x_0, x_1))_t
\]
\[
\quad \text{(since } M_t \text{ are general linear forms)}
\]
\[
= \dim_k (J^{[4,1]} \cap I^{[2]})_t \quad \text{for every } t \geq 0.
\]

Using equation (3.14) and the following exact sequence
\[
0 \rightarrow J^{[4,1]} \cap I^{[2]} \rightarrow J^{[4,1]} \rightarrow (J^{[4,1]} + I^{[2]})/I^{[2]} \rightarrow 0,
\]
we get that
\[
\dim_k J_4^{[4,2]} = \dim_k (J^{[4,1]} \cap I^{[2]})_4 = \dim_k (J^{[4,1]})_4 - \dim_k ((J^{[4,1]} + I^{[2]})/I^{[2]})_4
\]
\[
= 8 - 4 = 4.
\]

Using the same method as in equation (3.14), we obtain that
\[
\dim_k ((J^{[4,2]} + I^{[2]})/I^{[2]})_4 = \dim_k (J^{[4]} \cap (x_1, x_2) \cap (x_2, x_3) + (x_0, x_1))/(x_0, x_1)_4
\]
\[
= \dim_k (J^{[4]} \cap (x_2) \cap (x_2, x_3) + (x_0, x_1))/(x_0, x_1)_4 = \dim_k (J^{[4]} \cap (\bar{x}_2))/4 = \dim_k J_3^{[4]}
\]
\[
= 4.
\]

Note that
\[
J^{[4]} \cap I^{[4]} \subseteq (J^{[4]} \cap (x_1, x_2) \cap (x_2, x_3)) \cap (x_0, x_1)
\]
\[
= J^{[4,2]} \cap I^{[2]}.
\]

(3.16)
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Using equations (3.15) and (3.16), and the following exact sequence

\[
0 \rightarrow J^{[4,2]} \cap I^{[2]} \rightarrow J^{[4,2]} \rightarrow (J^{[4,2]} + I^{[2]})/I^{[2]} \rightarrow 0,
\]

we have

\[
\dim_k(J^{[4]} \cap I^{[4]})_4 \leq \dim_k J^{[4,2]} \cap I^{[2]} = \dim_k(J^{[4,2]})_4 - \dim_k((J^{[4,2]} + I^{[2]})/I^{[2]})_4 = 0.
\]

**Case 3.** Let \( s = 5 \).

Define

\[
L_1 = x_0, L_2 = x_1, L_3 = x_2, L_4 = x_3.
\]

Since the \( M_i \) are general linear forms, we have that

\[
\dim_k((J^{[5]} + I^{[2]})/I^{[2]})_4 = \left( \sum_{i=1}^{5} (M_1 \cdots M_i \cdots M_5) \right)_4 = 5 = \dim_k(R/(x_0, x_1)_4).
\]

Thus we know

\[
\dim_k((J^{[5]} + I^{[2]})/I^{[2]})_5 = 6,
\]

and by Proposition 3.3,

\[
\dim_k J^{[5]}_5 = 16.
\]

Using the following exact sequence

\[
0 \rightarrow J^{[5]} \cap I^{[2]} \rightarrow J^{[5]} \rightarrow (J^{[5]} + I^{[2]})/I^{[2]} \rightarrow 0,
\]

we obtain that

\[
(3.17) \quad \dim_k(J^{[5]} \cap I^{[2]})_5 = \dim_k J^{[5]}_5 - \dim_k((J^{[5]} + I^{[2]})/I^{[2]})_5 = 16 - 6 = 10.
\]

Define

\[
J^{[5,1]} := J^{[5]} \cap (x_1, x_2), \quad \text{and}
J^{[5,2]} := J^{[5]} \cap (x_1, x_2) \cap (x_2, x_3).
\]

By equation (3.17),

\[
(3.18) \quad \dim_k J^{[5]}_5^{[5,1]} = \dim_k(J^{[5]} \cap (x_1, x_2))_5 = \dim_k(J^{[5]} \cap (x_0, x_1))_5 = \dim_k(J^{[5]} \cap I^{[2]})_5 = 10.
\]
Moreover, note that

\[ \dim_k((J^{[5,1]} + I^{[2]})/I^{[2]})_5 \]
\[ = \dim_k((J^{[5,1]} + (x_0, x_1))/(x_0, x_1))_5 \]
\[ = \dim_k(J^{[5]} \cap (x_1, x_2) + (x_0, x_1))/(x_0, x_1))_5 \]
\[ = \dim_k(J^{[5]} \cap (\bar{x}_2))_5 \]
\[ = \dim_k(J^{[5]})_4 \]

(3.19)

\[ \dim_k J^{[5,2]}_5 = \dim_k J^{[5]}_5 \]
\[ = 5, \text{ and } \]
\[ \dim_k J^{[5,2]}_5 = \dim_k(J^{[5]} \cap (x_1, x_2) \cap (x_2, x_3))_5 \]
\[ = \dim_k(J^{[5]} \cap (x_1, x_2) \cap (x_0, x_1))_5 \]
\[ = \dim_k(J^{[5]} \cap (x_2) + (x_0, x_1))/(x_0, x_1))_5 \]
\[ = \dim_k(J^{[5]} \cap (\bar{x}_2))_5 \]
\[ (\text{since } M_i \text{ are general linear forms}) \]
\[ = \dim_k(J^{[5,1]} \cap I^{[2]})_5. \]

Using equations (3.18), (3.19) and the following exact sequence

\[ 0 \rightarrow J^{[5,1]} \cap I^{[2]} \rightarrow J^{[5,1]} \rightarrow (J^{[5,1]} + I^{[2]})/I^{[2]} \rightarrow 0, \]

we obtain that

\[ \dim_k J^{[5,2]}_5 = \dim_k(J^{[5,1]}_5) - \dim_k((J^{[5,1]} + I^{[2]})/I^{[2]})_5 \]
\[ = 10 - 5 = 5. \]

Note that

\[ \dim_k((J^{[5,2]} + I^{[2]})/I^{[2]})_5 \]
\[ = \dim_k(J^{[5]} \cap (x_1, x_2) \cap (x_2, x_3) + (x_0, x_1))/(x_0, x_1))_5 \]
\[ = \dim_k(J^{[5]} \cap (x_2) \cap (x_3, x_0, x_1))/(x_0, x_1))_5 \]
\[ = \dim_k(J^{[5]} \cap (x_2) + (x_0, x_1))/(x_0, x_1))_5 \]
\[ = \dim_k(J^{[5]} \cap (\bar{x}_2))_5 \]
\[ (\text{since } M_i \text{ are general linear forms}) \]

(3.21)

\[ = \dim_k(J^{[5]})_4 \]
\[ = \dim_k(J^{[5]})_4 \]
\[ = 5, \text{ and } \]
\[ \dim_k(J^{[5]} \cap I^{[5]})_5 \]
\[ \leq \dim_k(J^{[5]} \cap I^{[4]})_5 \]
\[ \leq \dim_k(J^{[5]} \cap (x_1, x_2) \cap (x_2, x_3) \cap (x_0, x_1))_5 \]
\[ = \dim_k(J^{[5,2]} \cap I^{[2]})_5. \]

Using equations (3.20) and (3.21), and the following exact sequence

\[ 0 \rightarrow J^{[5,2]} \cap I^{[2]} \rightarrow J^{[5,2]} \rightarrow (J^{[5,2]} + I^{[2]})/I^{[2]} \rightarrow 0, \]
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we have
\begin{align*}
\dim_k (J^{[5]} \cap I^{[5]})_5 & \leq \dim_k (J^{[5,2]} \cap I^{[2]})_5 \\
& = \dim_k (J^{[5,2]})_5 - \dim_k ((J^{[5,2]} + (x_0, x_1))/(x_0, x_1))_5 \\
& = 0,
\end{align*}
which completes the proof. \qed

Now we are ready to prove the main theorem. We first introduce the following theorem and proposition in [13].

**Theorem 3.5 ([13])**. Let $R = k[x_0, x_1, x_2] = \bigoplus_{i=0}^n R_i$ Let $X := X(t,s)$ be the union of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$ with $3 \leq t \leq 9$ and $s \geq t$. Then $R/I_X$ has generic Hilbert function.

**Proposition 3.6 (Proposition 4.1, [13])**. Let $X := X(s,s)$ be the union of two linear star-configurations in $\mathbb{P}^2$ of type $s \times s$ with $s \geq 6$. Then $(I_X)_s = \{0\}$.

**Theorem 3.7**. Let $R = k[x_0, x_1, \ldots, x_n]$ with $n \geq 3$ and $L_i, M_i$ be general linear forms in $R$ for $i = 1, 2, \ldots, s$ with $s \geq 3$. Let
\begin{align*}
I^{[s]} & := \sum_{i=1}^s (L_1 \cdots \hat{L}_i \cdots L_s), \\
J^{[s]} & := \sum_{i=1}^s (M_1 \cdots \hat{M}_i \cdots M_s).
\end{align*}
Then
\begin{align*}
\dim_k (I^{[s]} \cap J^{[s]})_s & = 0.
\end{align*}

**Proof.** We shall prove this theorem by induction on $n \geq 3$. First, by Lemma 3.4, the statement holds for $n = 3$ and $3 \leq s \leq 5$.

Now assume $n > 3$ and $3 \leq s \leq 5$. By Remark (b), the union of two star-configurations in $\mathbb{P}^n$ is a subscheme in $\mathbb{P}^n$ of codimension 2, and so we may assume $L = x_0$ is a nonzero divisor of $R/I^{[s]} \cap J^{[s]}$. Define
\begin{align*}
(I^{[s]} \cap J^{[s]}, L)/(L) & := \overline{(I^{[s]} \cap J^{[s]})} \\
\overline{L}_i & := (L_i + (x_0))/(x_0), \\
\overline{M}_i & := (M_i + (x_0))/(x_0), \\
\overline{I}^{[s]} & := \sum_{i=1}^s (\overline{L}_1 \cdots \overline{L}_i \cdots \overline{L}_s), \text{ and} \\
\overline{J}^{[s]} & := \sum_{i=1}^s (\overline{M}_1 \cdots \overline{M}_i \cdots \overline{M}_s).
\end{align*}
Since $I_s \cap J_s$ is not saturated in general, for $3 \leq s \leq 5$
\[ \dim_k (I_s \cap J_s)_s \leq \dim_k (\bar{I}_s \cap \bar{J}_s)_s = 0 \]  
(3.22)
(by Lemma 3.4 and induction on $n$)
\[ \Rightarrow \dim_k (I_s \cap J_s)_s = 0. \]
Furthermore, since $L$ is not a zero divisor of $I_s \cap J_s$, we get that
\[ (I_s \cap J_s)_s = \{0\} \]
for such $s$.

Now consider the case for $n \geq 3$ and $s \geq 6$. With the same notations as above, by Proposition 3.6,
\[ (\bar{I}_s \cap \bar{J}_s)_s = \{0\} \quad \text{for } n = 3 \text{ and } s \geq 6. \]
By the same arguments as in equation (3.22),
\[ (I_s \cap J_s)_s = \{0\} \quad \text{for } n = 3 \text{ and } s \geq 6. \]
Therefore, by induction on $n$, we show that
\[ (I_s \cap J_s)_s = \{0\} \quad \text{for } n \geq 3 \text{ and } s \geq 6, \]
which completes the proof.

As an immediate consequence of Proposition 3.3, Lemma 3.4, and Theorem 3.5 with Corollary 4.3 in [13], we obtain the following corollary.

**Corollary 3.8.**

Sec$_1$(Split$_s$(P$^n$)) has the expected dimension for $n \geq 2$ and $s \geq 3$. In particular,
\[ \dim \text{Sec}_1(\text{Split}_s(\mathbb{P}^n)) = \expdim \text{Sec}_1(\text{Split}_s(\mathbb{P}^n)) = 2ns + 1, \]
for $n \geq 3$ and $s \geq 3$.

**Proof.** First, by Corollary 4.3 in [13], the statement holds for $n = 2$ and $s \geq 3$.
Now suppose $n \geq 3$ and $s \geq 3$. Let $X := X^{(s,s)}$ be the union of two linear star-configurations $X_1$ and $X_2$ in $\mathbb{P}^n$ of type $s$, and let $I := I_{X_1} + I_{X_2}$.
By Theorem 3.7 and the following exact sequence
\[ 0 \to I_X \to I_{X_1} \oplus I_{X_2} \to I \to 0, \]
we have that
\[ \dim_k I_s = \dim_k (I_{X_1})_s + \dim_k (I_{X_2})_s = 2 \dim_k (I_{X_1})_s = 2ns + 2, \]
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and hence

\[
\expdim \text{Sec}_1(\text{Split}_s(\mathbb{P}^n)) = \min \{2 \times \dim(\mathbb{P}(R_1) \times \cdots \times \mathbb{P}(R_1)) + 1, \dim \mathbb{P}(R_s)\} \\
= \min \left\{ 2ns + 1, \binom{s+n}{n} - 1 \right\} \\
= 2ns + 1 \quad \text{(since } n \geq 3 \text{ and } s \geq 3) \\
= \dim_k I_s - 1 \\
= \dim \text{Sec}_1(\text{Split}_s(\mathbb{P}^n)) \quad \text{(by Corollary 2.6)},
\]

as we wished. \( \square \)

**Remark 3.9.** In [3], they showed that the secant variety

\[ \text{Sec}_{r-1}(\text{Split}_s(\mathbb{P}^n)) \]

has the expected dimension for \( 3(s-1) \leq n \) and \( s > 2 \) using Terracini’s Lemma (see [3] and [14]). Their results however do not cover the case of reducible plane curves. For this case, the author in [13] showed that the secant line varieties \( \text{Sec}_1(\text{Split}_s(\mathbb{P}^2)) \) still have the expected dimension. In Corollary 3.8 of this paper, we introduced another way (algebraic method) to prove that the secant line varieties \( \text{Sec}_1(\text{Split}_s(\mathbb{P}^n)) \) have the expected dimension for \( n \geq 3 \).

**References**


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