ON STABILITY OF NONLINEAR INTEGRO-DIFFERENTIAL SYSTEMS WITH IMPULSIVE EFFECT

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Abstract. In this paper we study the stability properties of solutions of nonlinear impulsive integro-differential systems by using an integral inequality under the stability of the corresponding variational impulsive integro-differential systems. Also, we give examples to illustrate our results.

1. Introduction

The qualitative theory of impulsive systems have attracted an increased interest of numerous researchers around the globe due to their important applications in many areas such as control theory, biology, physics and electronics, etc. (see [1, 11, 15, 20, 23]).

Rama Mohana Rao et al. [18] investigated the asymptotic stability properties of solutions of impulsive integro-differential equations of Volterra type in help of the construction of an equivalent differential system. Also, Lakshmikantham et al. [11] studied the stability properties of a nonlinear impulsive integro-differential system by using the corresponding variation of parameter formula and the comparison principle.

Pinto [16] introduced the notion of $h$-stability for differential systems without impulse effect with the intention of obtaining results about stability for weakly stable differential systems under some perturbations. The various notions of $h$-stability for differential systems given in [16, 17] include several types of known stability properties as uniform stability, exponential asymptotic stability [12] and uniform Lipschitz stability [4].

Choi and Koo [2] proved that two concepts of $h$-stability and $h$-stability in variation for nonlinear impulsive differential systems are equivalent via $t_{\infty}$-similarity of the associated variational impulsive systems and impulsive integral inequalities. Also, they characterized $h$-stability for nonlinear impulsive differential systems by using the notions of piecewise continuous auxiliary functions

Received December 1, 2019; Revised April 11, 2020; Accepted May 18, 2020.
2010 Mathematics Subject Classification. 34A37, 34D20, 34K20, 34K45, 45J05, 45M10.
Key words and phrases. Impulsive integro-differential system, variational impulsive system, $h$-stability, integral inequality.
and impulsive differential inequalities. Furthermore, they [3] obtained a converse $h$-stability theorem for the nonlinear impulsive systems by employing the notion of $t_\infty$-similarity of the associated impulsive variational systems and relations. Many authors [5–10,13,19,21,22,24] have studied the various types of the stability properties of solutions for integro-differential equations (or with impulsive effect). However, to the best of our knowledge, there are no papers published on the $h$-stability for nonlinear impulsive integro-differential systems.

Motivated by the above discussion, we develop an integral inequality and study the $h$-stability of solutions of nonlinear impulsive integro-differential systems which are related to equivalent impulsive differential systems by using an integral inequality.

2. Preliminaries

In this section we recall the basic notions and important theory of impulsive integro-differential systems which are used in this paper. For more information about the basic theory of impulsive systems and integro-differential systems, we mainly refer to some books [11,14].

Let $\mathbb{R}_+ = [0, \infty)$ and $\Omega$ be an open subset of the Euclidean space $\mathbb{R}^n$ with a convenient vector norm $|\cdot|$ containing the origin.

We consider the nonlinear integro-differential system with impulses at fixed times

$$
\begin{cases}
x'(t) = f(t, x(t)) + \int_{t_0}^t g(t, s, x(s))ds, & t \neq t_k, \\
\Delta x(t_k) = A_k x(t_k), & k \in \mathbb{N}, \\
x(t_0^+) = x_0 \in \Omega,
\end{cases}
$$

under the assumption that the following basic conditions hold:

(A1) $\Delta x(t) = x(t+0) - x(t-0)$ and $A_k$ is an $n \times n$ constant matrix for each $k \in \mathbb{N}$.

(A2) $\{t_k\}$ is a fixed strictly increasing sequence satisfying $0 \leq t_0$ and $\lim_{k \to \infty} t_k = \infty$.

(A3) The function $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is continuous and has a continuous partial derivative $f_x = \frac{\partial f}{\partial x}$ in $(t_{k-1}, t_k] \times \Omega, k \in \mathbb{N}$, and $f(t, 0) = 0$ for each $t \in \mathbb{R}_+$.

(A4) The function $g : \mathbb{R}^2_+ \times \Omega \to \mathbb{R}^n$ is continuous and has a continuous partial derivative $g_x$, and $g(t, s, 0) = 0$ for each $t \geq s \geq 0$.

(A5) The solution $x(t, t_0, x_0)$ of the system (1) which satisfies the initial condition $x(t_0+0, t_0, x_0) = x_0$ is defined in the interval $(t_0, \infty)$, and is left continuous.
We consider the variational impulsive system associated to zero solution of the system (1)

\[
\begin{align*}
\begin{cases}
z'(t) = f_z(t,0)z(t) + \int_{t_0}^{t} g_z(t,s,0)z(s)ds, & t \neq t_k, \\
\Delta z(t_k) = A_k z(t_k), & k \in \mathbb{N}, \\
z(t_0^+) = z_0 \in \mathbb{R}^n.
\end{cases}
\end{align*}
\]

Then using the mean value theorem, the system (1) is equivalent to the following form

\[
\begin{align*}
\begin{cases}
x'(t) = f_x(t,0)x(t) + \int_{t_0}^{t} g_x(t,s,0)x(s)ds + F(t,x(t)), & t \neq t_k, \\
\Delta x(t_k) = A_k x(t_k), & k \in \mathbb{N}, \\
x(t_0^+) = x_0 \in \Omega,
\end{cases}
\end{align*}
\]

where

\[
F(t,x) = \int_{0}^{1} [f_x(t,x\theta) - f_x(t,0)] d\theta x
\]

and

\[
G(t,s,x) = \int_{0}^{1} [g_x(t,s,x\theta) - g_x(t,s,0)] d\theta x.
\]

Let \(M_n(\mathbb{R})\) be the set of all \(n \times n\) matrices over \(\mathbb{R}\) and let \(PC(\mathbb{R}_+, M_n(\mathbb{R}))\) denote the class of piecewise continuous functions from \(\mathbb{R}_+\) to \(M_n(\mathbb{R})\) that have first kind discontinuities at the points \(t = t_k(k \in \mathbb{N})\) only and that are left-continuous at \(t = t_k\).

We consider the linear impulsive differential system with impulses at fixed times

\[
\begin{align*}
\begin{cases}
x'(t) = A(t)x(t), & t \neq t_k, \\
\Delta x(t_k) = A_k x(t_k), & k \in \mathbb{N},
\end{cases}
\end{align*}
\]

and its perturbed linear impulsive differential system

\[
\begin{align*}
\begin{cases}
x'(t) = A(t)x(t) + F(t), & t \neq t_k, \\
\Delta x(t_k) = A_k x(t_k) + B_k, & k \in \mathbb{N}, \\
x(t_0^+) = x_0 \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

under the assumption that the following conditions hold:

(B1) \(A \in PC(\mathbb{R}_+, M_n(\mathbb{R}))\) and \(F \in PC(\mathbb{R}_+, \mathbb{R}^n)\);

(B2) \(A_k \in M_n(\mathbb{R})\), \(\det(I + A_k) \neq 0\), and \(B_k \in \mathbb{R}^n\) for each \(k \in \mathbb{N}\). Here \(I\) is the identity matrix.
Also, we consider the impulsive matrix system

\[
\begin{cases}
X'(t) = A(t)X(t), \ t \neq t_k, \\
\Delta X(t_k) = A_kX(t_k), \ k \in \mathbb{N}.
\end{cases}
\]

Then we obtain the following result about the variation of parameters of the system (5).

**Lemma 2.1** ([20, Corollary 3]). Let \(X(t)\) be a fundamental matrix of the system (4) and \(X(t,s) = X(t)X^{-1}(s)\). Then any matrix solution \(X(t,t_0)\) of the system (6) with \(X(t_0,t_0) = I\) is given by

\[
X(t,t_0) = U(t,t_j)(I + A_j) \prod_{i=j-1}^{1} U(t_{i+1},t_i)(I + A_i)U(t_1,t_0),
\]

where \(U(t,s)\) is a solution of the matrix Cauchy problem

\[
\frac{dU}{dt} = A(t)U, \ U(s,s) = I.
\]

Also, any solution \(x(t) = x(t,t_0,x_0)\) of the system (5) with \(x(t_0,t_0,x_0) = x_0\) is given by the formula

\[
x(t) = X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s)F(s)ds + \sum_{t_0 < t_k < t} X(t,t_k^+)B_k, \ t \geq t_0.
\]

**Proof.** Let \(U(t,s)\) be a solution of the system (8). Then any solution \(X(t)\) of the system (6) can be represented as

\[
X(t) = U(t,t_{j+l})(I + A_{j+l})U(t_{j+l},t_{j+l-1})(I + A_{j+l-1})
\]

\[\cdots (I + A_j)U(t_j,t_0)X(t_0),
\]

\[0 \leq t_0 \leq t_1 < t_j < t_{j+1} < \cdots < t_{j+l} < t \leq t_{j+l+1}.
\]

In particular, for \(l = 0\) in (10), we have

\[
X(t) = U(t,t_j)(I + A_j) \prod_{i=j-1}^{1} U(t_{i+1},t_i)(I + A_i)U(t_1,t_0)X(t_0),
\]

\[0 \leq t_0 \leq t_1 < t_j < t \leq t_{j+1}.
\]

Then, the rest of proof can be proved in a similar manner as that of [20, pp. 47–49], so we omit the detail. \(\Box\)

We recall the notions of the \(h\)-stability which are analogous to the definitions firstly introduced by Pinto [17].
Definition 2.2. The system (1) is called $h$-stable (hS) if there exist $c \geq 1, \delta > 0$ and a positive bounded continuous function $h$ defined on $\mathbb{R}_+$ such that
\[ |x(t)| \leq c|x_0|h(t)h(t_0)^{-1}, \quad t \geq t_0, \]
for $|x_0| \leq \delta$ (here $h(t)^{-1} = \frac{1}{h(t)}$).

3. Main results

In this section we study the $h$-stability for nonlinear impulsive integro-differential systems by the help of an impulsive integral inequality under the $h$-stability of the corresponding variational integro-differential systems.

From Theorem 1.4.1 in [11], we can obtain the following integral inequality of Gronwall-Bellman type which is used in investigation of the stability for nonlinear impulsive differential systems.

Lemma 3.1. If a function $m \in PC(\mathbb{R}_+, \mathbb{R})$ satisfies
\[ m(t) \leq m(t_0) + \int_{t_0}^t \left[ \lambda_1(s)m(s) + \lambda_2(s) \int_{t_0}^s k(s,\tau)m(\tau)d\tau \right] ds, \]
where $\lambda_1, \lambda_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $k \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, then
\[ m(t) \leq m(t_0) \exp \left( \int_{t_0}^t \left[ \lambda_1(s) + \lambda_2(s) \int_{t_0}^s k(s,\tau)d\tau \right] ds \right), \quad t \geq t_0. \]

We consider the linear impulsive integro-differential system
\begin{equation}
\begin{cases}
    x'(t) = A(t)x(t) + \int_{t_0}^t K(t,s)x(s)ds + F(t), \quad t \neq t_k, \\
    \Delta x(t_k) = A_kx(t_k), \quad k \in \mathbb{N}, \\
    x(t_0^+) = x_0,
\end{cases}
\end{equation}
where $A \in PC(\mathbb{R}_+, M_n(\mathbb{R}))$, $K \in PC(\mathbb{R}_+^2, M_n(\mathbb{R}))$, $F \in PC(\mathbb{R}_+, \mathbb{R}^n)$ and $A_k$ is an $n \times n$ constant matrix such that $\det(I + A_k) \neq 0$ for each $k \in \mathbb{N}$.

If $F(t) = 0$ in (12), then the system (12) reduces to the homogenous impulsive integro-differential system
\begin{equation}
\begin{cases}
    x'(t) = A(t)x(t) + \int_{t_0}^t K(t,s)x(s)ds, \quad t \neq t_k, \\
    \Delta x(t_k) = A_kx(t_k), \quad k \in \mathbb{N}, \\
    x(t_0^+) = x_0.
\end{cases}
\end{equation}

In order to study the stability properties of solutions for impulsive integro-differential systems, we need the variation of parameters for the linear impulsive integro-differential system which is a slight modification of the resolvent kernel in [18, Theorem 3.1] (or see [21, p. 6]).
Lemma 3.2. Assume that there exists an \( n \times n \) matrix function \( L \in PC(\mathbb{R}^2_+, \, M_n(\mathbb{R})) \) such that \( L(t, s) \) exists and is continuous for \( t_k - 1 < s \leq t_k < t \) and satisfies

\[
\begin{align*}
\frac{\partial L(t, s)}{\partial t} &= A(t)L(t, s) + \int_s^t K(t, v)L(v, s)dv, \quad s \neq t_k, \quad t \neq t_k, \\
L(t^+, s) &= (I + A_k)L(t_k, s), \quad k \in \mathbb{N}, \\
L(t^+, t_0) &= I.
\end{align*}
\]

Then the fundamental matrix solution \( L(t, t_0) \) of the system (14) with \( L(t_0^+, t_0) = I \) can be represented as

\[
L(t, t_0) = L(t, t_{j+k})(I + A_{j+k})L(t_{j+k}, t_{j+k-1})(I + A_{j+k-1})
\cdots (I + A_j)L(t_j, t_0),
\]

\[
t_{j-1} < t_0 < t_{j+k} < t < t_{j+k+1}.
\]

Also, any solution \( x(t, t_0, x_0) \) of the system (12) can be written as

\[
x(t) = L(t, t_0)x_0 + \int_{t_0}^t L(t, s)F(s)ds, \quad t \geq t_0,
\]

where \( L(t, s) \) is the corresponding matrix solution of the system (14).

Samoilenko and Perestyuk [20] investigated the stability properties of solutions of the system (4). We obtain the following result by applying (7) in Lemma 2.1.

Theorem 3.3. Assume that \( \prod_{k=1}^{\infty} (1 + |A_k|) < \infty \) in (4) and there exists a positive bounded continuous function \( h : \mathbb{R}_+ \to \mathbb{R} \) such that

\[
|U(t, s)| \leq h(t)h(s)^{-1}, \quad t \geq s \geq t_0,
\]

where \( U(t, s) \) is a fundamental matrix solution of the system (8). Then the system (4) is \( h \)-stable.

Proof. Let \( X(t, t_0) \) be a fundamental matrix solution of the system (4) with \( X(t_0, t_0) = I \). Then it follows from Lemma 2.1 that

\[
|X(t, t_0)| = |U(t, t_k)(I + A_k) \prod_{i=k}^{1} U(t_{i+1}, t_i)(I + A_i)U(t_1, t_0)|
\leq h(t)h(t_k)^{-1}(1 + |A_k|) \prod_{i=k}^{1} h(t_{i+1})h(t_i)^{-1}(1 + |A_i|)h(t_1)h(t_0)^{-1}
\leq ch(t)h(t_0)^{-1}, \quad t_0 \leq t < t_k, \quad k \in \mathbb{N},
\]

where \( c = \prod_{k=1}^{\infty} (1 + |A_k|) \). Thus the system (4) is \( h \)-stable. This completes the proof.

Remark 3.4. We can obtain the following results as the special cases of Theorem 3.3 (see [20, Theorem 11]).
(i) If \( h(t) = e^{-ct} \) for a nonnegative constant \( c \), then the system (4) is exponentially stable.

(ii) If \( h(t) = c \), then the system (4) is bounded, i.e., there exists a nonnegative constant \( M \) such that

\[ |U(t, t_0)| \leq M, \quad t \geq t_0, \]

where \( U(t, t_0) \) is a fundamental matrix solution of the system (8).

(iii) If \( h(t) \to 0 \) as \( t \to \infty \), then the system (4) is asymptotically stable.

**Corollary 3.5.** Assume that \( \prod_{k=1}^{\infty} (1 + |A_k|) < \infty \) and \( \sum_{k=1}^{\infty} |B_k| < \infty \) in (5). If the system (4) is bounded and \( \int_{t_0}^{\infty} |F(s)|ds < \infty \) for fixed \( t_0 \in \mathbb{R}_+ \), then the system (5) is also bounded.

**Proof.** Let \( x(t, t_0, x_0) \) be any solution of the system (5). Since the system (4) is bounded, there exists a nonnegative constant \( M_1 \) such that

\[ |X(t, t_0)| \leq M_1, \quad t \geq t_0, \]

where \( X(t) \) is a fundamental matrix solution of the system (4). It follows from (9) in Lemma 2.1 that

\[ |x(t)| \leq |X(t, t_0)||x_0| + \int_{t_0}^{t} |X(t, s)||F(s)|ds + \sum_{t_0 < t_k < t} |X(t, t_k^+)||B_k| \]

\[ \leq M_1 \left( |x_0| + \int_{t_0}^{t} |F(s)|ds + \sum_{t_0 < t_k < t} |B_k| \right) \]

\[ \leq M(x_0), \quad t \geq t_0, \]

where \( M(x_0) = M_1 \left( |x_0| + \int_{t_0}^{\infty} |F(s)|ds + \sum_{k=1}^{\infty} |B_k| \right) < \infty \). This completes the proof. □

Next, we study the \( h \)-stability of solutions for nonlinear impulsive integro-differential systems by the means of an impulsive integral inequality of Gronwall-Bellman type.

**Theorem 3.6.** Assume that

(i) the system (2) is \( h \)-stable;

(ii) \( |F(t, x)| \leq \lambda_1(t)|x| \) and \( |G(t, s, x)| \leq \lambda_2(t) \int_0^t k(t, s)|x|ds \),

where \( \lambda_1, \lambda_2 \in C[\mathbb{R}_+, \mathbb{R}_+] \) and \( k \in C[\mathbb{R}_+^2, \mathbb{R}_+] \);

(iii) \( \int_{t_0}^{\infty} \left[ \lambda_1(s) + \lambda_2(s) \int_0^s k(s, \tau) \frac{h(s)}{M(x_0)} d\tau \right] ds < \infty \).

Then the system (1) is \( h \)-stable.

**Proof.** Let \( x(t) = x(t, t_0, x_0) \) be any solution of the system (1) with \( x(t_0^+) = x_0 \). Since the system (2) is \( h \)-stable, then there exist a positive constant \( c \) and a positive bounded continuous function \( h : \mathbb{R}_+ \to \mathbb{R} \) such that

\[ |L(t, s)| \leq ch(t)h(s)^{-1}, \quad t \geq s \geq t_0, \]

(16)
where \( L(t, s) \) is the corresponding fundamental matrix solution of the system (14). In view of (15) in Lemma 3.2 (see [21, p. 12]), any solution \( x(t) \) of the system (1) which is equivalent to the system (3) satisfies

\[
x(t) = L(t, t_0)x_0 + \int_{t_0}^{t} L(t, s) \left[ F(s, x(s)) + \int_{t_0}^{s} G(s, \tau, x(\tau)) d\tau \right] ds, \quad t \geq t_0.
\]

From the conditions (i)-(iii) and (16), it follows that

\[
| x(t) | \leq c h(t) h(t_0)^{-1} | x_0 | + \int_{t_0}^{t} c h(t) h(s)^{-1} \left[ \lambda_1(s) | x(s) | + \lambda_2(s) \int_{t_0}^{s} k(s, \tau) | x(\tau) | d\tau \right] ds, \quad t \geq t_0.
\]

Putting \( m(t) = \frac{| x(t) |}{| x_0 |} \), we obtain

\[
m(t) \leq c m(t_0) + c \int_{t_0}^{t} \left[ \lambda_1(s) m(s) + \lambda_2(s) \int_{t_0}^{s} k(s, \tau) \frac{h(\tau)}{h(s)} m(\tau) d\tau \right] ds, \quad t \geq t_0.
\]

Then, it follows from Lemma 3.1 that

\[
m(t) \leq c m(t_0) \exp \left( c \int_{t_0}^{t} \left[ \lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} k(s, \tau) \frac{h(\tau)}{h(s)} d\tau \right] ds \right), \quad t \geq t_0.
\]

Thus we obtain

\[
| x(t) | \leq d | x_0 | h(t) h(t_0)^{-1}, \quad t \geq t_0,
\]

where

\[
d = c \exp \left( c \int_{t_0}^{\infty} \left[ \lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} k(s, \tau) \frac{h(\tau)}{h(s)} d\tau \right] ds \right).
\]

Hence the system (1) is \( h \)-stable. This completes the proof. \( \square \)

4. Examples

In this section we give two examples to illustrate Lemma 3.2 and Theorem 3.3.

**Example 4.1** ([18]). We consider the following impulsive integro-differential equation

\[
\begin{aligned}
 x'(t) &= \frac{1}{2} x(t) - 9 \int_{t_0}^{t} e^{-7(t-s)} x(s) ds, \quad t \neq t_k, \\
 \Delta x(t_k) &= -\frac{1}{2} x(t_k), \quad k \in \mathbb{N}, \\
 x(t_k^+) &= x_0 \in \mathbb{R},
\end{aligned}
\]

(17)
where $A(t) = \frac{1}{7}$, $K(t, s) = -9e^{-7(t-s)}$, $A_k = -\frac{1}{2}$ for each $k \in \mathbb{N}$. Then the equation (17) is equivalent to the impulsive differential equation

$$
\begin{aligned}
x'(t) &= -\frac{11}{2}x(t) + L(t, t_0)x_0, \quad t \neq t_k, \\
\Delta x(t_k) &= -\frac{1}{2}x(t_k), \quad k \in \mathbb{N}, \\
x(t_0) &= x_0 \in \mathbb{R},
\end{aligned}
$$

(18)

where the solution $L(t, s)$ of the equation (14) as in Lemma 3.2 is given by

$$
L(t, s) = \begin{cases} 
6e^{-7(t-s)}, & t_{k-1} < s \leq t < t_k, \\
2e^{-7(t-s)} + e^{-7(t-t_k)} - \frac{1}{7}(t_k - s), & t_{k-1} < t < t_k < t_{k+1}, \\
-\frac{2}{3}e^{-7(t-s)} + \frac{1}{3}e^{-7(t-t_k)} - \frac{1}{7}(t_k - s) - \frac{1}{7}(t_{k+1} - t_k) & t_{k-1} < t < t_{k+1} < t < t_{k+2}, \\
\frac{2}{3}e^{-7(t-s)} - \frac{1}{3}e^{-7(t-t_k)} - \frac{1}{7}(t_k - s) - \frac{1}{7}(t_{k+1} - t_k) & t_{k-1} < t < t_{k+1} < t < t_{k+2}, \\
\frac{2}{3}e^{-7(t-s)} - \frac{1}{3}e^{-7(t-t_k)} - \frac{1}{7}(t_k - s) - \frac{1}{7}(t_{k+1} - t_k), & t_{k-1} < s < t < t_{k+1} < t < t_{k+2}, \\
\end{cases}
$$

(19)

Thus any solution $x(t)$ of the equation (18) with $x(t_0, t_0^+, x_0) = x_0$ is given by

$$
x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)L(s, t_0)x_0ds
$$

$$
= e^{-\frac{11}{2}(t-t_0)} \prod_{t_0 < i < t} (1 - \frac{1}{2})x_0
$$

$$
+ \int_{t_0}^{t} e^{-\frac{11}{2}(t-s)} \prod_{s < i < t} (1 - \frac{1}{2})L(s, t_0)x_0ds, \quad t \geq t_0,
$$

where $L(t, s)$ is given by (19) and $X(t, t_0) = e^{-\frac{11}{2}(t-t_0)} \prod_{t_0 < i < t} (1 - \frac{1}{2})$ is a fundamental matrix of the impulsive differential equation

$$
\begin{aligned}
x'(t) &= -\frac{11}{2}x(t), \quad t \neq t_k, \\
\Delta x(t_k) &= -\frac{1}{2}x(t_k), \quad k \in \mathbb{N}, \\
x(t_0^+) &= x_0 \in \mathbb{R},
\end{aligned}
$$

(20)

Then we obtain

$$
|x(t)| \leq |X(t, t_0)x_0| + \int_{t_0}^{t} |X(t, s)||L(s, t_0)||x_0|ds
$$

$$
= e^{-\frac{11}{2}(t-t_0)} \prod_{t_0 < i < t} (1 - \frac{1}{2})|x_0| + \int_{t_0}^{t} e^{-\frac{11}{2}(t-s)} \prod_{s < i < t} (1 - \frac{1}{2})|L(s, t_0)||x_0|ds
$$

$$
\leq |x_0| e^{-\frac{2}{3}(t-t_0)} + \frac{6}{11} \left(1 - e^{-\frac{11}{2}(t-t_0)}\right)
$$
\[ L \leq L |x_0|, \quad t \geq t_0, \]
where \( L = \frac{23}{22} \). That is, the equation (17) is uniformly Lipschitz stable.

**Example 4.2.** We consider the following impulsive differential equation:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t), \quad t \neq t_k, \\
\Delta x(t_k) &= A_k x(t_k), \quad k \in \mathbb{N}, \\
x(t_0^+) &= x_0 \in \mathbb{R},
\end{align*}
\]

(21)

where \( A(t) = -\frac{11}{2} \) and \( A_k = -\frac{1}{2^k} \) for each \( k \in \mathbb{N} \). Let \( U(t,s) \) be a solution of the matrix equation

\[
\frac{dU}{dt} = A(t)U = -\frac{11}{2} U, \quad U(s,s) = I.
\]

(22)

Then there exists a positive bounded continuous function \( h : \mathbb{R}_+ \rightarrow \mathbb{R} \) such that

\[
|U(t,s)| = |e^{-\frac{11}{2}(t-s)}| \\
\leq ch(t)h(s)^{-1}, \quad t \geq s \geq t_0,
\]

where \( c = 1 \) and \( h(t) = e^{-\frac{11}{2}t} \). Also we easily see that

\[
\prod_{k=1}^{\infty} (1 + |A_k|) \leq \exp \left( \sum_{k=1}^{\infty} |A_k| \right) \\
= \exp \left( \sum_{k=1}^{\infty} \frac{1}{2^k} \right) < \infty.
\]

Since all conditions of Theorem 3.3 are satisfied, the equation (21) is \( h \)-stable by Theorem 3.3.

In fact, for any solution \( x(t,t_0,x_0) \) of the equation (21), we have

\[
\begin{align*}
|x(t,t_0,x_0)| &= |X(t_0)x_0| \\
&= |U(t_k)(I + A_k) \prod_{i=k}^{1} U(t_{i+1},t_i)(I + A_i)U(t_1,t_0)x_0| \\
&\leq |x_0|h(t)h(t_k)^{-1}(1 + |A_k|) \prod_{i=k}^{1} h(t_{i+1})h(t_i)^{-1}(1 + |A_i|)h(t_1)h(t_0)^{-1} \\
&\leq d|x_0|h(t)h(t_0)^{-1}, \quad t_0 \leq t_k < t \leq t_{k+1}, \quad k \in \mathbb{N},
\end{align*}
\]

where \( d = \prod_{k=1}^{\infty} (1 + |A_k|) = \prod_{k=1}^{\infty} (1 + \frac{1}{2^k}) < \infty. \)
Acknowledgment. This work was supported by research fund of Chungnam National University in 2018. The authors are thankful to the anonymous referee for his/her valuable comments to improve this paper.

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