Stability of a Generalized Quadratic Type Functional Equation

일반화된 2차형 범함수 방정식의 안정성

김미혜
충북대학교 전기전자및컴퓨터공학부 초빙교수

황인성
충북대학교 수학과 대학원

Mi-Hye Kim
Invited Professor, School of Electrical & Computer Engineering, Chungbuk National Univ.
In-Sung Hwang
Candidate for Master, Dep. of Math., Chungbuk National Univ.

중심어 : stability, quadratic functional equation

요 약

함수 방정식은 연구원들이 함수 자체의 정확한 형태를 가정하지 않고 단순히 기본적인 함수의 성질만을 언급하는 한정적이지 않은 방정식을 통하여 일반적인 관점의 수학적 형상화를 공식화하는데 매우 중요한 구실을 하기 때문에 실험적인 학문에서 유용하다. 그러한 많은 함수 방정식 가운데에서 이 논문은 다소 일반화된 2차 함수 방정식을 선택해 해를 구하며 이 방정식의 안정성을 증명한다.

$$a^2f\left(\frac{x+y}{a}\right)+b^2f\left(\frac{x-y}{b}\right)=2f(x)+2f(y)$$

Abstract

Functional equations are useful in the experimental science because they play very important role for researchers to formulate mathematical models in general terms, through some not very restrictive equations that only stipulate basic properties of functions showing in these equations, without postulating the exact forms of such functions.

Of lots of such functional equations, in this paper we adopt and solve some generalized quadratic functional equation

$$a^2f\left(\frac{x+y}{a}\right)+b^2f\left(\frac{x-y}{b}\right)=2f(x)+2f(y)$$

I. Introduction

Functional equations are a useful tool for narrowing the possible models for a phenomenon in that at least one more not very restrictive equations can formulate a model and when paired with an empirical or logical constraint of a general character those equations lead to precise quantitative relationships. In this paper we will deal one of the functional equations problem.

In 1940, S. M. Ulam [9] gave a wide ranging talk before a mathematical. Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exists a $\delta>0$ such that if a function $h: G_1\to G_2$ satisfies the inequality $d(h(xy), h(x)h(y))<\delta$ for all $x, y\in G_1$ then there is a homomorphism $H: G_1\to G_2$ with $d(h(x), H(x))<\varepsilon$ for all $x\in G_1$?

For the case where the answer is affirmative, the functional equation for homomorphisms will be called stable. The first result concerning the stability of functional equations was presented by D. H. Hyers [1]. He has excellently answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately liner mappings was given by Th. M. Rassias. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors.

The quadratic function $f(x) = x^2$ is **a solution of the functional equation**

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

So, every solution of the functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is said to be a **quadratic function**. S. H. Lee[3] proved the stability of the equation

$$a^2 f\left(\frac{x+y}{a}\right) + a^2 f\left(\frac{x-y}{a}\right) = 2f(x) + 2f(y).$$

In this paper we deal with a generalized quadratic functional equation

$$a^2 f\left(\frac{x+y}{a}\right) + b^2 f\left(\frac{x-y}{b}\right) = 2f(x) + 2f(y)$$

where a and b are nonzero real constants.

In Section 2 we solve a generalized quadratic functional equation. In Section 3 we prove the stability of a generalized quadratic functional equation. Throughout this paper a and b are nonzero real constants.

II. A solution of a generalized quadratic functional equation

Throughout this section X and Y will be real linear spaces. Given a function $f: X \to Y$, consider the following equation

(2.1) $$a^2 f\left(\frac{x+y}{a}\right) + b^2 f\left(\frac{x-y}{b}\right) = 2f(x) + 2f(y)$$

for all $x,\ y \in X$.

Theorem 2.1. If a function $f: X \to Y$ satisfies (2.1) for all $x,\ y \in X$, then $f(x) - f(0)$ is quadratic.

proof. We consider first the case $a^2 + b^2 - 4 \neq 0$. Putting $x = y = 0$ in (2.1) we have

$$a^2 f(0) + b^2 f(0) = 4f(0).$$

Hence $f(0) = 0$ since $a^2 + b^2 - 4 \neq 0$.

Putting $y = x$ in (2.1) we have

(2.2) $$a^2 f\left(\frac{2x}{a}\right) = 4f(x)$$

for all $x \in X$. Putting $y = 0$ in (2.1) we have

(2.3) $$a^2 f\left(\frac{x}{a}\right) + b^2 f\left(\frac{x}{b}\right) = 2f(x)$$

for all $x \in X$. Putting $x = 0$ and $y = x$ in (2.1) we have

(2.4) $$a^2 f\left(\frac{x}{a}\right) + b^2 f\left(\frac{-x}{b}\right) = 2f(x)$$

for all $x \in X$. Subtracting (2.4) to (2.3) we have

$$b^2 f\left(\frac{x}{b}\right) = b^2 f\left(\frac{-x}{b}\right)$$

for all $x \in X$. Hence we obtain $f(x) = f(-x)$ for all $x \in X$.

Putting $y = -x$ in (2.1) we have

(2.5) $$b^2 f\left(\frac{2x}{b}\right) = 4f(x)$$

for all $x \in X$. Putting $x = 0$ and $y = 2x$ in (2.1) we have

(2.6) $$a^2 f\left(\frac{2x}{a}\right) + b^2 f\left(\frac{-2x}{b}\right) = 2f(2x)$$

for all $x \in X$. Using (2.2), (2.5), (2.6) and evenness of f we have

(2.7) $$f(2x) = 4f(x)$$

for all $x \in X$. By (2.2), (2.5), (2.7), we have

(2.8) $$a^2 f\left(\frac{x}{a}\right) = f(x) = b^2 f\left(\frac{x}{b}\right)$$

for all $x \in X$. From (2.1) and (2.8) we have

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x,\ y \in X$.

Now, we prove the case $a^2 + b^2 - 4 = 0$.

Let $Q(x) = f(x) - f(0)$ for all $x \in X$.

Then $Q(0) = 0$ and Q satisfies (2.1).

As a similar way to the case $a^2 + b^2 - 4 \neq 0$, we have Q is quadratic. □

III. Stability of a generalized quadratic functional equation

Let R^+ denote the set of nonnegative real numbers. Recall that a function $H: R^+ \times R^+ \to R^+$ is homogeneous of degree $p > 0$ if it satisfies $H(tu, tv) = t^p H(u, v)$ for all nonnegative real numbers t, u and v. Throughout this section X and Y will be a real normed space and a real Banach space, respectively. We may assume that H is homogeneous of degree p. Given a function $f: X \to Y$, we set

$$Df(x,y) = a^2 f\left(\frac{x+y}{a}\right) + b^2 f\left(\frac{x-y}{b}\right) - 2f(x) - 2f(y).$$

Theorem 3.1. Assume that $\delta \geq 0$, $p \in (0, \infty) \setminus \{2\}$ and $\delta = 0$ when $p > 2$. Let an even function $f: X \to Y$ satisfy

(3.1) $$\|Df(x,y)\| \leq \delta + H(\|x\|, \|y\|)$$

for all $x, y \in X$ and $f(0) = 0$. Then there exists a unique quadratic function $Q: X \to Y$ such that

(3.2) $$\|f(x) - Q(x)\| \leq \frac{1}{2}\delta + \frac{1}{2|4-2^p|} h(x)$$

for all $x \in X$, where

$$h(x) = 2H(\|x\|, \|x\|) + H(\|2x\|, 0).$$

Proof. Putting $y = x$ in (3.1) we have

(3.3) $$\left\|a^2 f\left(\frac{2x}{a}\right) - 4f(x)\right\| \leq \delta + H(\|x\|, \|x\|)$$

for all $x \in X$. Putting $x = 2x$ and $y = 0$ in (3.1) we have

(3.4) $$\left\|a^2 f\left(\frac{2x}{a}\right) + b^2 f\left(\frac{2x}{b}\right) - 2f(2x)\right\| \leq \delta + H(\|2x\|, 0)$$

for all $x \in X$. Putting $y = -x$ in (3.1) we have

(3.5) $$\left\|b^2 f\left(\frac{2x}{b}\right) - 4f(x)\right\| \leq \delta + H(\|x\|, \|x\|)$$

for all $x \in X$ since f is even. From (3.3), (3.4) and (3.5) we have

$$\|8f(x) - 2f(2x)\| \leq 3\delta + h(x)$$

for all $x \in X$, where

$$h(x) = 2H(\|x\|, \|x\|) + H(\|2x\|, 0).$$

Hence, we have

(3.6) $$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{3}{8}\delta + \frac{1}{8}h(x)$$

for all $x \in X$. We divide the remaining proof by two cases.

(1) The case $0 < p < 2$. Using (3.6) we have

(3.7) $$\left\|\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}}\right\| = \frac{1}{4^n}\left\|f(2^n x) - \frac{f(2 \cdot 2^n x)}{4}\right\| \leq \frac{1}{4^n}\frac{3}{8}\delta + \frac{1}{8}2^{(p-2)n}h(x)$$

for all $x \in X$ and all positive integers n. From (3.6) and (3.7) we have

(3.8) $$\left\|\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}\right\| \leq \sum_{k=m}^{n-1}\frac{1}{4^k}\cdot\frac{3}{8}\delta + \sum_{k=m}^{n-1}\frac{1}{8}2^{(p-2)k}h(x)$$

for all $x \in X$ and all nonnegative integers m and n with $m < n$.

This show that $\left\{\frac{f(2^n x)}{4^n}\right\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a function

$$Q: X \to Y \text{ by } Q(x) = \lim_{n\to\infty}\frac{f(2^n x)}{4^n}$$

for all $x \in X$. We have $Q(0) = 0$ and

$$\|DQ(x,y)\| = \lim_{n\to\infty} 4^{-n}\|DQ(2^n x, 2^n y)\| \leq \lim_{n\to\infty}(4^{-n}\delta + 2^{n(p-2)})H(\|x\|, \|y\|) = 0$$

for all $x, y \in X$. By Theorem 2.1, it follows that Q is quadratic. Putting $m = 0$ in (3.8) and letting $n \to \infty$ we have (3.2). Now, let $Q': X \to Y$ be another quadratic function satisfying (3.2). Then we have

$$\begin{aligned}\|Q(x)-Q'(x)\| &= 4^{-n}\|Q(2^n x)-Q'(2^n x)\| \\ &\le 4^{-n}(\|Q(2^n x)-f(2^n x)\| \\ &\quad + \|Q'(2^n x)-f(2^n x)\|) \\ &\le 4^{-n}\delta + \frac{2^{n(p-2)}}{|4-2^p|}h(x)\end{aligned}$$

for all $x\in X$ and all positive integers n.

Since $\lim_{n\to\infty}\left(4^{-n}\delta + \frac{2^{n(p-2)}}{|4-2^p|}h(x)\right) = 0,$ we can conclude the $Q(x) = Q'(x)$ for all $x\in X$. This proves the uniqueness of Q.

(2) The case $p>2$. Replacing x by $\frac{x}{2}$ in (3.6) and multiplying both sides of (3.6) by 4 we have

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le 2^{-p-1}h(x) \tag{3.9}$$

for all $x\in X$. Using (3.9) we have

$$\|4^n f(2^{-n}x) - 4^{n+1}f(2^{-(n+1)}x)\| \le 2^{-p-1}2^{(2-p)n}h(x) \tag{3.10}$$

for all $x\in X$ and all positive integers n. From (3.9) and (3.10) we have

$$\|4^n f(2^{-n}x) - f(x)\| \le \sum_{k=0}^{n-1} 2^{-p-1}2^{k(2-p)}h(x)$$

for all $x\in X$ and all positive integers n. The rest of the proof is similar to the corresponding part of the case $0<p<2$. □

Theorem 3.2. Assume that $\delta\ge 0$. Let an odd function $f: X\to Y$ satisfy

$$\|Df(x,y)\| \le \delta + H(\|x\|, \|y\|) \tag{3.11}$$

for all $x, y\in X$. Then

$$\|f(x)\| \le \frac{1}{b^2}\delta + \frac{1}{b^2}h(x) \tag{3.12}$$

for all $x\in X$, where

$$h(x) = H\left(\left\|\frac{b}{2}x\right\|, \left\|\frac{b}{2}x\right\|\right)$$

Proof. Putting $y=-x$ in (3.11) we have

$$\left\|b^2 f\left(\frac{2x}{b}\right)\right\| \le \delta + H(\|x\|, \|x\|) \tag{3.13}$$

for all $x\in X$. Replacing x by $\frac{b}{2}x$ in (3.13) and then dividing both sides of its result by b^2 yields (3.12). □

Theorem 3.3. Let $\delta\ge 0$ and $p\in(0,\infty)\setminus\{2\}$. Assume that $\delta=0$ if $p>2$ and

$$\|(a^2+b^2-4)f(0)\| = 0 \text{ if } p>2.$$

If a function $f: X\to Y$ satisfies (3.1) for all $x, y\in X$ then there exists a unique quadratic function $Q: X\to Y$ such that

$$\begin{aligned}&\|f(x)-f(0)-Q(x)\| \\ &\le \left(\frac{1}{2}+\frac{1}{b^2}\right)\delta + \frac{1}{2}(\|(a^2+b^2-4)f(0)\|) \\ &\quad + \frac{1}{2(|4-2^p|)}h_1(x) + \frac{1}{b^2}h_2(x)\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\left\|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\right\| \\ &\le \frac{1}{2(|4-2^p|)}h_1(x) \\ &\quad + \frac{1}{2}(\delta + \|(a^2+b^2-4)f(0)\|)\end{aligned} \tag{3.15}$$

and

$$\left\|\frac{f(x)-f(-x)}{2}\right\| \le \frac{1}{b^2}\delta + \frac{1}{b^2}h_2(x) \tag{3.16}$$

for all $x\in X$, where

$$h_1(x) = 2H(\|x\|, \|x\|) + H(\|2x\|, 0)$$

and

$$h_2(x) = H\left(\left\|\frac{b}{2}x\right\|, \left\|\frac{b}{2}x\right\|\right).$$

Proof. Let $q_1(x) = \frac{f(x)+f(-x)}{2}$ for all $x\in X$. Then $q_1(0) = f(0)$, $q_1(-x) = q_1(x)$ and $\|Dq_1(x,y)\| \le \delta + H(\|x\|, \|y\|)$ for all $x, y\in X$. Let $q(x) = q_1(x) - q_1(0)$ for all $x\in X$. Then $q(0)=0$, $q(-x)=q(x)$ and

$$\| Dq(x, y) \| = \| Dq_1(x, y) - (a^2 + b^2 - 4) f(0) \|$$
$$\le \| Dq_1(x, y) \| + \| (a^2 + b^2 - 4) f(0) \|$$
$$\le \delta + \|(a^2 + b^2 - 4) f(0)\| + H(\|x\|, \|y\|)$$

for all $x, y \in X$.

By Theorem 3.1, there exists a unique quadratic function $Q: X \to Y$ satisfying (3.15).

Let $g(x) = \dfrac{f(x) - f(-x)}{2}$ for all $x \in X$.

Then $g(-x) = -g(x)$ and

$$\| Dg(x, y) \| \le \delta + H(\|x\|, \|y\|)$$

for all $x, y \in X$. By Theorem 3.2, we have (3.16). Clearly, we have (3.14) for all $x \in X$.

Define a mapping $H: R^+ \times R^+ \to R^+$ by

$$H(a, b) = \theta(a^p + b^p).$$

Then H is homogeneous of the degree p. Thus we have the following corollaries. □

Corollary 3.4. Assume that $\delta \ge 0$, $p \in (0, \infty) \setminus \{2\}$ and $\delta = 0$ when $p > 2$. Let an even function $f: X \to Y$ satisfy

$$\| Df(x, y) \| \le \delta + \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and $f(0) = 0$. Then there is a unique quadratic function $Q: X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2}\delta + \frac{4 + 2^p}{2(4 - 2^p)}\theta \|x\|^p$$

for all $x \in X$.

Corollary 3.5. Assume that $\delta \ge 0$. Let an odd function $f: X \to Y$ satisfy

$$\| Df(x, y) \| \le \delta + \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then

$$\| f(x) \| \le \frac{1}{b^2}\delta + \frac{b^{p-2}}{2^{p-1}}\theta \|x\|^p$$

for all $x \in X$.

Corollary 3.6. Let $\delta \ge 0$ and $p \in (0, \infty) \setminus \{2\}$.

Assume that $\delta = 0$ if $p > 2$ and

$$\|(a^2 + b^2 - 4) f(0)\| = 0 \text{ if } p > 2.$$

If a function $f: X \to Y$ satisfies

$$\|Df(x, y)\| \le \delta + \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then there exists a unique quadratic function $Q: X \to Y$ such that

$$\| f(x) - f(0) - Q(x) \|,$$
$$\le \left(\frac{1}{2} + \frac{1}{b^2}\right)\delta + \frac{1}{2}(\|(a^2 + b^2 - 4) f(0)\|)$$
$$+ \left(\frac{4 + 2^p}{2(|4 - 2^p|)} + \frac{b^{p-2}}{2^{p-1}}\right)\theta \|x\|^p$$

$$\left\| \frac{f(x) + f(-x)}{2} - f(0) - Q(x) \right\|$$
$$\le \frac{1}{2}(\delta + \|(a^2 + b^2 - 4) f(0)\|)$$
$$+ \frac{4 + 2^p}{2|4 - 2^p|}\theta \|x\|^p$$

and for all $x \in X$.

$$\left\| \frac{f(x) - f(-x)}{2} \right\| \le \frac{1}{b^2}\delta + \frac{b^{p-2}}{2^{p-1}}\theta \|x\|^p$$

IV. Conclusion

So far we investigated a generalized quadratic functional equation

$$a^2 f\left(\frac{x+y}{a}\right) + b^2 f\left(\frac{x-y}{b}\right) = 2f(x) + 2f(y)$$

where a and b are nonzero real constants..

We also solve a generalized quadratic functional equation and then prove the stability of a generalized quadratic functional equation.

This functional equation can be used for a variety of applications in some areas of the behavioral sciences such as sensory psychology(psychophysics), utility theory under uncertainty, and aggregation of inputs where proves the stability of this equation. and outputs in an economic or social context.

It is expected that more functional equations in the various forms such as exponential, logarithmic and multiplicative functional equations would be applicable soon.

REFERENCES

[1] D. H. Hyers, "On the stability of the linear functional equation," Proc. Nat. Acad. Sci. U. S. A. Vol.27, pp.222-224, 1941.

[2] D. H. Hyers and Th. M. Rassias, "Approximate homomorphisms," Aeq. Math. Vol.44, pp.125-153, 1992.

[3] S. H. Lee, "Stability of some generalized quadratic functional equation," J. Chungcheong Math. Soc. Vol.14, pp.1-6, 2001.

[4] S. M. Jung, "Hyers-Ulam-Rassias stability of Jensen's equation and its application," Proc. Amer. Math. Soc. Vol.126, pp.3137-3143, 1998.

[5] S. M. Jung, "Quadratic functional equation of Pexider type," Internat. J. Math. & Math. Sci. Vol.24, pp.351-359, 2000.

[6] Y. H. Lee and K. W. Jun, "A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation," J. Math. Anal. Appl. Vol.238, pp.305-315, 1999.

[7] J. C. Parnami and H. L. Vasudeva, "On Jensen's functional equation," Aeq. Math. Vol.43, pp.211-218, 1992.

[8] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," Proc. Amer. Math. Soc. Vol.72, pp.297-300, 1978.

[9] T. Trif, "Hyers-Ulam-Rassias stability of a Jensen type functional equation," J. Math. Anal. Appl. Vol.250, pp.579-588, 2000.

[10] S. M. Ulam, Problems in Mordern Mathematics, Chapter VI, Science Editions, Wiley, New York 1964.

[11] Y. W. Lee and S. H. Park, "On the Hyers-Ulam-Rassias stability of a quadratic functional equation," Korean J. Comput. & Appl. Math.(Series A) Vol.9, pp.371-380, 2002.

김 미 혜(Mi-Hye Kim) 정회원

2001년 2월 : 충북대학교 수학과 (이학박사)
2001년 4월 ~ 현재 : 충북대학교 전기전자 및 컴퓨터 공학부 초빙교수
<관심분야> : 퍼지 이론, 금융 수학

황 인 성(In-Sung Hwang) 정회원
2001년 3월 ~ 현재 : 충북대학교 수학과 대학원 재학