ON ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF \textit{LNQD} RANDOM VARIABLES

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Abstract. Let \( \{X_{ni}, 1 \leq i \leq n, n \geq 1\} \) be a sequence of \textit{LNQD} which are dominated randomly by another random variable \( X \). We obtain the complete convergence and almost sure convergence of weighted sums \( \sum_{i=1}^{n} a_{ni}X_{ni} \) for \textit{LNQD} by using a new exponential inequality, where \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) is an array of constants. As corollary, the results of some authors are extended from i.i.d. case to not necessarily identically \textit{LNQD} case.

1. Introduction

Let \( \{X_{ni}, 1 \leq i \leq n, n \geq 1\} \) be a sequence of random variables (r.v.'s) with \( EX_n = 0 \), and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be an array of real numbers. Many authors was studied the almost sure convergence of weighted sums \( \sum_{i=1}^{n} a_{ni}X_{i} \) when \( \{X, X_n, n \geq 1\} \) are assumed to be independent and identically distributed (i.i.d.) r.v.'s (see Bai and Cheng (2000), Sung (2001), Cuzick (1995), Chow and Lai (1973) among others). In addition, Bai and Cheng (2000) proved the strong law of large numbers \( \sum_{i=1}^{n} a_{ni}X_{i}/b_n \to 0 \text{ a.s.} \) when \( \{X, X_n, n \geq 1\} \) is a sequence of i.i.d. r.v.'s with \( EX = 0 \) and

\[ E[\exp(h|X|^\gamma)] < \infty \quad \text{for some} \quad h > 0 \ (\gamma > 0), \tag{1.1} \]

and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) is an array of real numbers satisfying

\[ A_\alpha = \limsup_{n \to \infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n}^\alpha = \sum_{i=1}^{n} |a_{ni}|^\alpha/n \tag{1.2} \]
for some $1 < \alpha < 2$, where $b_n = n^{1/\alpha} \log n^{1/\gamma + \gamma(\alpha-1)/\alpha(1+\gamma)}$. Sung (2001) extended result of Bai and Cheng (2000) and obtained another almost sure limiting law when condition (1.1) is replaced by stronger condition

$$E[\exp(h|X|^\gamma)] < \infty \quad \text{for any} \quad h > 0 \quad (\gamma > 0).$$

In this case, $b_n = n^{1/\alpha} \log n^{1/\gamma}$ if $0 < \gamma \leq 1$.

We extended the result of Sung(2001) by using a new exponential inequality of LNQD r.v.'s with a below concepts. We first recall the definitions and lemmas of negatively associated, negative quadrant dependent and linearly negative quadrant dependent random variables.

**Definition 1.1(Alam and Saxena (1981)).** A finite sequence of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets $A_1, A_2$ of $\{1, 2, \cdots, n\}$,

$$\text{Cov}\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0,$$

whenever $f$ and $g$ are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ is NA if every finite subcollection is NA.

This definition is introduced by Alam and Saxena (1981). Many authors derived several important properties about NA sequences and also discussed some applications in the area of statistics, probability, reliability and multivariate analysis. Compared to positively associated random variables, the study of NA random variables has received less attention in the literature. Readers may refer to Karlin and Rinott(1980), Joag-Dev and Proschan(1983), Matula(1992) and Roussas(1994) among others. Recently, some authors focussed on the problem of limiting behavior of partial sums of NA sequences(see, Su and Qin(1997), Shao and Su(1999), Liang(2000), Liang et al(2004), and Baek et al(2005)).

**Definition 1.2(Lehmann (1966)).** Two random variables $X$ and $Y$ are said to be negative quadrant dependent (NQD) if for any $x, y \in \mathbb{R}$,

$$P(X < x, Y < y) \leq P(X < x)P(Y < y).$$

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise NQD if $X_i$ and $X_j$ are NQD for all $i, j \in \mathbb{N}^+$ and $i \neq j$.

**Lemma 1.1(Lehmann (1966)).** Let $X$ and $Y$ be NQD random variables, then (a) $EXY \leq EXEY$, (b) $P(X < x, Y < y) \leq P(X < x)P(Y < y)$. 


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x) \( P(Y < y) \), and (c) If \( f \) and \( g \) are both nondecreasing (or both nonincreasing) functions, then \( f(X) \) and \( g(Y) \) are NQD.

**Definition 1.3 (Newman (1984)).** A sequence \( \{X_n, n \geq 1\} \) of random variables is said to be linearly negative quadrant dependent (LNQD) if for any disjoint subsets \( A, B \subset \mathbb{N}^+ \) and positive \( r_j/s \),

\[
\sum_{k \in A} r_k X_k \text{ and } \sum_{j \in B} r_j X_j \text{ are } NQD.
\]

**Lemma 1.2.** Let \( \{X_n, n \geq 1\} \) be a sequence of LNQD random variables with \( EX_n = 0 \) for each \( n \geq 1 \), then for any \( t > 0 \),

\[
Ee^{\sum_{i=1}^{n} X_i} \leq \prod_{i=1}^{n} Ee^{tX_i} \leq e^{t^2/2\sum_{i=1}^{n} EX_i^2 e^{t|X_i|}}
\]

**Proof.** Noticing that \( tX_i \) and \( \sum_{j=i+1}^{n} tX_j \) are LNQD, we know by Definition 1.3, \( e^{tX_i} \) and \( e^{\sum_{j=i+1}^{n} X_j} \) are also LNQD for \( i = 1, 2, \ldots, n-1 \). We will prove the first inequality by mathematical induction that

\[
Ee^{\sum_{i=1}^{n} X_i} \leq \prod_{i=1}^{n} Ee^{tX_i}. \tag{1.4}
\]

First, we observe that

\[
Ee^{t(X_1+X_2)} \leq Ee^{tX_1} Ee^{tX_2} = \prod_{i=1}^{2} Ee^{tX_i}.
\]

Where the inequality follows from Lemma 1.1. Thus, (1.4) is true for \( i = 2 \). Assume now that the statement is true for \( i = k \). We will show that it is true for \( i = k + 1 \).

\[
Ee^{\sum_{i=1}^{k+1} X_i} = E \left( e^{t \sum_{i=1}^{k} X_i} e^{tX_{k+1}} \right)
\]

\[
\leq Ee^{\sum_{i=1}^{k} X_i} Ee^{tX_{k+1}}
\]

\[
\leq \prod_{i=1}^{k} Ee^{tX_i} Ee^{tX_{k+1}} = \prod_{i=1}^{k+1} Ee^{tX_i}.
\]
Finally, we will prove the second inequality that

$$\prod_{i=1}^{n} E e^{t X_i} \leq e^{t^2/2 \sum_{i=1}^{n} E X_i^2 e^{t|X_i|}}.$$  

For all $x \in \mathbb{R}$, taking $e^x \leq 1 + x + x^2/2e^{|x|}$ and $E X_i = 0$, we have

$$E e^{t X_i} \leq 1 + t E X_i + t^2/2 E X_i^2 e^{t|X_i|} = 1 + t^2/2 E X_i^2 e^{t|X_i|} \leq e^{t^2/2 E X_i^2 e^{t|X_i|}} \text{, by } 1 + x \leq e^x.$$  

Thus, we get that

$$\prod_{i=1}^{n} E e^{t X_i} \leq e^{t^2/2 \sum_{i=1}^{n} E X_i^2 e^{t|X_i|}}.$$  

From the above definition, it is immediate that $NA$ implies $LNQD$. Newman(1984) introduced the concepts of $LNQD$ r.v.’s and many authors derived several important properties about $LNQD$ r.v.’s and also discussed some applications in several areas (see Newman(1984), Cai and Roussas(1997), Wang and Zhang(2006), Ko et al.(2007) among others).

The main purpose of this paper is to extend results of Sung(2001) for i.i.d. case to not necessarily identically distributed the case of linearly negative quadrant dependent r.v.’s, which contains independent and negatively associated random variables as special cases. First, we shall study the limit properties of weighted sums of $LNQD$ r.v.’s by using a exponential inequalities, which are dominated randomly by another random variable $X$. In particular, we shall consider the case when $\{X_i, 1 \leq i \leq n, n \geq 1\}$ are $LNQD$ r.v.’s with $P(|X_i| > x) \leq cP(|X| > x)$ for all $i$ and $x \geq 0$. As corollary, the results of some authors are extended from i.i.d. case to not necessarily identically distributed $LNQD$ setting. Throughout this paper, $a_{ni} = a_{ni}^+ - a_{ni}^-$, where $a_{ni}^+ = \max(a_{ni}, 0)$, $a_{ni}^- = \max(-a_{ni}, 0)$, $c$ denote the positive constant whose values are unimportant and may vary at different place.

2. Main results

We will deal with the complete convergence and almost sure convergence for weighted sums of $LNQD$ r.v.’s by using exponential inequalities in this Chapter.
Theorem 2.1. Let \( \{X_{ni}, 1 \leq i \leq n, \ n \geq 1\} \) be an array of rowwise LNQD random variables with \( EX_{ni} = 0 \) and \( P(|X_{ni} > x|) \leq cP(|X| > x) \) for all \( i > 1 \) and \( x \geq 0 \). Assume that \( \{a_{ni}, 1 \leq i \leq n, \ n \geq 1\} \) is an array of constants, and that \( X_{ni}^2 \sum_{i=1}^{n} a_{ni}^2 \leq v_n |X_i|^\delta / \log n \) a.s. for some \( \delta > 0 \) and some sequence \( \{v_n\} \) of constants such that \( v_n \to 0 \).

(a) Let \( |a_{ni}X_{ni}| \leq u_n |X_i|^\beta / \log n \) a.s. for some \( 0 < \beta \leq \gamma \) and some sequence \( \{u_n\} \) of constants such that \( u_n \to 0 \). If \( E(e^{h|X|^{\gamma}}) < \infty \) for some \( h > 0 \) (\( \gamma > 0 \)), then

\[
\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni} X_{ni} | > \epsilon) < \infty \text{ for any } \epsilon > 0.
\] (2.1)

(b) Let \( |a_{ni}X_{ni}| \leq c|X_i|^\beta / \log n \) a.s. for some \( 0 < \beta \leq \gamma \) and some constant \( c \geq 0 \). If \( E(e^{h|X|^{\gamma}}) < \infty \) for any \( h > 0 \) (\( \gamma > 0 \)), then (2.1) remains true. Furthermore, both (a) and (b) imply \( \sum_{i=1}^{n} a_{ni} X_{ni} \to 0 \) a.s. as \( n \to \infty \).

Proof. We prove only (a), the proof of (b) is similar to the proof of (a). It suffices to show that

\[
\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^+ X_{ni} | > \epsilon) < \infty \text{ for any } \epsilon > 0,
\] (2.2)

\[
\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^- X_{ni} | > \epsilon) < \infty \text{ for any } \epsilon > 0.
\] (2.3)

We prove only (2.2), the proof of (2.3) is analogous. To prove (2.2), we need only to prove that

\[
\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^+ X_{ni} | > \epsilon) < \infty \text{ for any } \epsilon > 0,
\] (2.4)

\[
\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^+ X_{ni} | < -\epsilon) < \infty \text{ for any } \epsilon > 0.
\] (2.5)

Note that \( \{a_{ni}^+ X_{ni}, 1 \leq i \leq n, n \geq 1\} \) is still an array of rowwise LNQD random variables by Definition 1.3. Thus, by Lemma 1.2 and \( |x|^\delta \leq O(e^{(h/2)|x|^{\beta}}) \) for all \( x \in \mathbb{R} \) and taking \( t = M \log n/\epsilon \), where \( M \) is large constant, we have

\[
\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^+ X_{ni} | > \epsilon) < \infty.
\]
Next, by replacing $X_{ni}$ by $-X_{ni}$ from (2.4) and noticing $\{a_{ni}^-(-X_{ni})|1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise LNQD random variables, we know that

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}^+ X_{ni} < -\varepsilon\right) < \infty \text{ for any } \varepsilon > 0.$$ 

Hence, the result follows by (2.4) and (2.5). The proof is complete.

**Theorem 2.2.** Let $0 < p < 2$ and let $\{X_{ni}|1 \leq i \leq n, n \geq 1\}$ be an array of rowwise LNQD random variables with $E X_{ni} = 0$. Assume that $\{a_{ni}|1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying $\max_{1 \leq i \leq n} |a_{ni}| = O(n^{-1/p})$. If $|X_{ni}| \leq M$, where $M$ is a positive constant, then

$$\sum_{i=1}^{n} a_{ni} X_{ni} \to 0 \text{ completely as } n \to \infty.$$ 

**Proof.** As for the proof of Theorem 2.1, it suffices to show that

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}^+ X_{ni} > \varepsilon\right) < \infty \text{ for any } \varepsilon > 0.$$ 

Without loss of generality, we assume that

$$0 < a_{ni}^+ \leq n^{-1/p}, \text{ for } 1 \leq i \leq n, \ n \geq 1,$$

We also know that $\{a_{ni}^+ X_{ni}|1 \leq i \leq n, \ n \geq 1\}$ is still an array of rowwise LNQD random variables and $|a_{ni}^+ X_{ni}| \leq n^{-1/p} M$ and $E a_{ni}^+ X_{ni} = 0$.
Hence, taking \( t = 1/\varepsilon \log n \), we have that
\[
\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^+ X_{ni} > \varepsilon)
= \sum_{n=1}^{\infty} P(e^{\varepsilon t n^{1/p-1/2}/M} \sum_{i=1}^{n} a_{ni}^+ X_{ni} > e^{\varepsilon^2 t^2 n^{1/p-1/2}/M})
\leq \sum_{n=1}^{\infty} e^{-\varepsilon^2 t^2 n^{1/p-1/2}/M} e^{\varepsilon t n^{1/p-1/2}/M} \prod_{i=1}^{n} \sum_{h}^{\infty} E(e^{\varepsilon t n^{1/p-1/2}/M} a_{ni}^+ X_{ni})
\leq \sum_{n=1}^{\infty} e^{-\varepsilon^2 t^2 n^{1/p-1/2}/M - \varepsilon t n^{1/p-1/2}/M} \sum_{i=1}^{n} E(X_{ni} a_{ni}^+) e^{\varepsilon t n^{1/p-1/2}/M} \sum_{h}^{\infty} E(e^{\varepsilon t n^{1/p-1/2}/M} a_{ni}^+ X_{ni})
\leq e \sum_{n=1}^{\infty} e^{-2 \varepsilon^2 n^{1/p-1/2}/M} e^{\varepsilon t n^{1/p-1/2}/M} < \infty,
\]
since \( 0 < p < 2 \) and \( 1/p - 1/2 > 0 \). The proof is complete.

**Theorem 2.3.** Let \( \{X_{ni}, 1 \leq i \leq n, n \geq 1\} \) be an array of rowwise LNQD random variables with \( EX_{ni} = 0 \) and \( P(|X_{ni}| > x) \leq cP(|X| > x) \) for all \( i \geq 1 \) and \( x > 0 \). Assume that \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) is an array of constants satisfying (1.2).

(a) If \( E(e^{h|X|^\gamma}) < \infty \) for any \( h > 0 \) and \( 0 < \gamma \leq 1 \), then
\[
\sum_{i=1}^{n} a_{ni} X_{ni} / n^{1/\alpha} (\log n)^{1/\gamma} \to 0 \text{ a.s. as } n \to \infty.
\]

(b) If \( E(e^{h|X|^\gamma}) < \infty \) for some \( h > 0 \) and \( \gamma > 0 \), then
\[
\sum_{i=1}^{n} a_{ni} X_{ni} / n^{1/\alpha} (\log n)^{(1/\gamma) + \delta} \to 0 \text{ a.s. as } n \to \infty,
\]
where \( \delta = 1 - 1/\gamma - (\gamma - 1)/(1 + \alpha \gamma - \alpha) \).

**Proof.** We prove only (a), the proof of (b) is similar to the proof of (a). Let \( X_{ni} = X'_{ni} + X''_{ni} \) with \( X'_{ni} = X_{ni} I(X_{ni} \leq (\log n)^{1/\gamma}) + (\log n)^{1/\gamma} I(X_{ni} > (\log n)^{1/\gamma}) \). Then \( \{a_{ni} X'_{ni} | 1 \leq i \leq n, n \geq 1\} \) and \( \{a_{ni} X''_{ni} | 1 \leq i \leq n, n \geq 1\} \) are still an array of rowwise LNQD sequences.
by the Definition 1.3 of $X'_{ni}$ and $X''_{ni}$, respectively, and noticing that $EX'_{ni} + EX''_{ni} = 0$, we have that
\[
\sum_{i=1}^{n} a_{ni} X_{ni} / n^{1/\alpha} (\log n)^{1/\gamma}
\]
\[
= \sum_{i=1}^{n} a_{ni} (X'_{ni} - EX'_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma}
+ \sum_{i=1}^{n} a_{ni} (X''_{ni} - EX''_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma}
\]
\[=: I_1 + I_2 \]

As to $I_2$, according to Markov’s inequality and conditions of Theorem 2.3, it follows that
\[
P(I_2 > \varepsilon)
\]
\[
\leq e^{-\varepsilon t E t \sum_{i=1}^{n} a_{ni} (X'_{ni} - EX'_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma}}
\]
\[
\leq e^{-\varepsilon t n \prod_{i=1}^{n} E e^{ta_{ni} (X'_{ni} - EX'_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma}}}
\]
\[\leq c A_{a,n} E X^2 / (\log n)^{2/\gamma} \to 0 \text{ as } n \to \infty,\]

and
\[
\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni} (X''_{ni} - EX''_{ni}) > \varepsilon n^{1/\alpha} (\log n)^{1/\gamma}) < \infty.
\]

Thus, by the Borel-Cantelli Lemma, we obtain that
\[
\sup_{i=1}^{n} a_{ni} (X''_{ni} - EX''_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma} \to 0 \text{ as } n \to \infty. \quad (2.6)
\]

Next, as to $I_1$,
\[
P(I_1 > \varepsilon) \leq e^{-\varepsilon t \sum_{i=1}^{n} a_{ni} (X'_{ni} - EX'_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma}}
\]
\[
\leq e^{-t} \prod_{i=1}^{n} E e^{ta_{ni} (X'_{ni} - EX'_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma}}. \quad (2.7)
\]
Hence, by (2.7) and (2.8) and taking $t = \log n$, we have that

$$P(I_1 > \varepsilon) \leq e^{-ct} \sum_{i=1}^n a_{ni} (X_{ni} - E_{ni}) / n^{1/\alpha} (\log n)^1/\gamma$$

$$\leq e^{-ct} \prod_{i=1}^n E_{t_{ni}} (X_{ni} - E_{ni}) / n^{1/\alpha} (\log n)^1/\gamma$$

$$\leq ce^{-log n} E(|I|^{\gamma})$$

$$\leq cn^{-1}.$$
Thus, by (2.6) and (2.9), the proof is complete.

We extended results of Sung(2001), and Chow and Lai(1973) from i.i.d. case to not necessary identically distributed LNQD case.

**Corollary 2.1.** Let \( \{X_{ni}, 1 \leq i \leq n, n \geq 1\} \) be a sequence of LNQD random variables with \( EX_{ni} = 0 \) and \( P(|X_{ni}| > x) \leq cP(|X| > x) \) for all \( i \geq 1 \) and \( x \geq 0 \).

(a) Assume that \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) is an array of constants satisfying (1.2) for \( 1 < \alpha \leq 2 \). If \( E(e^{h|X|^\gamma}) < \infty \) for some \( h > 0 \) and \( b_n = n^{1/\alpha}(\log n)^{1/\gamma + \alpha + \beta} \) for \( \gamma > 1 \) and \( \beta > 0 \), then
\[
\sum_{i=1}^{n} a_{ni} X_{ni} / b_n \to 0 \text{ a.s. as } n \to \infty.
\]

(b) Assume that \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) is an array of constants satisfying \( \limsup_{n \to \infty} \sum_{i=1}^{n} a_{ni}^2 < \infty \). If \( E(e^{h|X|}) < \infty \) for all \( h > 0 \) and \( b_n = \log n \), then
\[
\sum_{i=1}^{n} a_{ni} X_{ni} / b_n \to 0 \text{ a.s. as } n \to \infty.
\]

**Proof of Corollary 2.1.** When weights \( a_{ni} \) satisfy (1.2), taking for \( 1 < \alpha \leq 2 \), \( |a_{ni}|/n^{1/\alpha} \leq (\sum_{i=1}^{n} |a_{ni}|^\alpha)^{1/\alpha}/n^{1/\alpha} = A_{\alpha,n}^1 \) and \( \sum_{i=1}^{n} a_{ni}^2/n^{2/\alpha} \leq (\sum_{i=1}^{n} |a_{ni}|^2)^{2/\alpha}/n^{2/\alpha} = A_{\alpha,n}^2 \), we can obtain the result of Corollary 2.1 by using Theorem 2.3, and the proof of (b) is similar to the proof (a).

**References**


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