준노름 퍼지적분의 비 선형성

Non-Linearity of the Seminormed Fuzzy Integral

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요 약

Fuzzy 측도 공간 $(X, \mathcal{F}, g)$ 에서 0과 1사이의 임의의 상수 a와 가측 함수 h와 가측 집합 A에 관하여, t-준 노름이 특별히 두 변수 곱하기일 경우

$$\int_A a \cdot h(x) \top g = a \cdot \int_A h(x) \top g$$

이 성립함을 보였다. 또한, 준 노름 fuzzy 적분이 선형성을 갖는 필요충분 조건이 {0.1}-족 집합임을 가지고 다음을 증명하였다.

$$af + bh \in L^o(x) \Rightarrow \int_A (af + bh) \top g = a \int_A f \top g + b \int_A h \top g;$$

이 성립할 필요 충분 조건은 g가 모든 가측 집합 A 에 대해 0과 1값만 갖는 확률 측도일 경우이다.

Abstract

Let $(X, \mathcal{F}, g)$ be a fuzzy measure space. Then for any $h \in L^0(X)$, $a \in [0, 1]$, and $A \in \mathcal{F}$

$$\int_A a \cdot h(x) \top g = a \cdot \int_A h(x) \top g$$

with the t-seminorm $\top(x, y) = xy$. And we prove that the seminormed fuzzy integral has some linearity properties only for {0,1}-classes of fuzzy measure as follow;

For any f, h $\in L^0(x)$, any $a, b \in R_+$:

$$af + bh \in L^o(x) \Rightarrow \int_A (af + bh) \top g = a \int_A f \top g + b \int_A h \top g;$$

if and only if g is a probability measure fulfilling g(A) $\in$ {0, 1} for all A $\in \mathcal{F}$.

I. Introduction

Sugeno [8] defined a fuzzy measure as a measure having the monotonicity instead of additivity. Since fuzzy measure does not satisfy countable additive, it gained recognition of its practical value. And a fuzzy integral which is an integral with respect to fuzzy measure is applied to make a synthetic evaluation about arbitrary objects. The concept of the seminormed fuzzy integral which is generalized fuzzy integral was proposed by Suarez and Gill [6],[7]. In general the fuzzy integral is not linear as consequence of the non-additivity of the fuzzy measure. Klement and Ralescu [3] showed that the fuzzy integral has some linearity properties only for small classes of fuzzy measures. In this paper, we will generalize above-mentioned property.

II. Preliminaries

We recall some notions and notations which will be used in this paper, and investigate elementary properties of fuzzy measure and seminormed fuzzy integral.

Let X be a nonempty set, $\mathcal{F}$ be a σ-algebra of subsets of X, and $g : \mathcal{F} \rightarrow [0, 1]$ be a set function.

A set function $g : \mathcal{F} \to [0,1]$ is called a fuzzy measure if

(1) $g(\emptyset) = 0$ (vanishing at $\emptyset$) ;

(2) $A \in \mathcal{F}, B \in \mathcal{F}$, and $A \subset B$ imply $g(A) \subset g(B)$ (monotonicity) ;

(3) $A_n \subset \mathcal{F}$, $A_1 \subset A_2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ imply $\lim_{n\to\infty} g(A_n) = g(\bigcup_{n=1}^{\infty} A_n)$ (continuity from below) ;

(4) $A_n \subset \mathcal{F}$, $A_1 \supset A_2 \supset \cdots$, and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$ imply $\lim_{n\to\infty} g(A_n) = g(\bigcap_{n=1}^{\infty} A_n)$ (continuity from above).

We $(X, \mathcal{F}, g)$ call a fuzzy measure space if g is a fuzzy measure on a measurable space $(X, \mathcal{F})$. The main difference between fuzzy measures and classical measures is the lack of additivity of the former. However each classical measure is a fuzzy measure. Since the fuzzy measure lose additivity in general, they appear much looser than the classical measures.

A real - valued function $h: X \to [0,1]$ is $\mathcal{F}$-measurable with respect to $\mathcal{F}$ and Ω (measurable, for short, if there is no confusion likely) if $\{h^{-1}(B) = x \mid h(x) \in B\} \in \mathcal{F}$ for any $B \in \Omega$, where Ω is the σ-algebra of Borel subsets of $[0,1]$.

The definition of measurability of function is the same as in the theory of Lebesgue integrals.

From now on, let us consider the set $L^0(X) = \{h: X \to [0,1] \mid h$ is measurable with respect to $\mathcal{F}$ and $\Omega\}$, where Ω is the usual σ-algebra of Borel subsets of $[0,1]$. For any given $h \in L^0(X)$, we write $H_\alpha = \{x \mid h(x) \geq \alpha\}$, where $\alpha \in [0,1]$.

Let $A \in \mathcal{F}$, $h \in L^0(X)$. The **fuzzy integral** of h with respect to g, which denote by $\int_A h\,dg$, is defined by

$$\int_A h\,dg = \sup_{\alpha\in[0,1]} [\alpha \wedge g(A \cap H_\alpha)].$$

When $A = X$, the fuzzy integral may be denote by $\int h\,dg$. Sometimes the fuzzy integral is also called Sugeno's integral in the literature.

A t-seminorm is a function $\top: [0,1] \times [0,1] \to [0,1]$ which satisfies

(1) For each $x \in [0, 1]$,

$\top(x, 1) = \top(1, x) = x$;

(2) For each $x_1, x_2, x_3, x_4 \in [0,1]$, if $x_1 \le x_3$, $x_2 \le x_4$, then $\top(x_1, x_2) \le \top(x_3, x_4)$.

Example 2.1. The following functions are t-seminorms ;

(1) $\top(x, y) = x \wedge y$

(2) $\top(x, y) = xy$

(3) $\top(x, y) = 0 \vee (x+y-1)$

Let $\top$ be a t-seminorm. For all $h \in L^0(X)$, the **seminormed fuzzy** integral of h over $A \in \mathcal{F}$ to fuzzy measure g is defined as

$$\int_A h \top g = \sup_{\alpha\in[0,1]} \top[\alpha, g(A \cap H_\alpha)]$$

In what follows, $\int_X h \top g$ will be denote $\int h \top g$ for short. The seminormed fuzzy integral contains as a particular case the fuzzy integral of Sugeno with $\top(x, y) = x \wedge y$.

Example 2.2. [An application of seminormed fuzzy integral]

Consider the employment problem of a company. A company decides to recruit one new person. Assume that the quality factors of the company considers are the computer quality, the ability in English, and the oral test.

We denote these factors by $C, E,$ and T respectively; hence we can set $X=\{C, E, T\}$.

Assume further that following set function g is employed as an importance measure (a fuzzy measure) :

$$g(C)=0.7, g(E)=0.1, g(T)=0, g(C,E)=0.9,$$
$$g(C,T)=0.8, g(E,T)=0.3, g(X)=1, g(\phi)=0.$$

Suppose that two examine A and B attained points as follows :

$$A(C)=0.9,\ A(E)=0.6,\ A(T)=0.1,$$
$$B(C)=0.4,\ B(E)=0.6,\ B(T)=0.6.$$

If we take $\top(x,y) = 0\vee(x+y-1)$. Then the synthetic evaluations of two examine are calculated as follows ;

$$\begin{aligned} E_A &= \int A \top g \\ &= \sup_{\alpha\in[0,1]} \top(\alpha,\ g(A_\alpha)) \\ &= \sup_{\alpha\in[0,0.1]}[0 \vee (\alpha+g(X)-1)] \\ &\quad \vee \sup_{\alpha\in[0.1,0.6]}[0\vee(\alpha+g(C,E)-1)] \\ &\quad \vee \sup_{\alpha\in[0.6,1]}[0\vee(\alpha+g(C)-1)] \\ &= 0.6. \end{aligned}$$

And

$$\begin{aligned} E_B &= \int B \top g \\ &= 0.3. \end{aligned}$$

Assume that an examine will be employed if he gets more than 0.5 points by seminormed fuzzy integrals. Then we conclude that A is employed but B is not.

Remark. Seminormed fuzzy integrals (including fuzzy integrals) can be applied to evaluate the synthetic values of certain object as we considered in Example 2.1. For each situation, one can choose a suitable t-seminorm.

Let $\mathcal{F}$ be a collection of subsets of X, and $F_1\in\mathcal{F}$, $F_2\in\mathcal{F}$. A set function g is called the **fuzzy additive** on $\mathcal{F}$ if

$$g(F_1\cup F_2)=g(F_1)\vee g(F_2).$$

We shall prove Fuzzy Beppo Levi's Theorem in which we use the supremum instead of addition in the expression ;

Theorem 2.3. (Fuzzy Beppo Levi's Theorem). Let $(X,\mathcal{F},g)$ be a fuzzy measure space. If g be fuzzy additive, and $\top$ be a continuous t-seminorm, then

$$\int_A \Big(\bigvee_{n=1}^{\infty} h_n\Big)\top g = \bigvee_{n=1}^{\infty} \int_A h_n \top g$$

for $h_n\in L^0(x)$, $n=1, 2, 3, \cdots$.

Proof. We may assume that A = X without loss of generality.

Let $h=\bigvee_{n=1}^{\infty} h_n$, and $H_\alpha=\{x\mid h(x)\geq\alpha\}$.

Then $H_\alpha=\bigcup_{n=1}^{\infty} H_\alpha^n$, where $H_\alpha^n=\{x\mid h_n(x)\geq\alpha\}$.

The fuzzy additivity of g and the continuity of $\top$ yield the following

$$\begin{aligned} \int \Big(\bigvee_{n=1}^{\infty} h_n\Big)\top g &= \sup_{\alpha\in[0,1]} \top(\alpha, g(H_\alpha)) \\ &= \sup_{\alpha\in[0,1]} \top\Big(\alpha, g\Big(\bigcup_{n=1}^{\infty} H_\alpha^n\Big)\Big) \\ &= \sup_{\alpha\in[0,1]} \top\Big(\alpha, \bigvee_{n=1}^{\infty} g(H_\alpha^n)\Big) \\ &= \sup_{\alpha\in[0,1]} \bigvee_{n=1}^{\infty} \top(\alpha, g(H_\alpha^n)) \\ &= \bigvee_{n=1}^{\infty} \sup_{\alpha\in[0,1]} \top(\alpha, g(H_\alpha^n)) \\ &= \bigvee_{n=1}^{\infty} \int h_n \top g. \quad \square \end{aligned}$$

III. Fuzzy Linearity

We should note that, in general, the fuzzy integral lacks some important properties Lebesgue's integral possesses.

For instance Lebesgue's integral has linearity, but the fuzzy integral does not. We can see this in the following example.

Example 3.1. Let $(X=[0,1], \mathcal{F}, g)$ be the Lebesgue measure space.

(1) If we take $h(x)=x$ for any $x\in X$, and $a=\frac{1}{2}$, then we have

$$\int ah\,dg = \int \frac{x}{2}\,dg$$

$$= \sup_{\alpha\in[0,1]} \alpha\wedge(1-2\alpha)=\frac{1}{3}$$

and

$$a\int h\,dg=\frac{1}{2}\int x\,dg=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}.$$

Consequently, we have

$$\int ah\,dg\neq a\int h\,dg.$$

(2) If we take $h(x)=\sqrt{x}$, $a=\frac{1}{2}$, $\top(x,y)=0\vee(x+y-1)$, then we have

$$\int ah\top g = \sup_{\alpha\in[0,1]} \top(\alpha,(1-4\alpha^2))$$

$$=\frac{1}{16},$$

but

$$a\int h\top g = \frac{1}{2}\sup_{\alpha\in[0,1]} \top(\alpha,(1-\alpha^2))$$

$$=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}.$$

Hence

$$\int ah\top g \neq a\int h\top g.$$

Naturally, the linearity heavily depends on the t-seminorm $\top$. For example, we can show that a scalar multiple of a fuzzy integral behaves nicely with $\top(x,y)=xy$.

Theorem 3.2 Let $(X,\mathcal{F},g)$ be a fuzzy measure space. Then for any $h\in L^0(X)$, $a\in[0,1]$, and $A\in\mathcal{F}$,

$$\int_A a\cdot h(x)\top g = a\cdot\int_A h(x)\top g$$

with the t-seminorm $\top(x,y)=xy$.

Proof. We denote

$$H_\alpha=\{x\in[0,1]: h(x)\geq\alpha\}$$

$$=h^{-1}[\alpha,\infty)\cap[0,1],$$

$$H^*_\alpha=\{x\in[0,1]: a\cdot h(x)\geq\alpha\}$$

$$=h^{-1}[\frac{\alpha}{a},\infty)\cap[0,1].$$

Then $H^*_\alpha=H_{\alpha/a}$. Therefore, since $g(H_\beta)=0$ for $x\in(1,1/a]$

$$\int_A a\cdot h(x)\top g = \sup_{\alpha\in[0,1]} \top(\alpha, g(H^*_\alpha\cap A))$$

$$= \sup_{\alpha\in[0,1]} \alpha\cdot g(H^*_\alpha\cap A)$$

$$= \sup_{\alpha\in[0,1]} \alpha\cdot g(H_{\alpha/a}\cap A)$$

$$= \sup_{\alpha\in[0,1]} a\cdot\alpha/a\cdot g(H_{\alpha/a}\cap A)$$

$$= a\cdot\sup_{\alpha\in[0,1]} \alpha/a\cdot g(H_{\alpha/a}\cap A)$$

$$= a\cdot\sup_{\beta\in[0,\frac{1}{a}]} \beta\cdot g(H_\beta\cap A)$$

$$= a\cdot\sup_{\beta\in[0,1]} \beta\cdot g(H_\beta\cap A)$$

$$= a\cdot\sup_{\beta\in[0,1]} \top(\beta, g(H_\beta\cap A))$$

$$= a\cdot\int_A h(x)\top g.$$

This finished the proof. □

Remark.3.3 Even if $h, k\in L^0(X)$ and the t-seminorm $\top$ satisfy the hypotheses in Theorem 3.2 ,

$$\int (h+k)\top g\neq\int h\top g+\int k\top g$$

in general. For example, if $h(x)=\frac{1}{2}x$, $k(x)=\frac{1}{2}$ and g is the Lebesgue measure, then

$$\int h\top g=\frac{1}{8}, \quad \int k\top g=\frac{1}{2},$$

but

$$\int (h+k)\top g=\frac{1}{2}$$

In [3], Klement and Ralescu showed that the fuzzy integral has some linearity properties only for small classes of fuzzy measures. Analogously, seminormed fuzzy integrals satisfy linearity properties only for small classes of fuzzy measures. The argument in [3] works for the following statement with little adjustment.

Theorem 3.4 Let $(X, \mathcal{F}, g)$ be a fuzzy measure space. Then the following statements are equivalent :

(1) For any $f, h\in L^0(X)$, any $a, b\in R_+$:

$[af+bh\in L^0(X)\Rightarrow$

$\int (af+bh)\top g = a\int_A f\top g+b\int_A h\top g]$;

(2) g is a probability measure fulfilling $g(A)\in\{0,1\}$ for all $A\in\mathcal{F}$.

Proof. $(1)\Rightarrow(2)$

We may assume that $A=X$ without loss of generality.

Let $H^*_\alpha=\{x\in[0,1] \mid a h(x)\geq\alpha\}$. Suppose, to get the contradiction, there exists a number α_0, $0<\alpha_0<1$, and a set $A\in\mathcal{F}$ such that $g(A)=\alpha_0$.

Then we choose $h=\frac{1}{2}\alpha_0\aleph_A$ and $a=1+\frac{1}{\alpha_0}$.

Now we obtain

$$g(H_\alpha)=\begin{cases}1 & \text{if } \alpha=0,\\ \alpha_0 & \text{if } 0<\alpha\leq\frac{1}{2}\alpha_0\\ 0 & \text{if } \frac{1}{2}\alpha_0<\alpha\end{cases}$$

and therefore

$$\int h\top g = \top(0,1)\vee\sup_{\alpha\in(0,\frac{1}{2}\alpha_0]}\top(\alpha,\alpha_0)$$

$$\vee\sup_{\alpha\in(\frac{1}{2}\alpha_0,1]}\top(\alpha,0)$$

$$=\top(\frac{1}{2}\alpha_0,\alpha_0).$$

On the other hand,

$$g(H^*_\alpha)=\begin{cases}1 & \text{if } \alpha=0,\\ \alpha_0 & \text{if } 0<\alpha\leq\frac{1}{2}(1+\alpha_0),\\ 0 & \text{if } \frac{1}{2}(1+\alpha_0)<\alpha,\end{cases}$$

from which

$$\int ah\top g$$

$$=\top(0,1)\vee\sup_{\alpha\in(0,\frac{1}{2}(1+\alpha_0)]}\top(\alpha,\alpha_0)$$

$$\vee\ \sup_{\alpha\in(\frac{1}{2}\alpha_0,1]}\top(\alpha,0)$$

$$=\top(\frac{1}{2}(1+\alpha_0),\ \alpha_0).$$

Since

$$\int ah\top g=\top(\frac{1}{2}(1+\alpha_0),\alpha_0)\leq\alpha_0<1,$$

$$a\int h\top g=(1+\frac{1}{\alpha_0})\ \top(\frac{1}{2}\alpha_0,\alpha_0)$$

$$\leq\ (1+\frac{1}{\alpha_0})\alpha_0$$

$$=\alpha_0+1>1,$$

we have

$$\int ah\top g\neq a\int h\top g.$$

Hence $g(A)\in\{0,1\}$ for all $A\in\mathcal{F}$.

In order to show that g is countably additive, we first choose sets $A, B\in\mathcal{F}$ such that $A\cap B=\emptyset$. Then we have

$$g(A\cup B)=\int\aleph_{A\cup B}\top g$$

$$=\int\aleph_A\top g+\int\aleph_B\top g$$

$$=g(A)+g(B).$$

And let $\{A_n\}$ be a sequence such that $A_i \cap A_j = \emptyset$ if $i \neq j$.

Then

$$A_1 \subset A_1 \cup A_2 \subset A_1 \cup A_2 \cup A_3 \subset \cdots$$

By the continuity from below of the fuzzy measure,

$$g(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} g(\bigcup_{k=1}^{n} A_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} g(A_k)$$

$$= \sum_{k=1}^{\infty} g(A_k)$$

Hence g is countably additive.

(2)⇒(1)

Fix $h \in L^0(X)$ and $a > 0$ such that $a h \in L^0(X)$ (for $a = 0$ we obviously have $\int ah \top g = 0 = a \int h \top g$) and define

$$\beta_0 = \sup\{\alpha \in [0,1] \mid g(H_\alpha) = 1\}.$$

Then obviously

$$\int h \top g = \sup_{\alpha \in [0,1]} \top(\alpha, g(H_\alpha))$$

$$= \sup_{\alpha \in [0,\beta_0]} \top(\alpha, 1)$$

$$= \beta_0.$$

Hence

$$\int a h \top g = \sup \top(\alpha, g(H_\alpha^*))$$

$$= \sup\{\alpha \in [0,1] \mid g(H_\alpha^*) = 1\}$$

$$= \sup\{\frac{\alpha}{a} \cdot a \in [0,1] \mid g(H_{\frac{\alpha}{a}}) = 1\}$$

$$= a \cdot \sup\{\frac{\alpha}{a} \in [0,1] \mid g(H_{\frac{\alpha}{a}}) = 1\}$$

$$= a \cdot \beta_0$$

$$= a \cdot \int h \top g.$$

It remains to prove that

$$\int (f+h) \top g = \int f \top g + \int h \top g.$$

Let $f + h = k$, $F_\alpha = \{x \mid f(x) \geq \alpha\}$,

$$H_\beta = \{x \mid h(x) \geq \beta\},$$

and

$$K_\gamma = \{x \mid (f+h)(x) \geq \gamma\}.$$

We can choose $f, h \in L^0(X)$ such that $k \in L^0(X)$.

Then for all $\alpha, \beta, \gamma \in [0, 1]$

$$F_\alpha \cap H_\beta \subset K_\gamma \subset F_\alpha \cup H_\beta.$$

Now put

$$\int f \top g = \alpha_0,\ \int h \top g = \beta_0,\ \int (f+h) \top g = \gamma_0,$$

and since $g(A) \in \{0,1\}$,

$$\alpha_0 = \sup\{\alpha \in [0,1] \mid g(F_\alpha) = 1\},$$

$$\beta_0 = \sup\{\beta \in [0,1] \mid g(H_\beta) = 1\},$$

$$\gamma_0 = \sup\{\gamma \in [0,1] \mid g(K_\gamma) = 1\}.$$

By the continuity from above of the fuzzy measure, we get

$$g(F_{\alpha_0}) = g(H_{\beta_0}) = g(K_{\gamma_0}) = 1.$$

Taking into account that $g(A) = g(B) = 1$ implies $g(A \cap B) = 1$, we conclude that $g(K_{\alpha_0 + \beta_0}) = 1$, which implies $\alpha_0 + \beta_0 \leq \gamma_0$.

Now assume $\gamma_0 > \alpha_0 + \beta_0$, thus

$$\gamma_0 = (\alpha_0 + \varepsilon) + (\beta_0 + \varepsilon) \quad \text{for some} \quad \varepsilon > 0.$$

But this means,

$$1 = g(K_{\gamma_0})$$

$$\leq g(\{F_{\alpha_0+\varepsilon}\} \cup \{H_{\beta_0+\varepsilon}\})$$

$$\leq g(F_{\alpha_0+\varepsilon}) + g(H_{\beta_0+\varepsilon})$$

$$= 0.$$

This is a contradiction. Hence $\alpha_0 + \beta_0 = \gamma_0$.

Therefore

$$\int (f+h) \top g = \int f \top g + \int h \top g.$$

The proof is complete. □

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