Fuzzy Measures Defined by the Semi-Normed Fuzzy Integrals

준 노름 퍼지 적분에 의해 정의된 퍼지 측도

김미혜
충북대학교 전기전자및컴퓨터공학부 초빙교수

이순석
충북대학교 컴퓨터정보통신연구소

Mi-Hye Kim
Invited Professor, School of Electrical & Computer Engineering, Chungbuk National Univ.
Soon-Seok Lee
RICIC Chungbuk National University

중심어 : Fuzzy measure, semi-normed fuzzy integral

요 약

본 논문에서는 t 준노름이 연속인 경우 이미 주어진 퍼지 측도에 관한 측정 가능한 함수의 준 노름 퍼지 적분을 이용하여 퍼지 측도를 정의하는 방법에 대해서 조사했다.

즉 $(X, \mathcal{F}, g)$이 퍼지 측도 공간이라고 하고 $h \in L^0(X)$, 이며 $\top$는 연속 t 준노름이라 하자. 그러면 임의의 $A \in \mathcal{F}$에 대해 $\nu(A) = \int_A h \top g$에 의하여 정의된 집합치 함수 ν는 $(X, \mathcal{F})$상에서 퍼지 측도이다.

Abstract

In this paper, we investigate for how to define a fuzzy measure by using the semi-normed fuzzy integral of a given measurable function with respect to another given fuzzy measure when t-seminorm is continuous.

Let $(X, \mathcal{F}, g)$ be a fuzzy measure space, $h \in L^0(X)$, and $\top$ be a continuous t-seminorm.

Then the set function ν defined by $\nu(A) = \int_A h \top g$ for any $A \in \mathcal{F}$ is a fuzzy measure on $(X, \mathcal{F})$.

I. Introduction

In recent years, artificial intelligence, behavioral science, and human engineering, etc. which originated in cybernetics have found many applications in all fields of engineering. Together with this tendency, a variety of problems on human subjectivity which was studied first mainly in psychology have become problems in engineering.

Problems on human beings have caught a general interest also in the field of systems engineering, where it is often pointed out that control systems shoud be regarded as essentially man-machine systems. One of the reasons for these facts would be that human abilities of judgement, analogy based on exprience,and adaptation to any unfamiliar environment,etc. have become again considered important compared with computers.

Concerning subjectivity among the characteristics of men which are superior to those of machines, L . A . Zadeh presented in 1965 the concept of fuzzy sets [16], which has given us a powerful means to deal with subjectivity by methods of mathematics as well as engineering.

Since his proposal, fuzzy sets theory have been widely applied in the fields of automata [2],[5],[7], linguistics [13],[16], algorithm [8], pattern recognition [15], and so on [12].

Sugeno [12] defined a fuzzy measure as a measure having the monotonicity instead of additivity. Since fuzzy measure does not satisfy countable additive, it gained recognition of its practical value. And a fuzzy integral which is an integral with respect to fuzzy measure is applied to make a synthetic evaluation about arbitrary objects. The concept of the seminormed fuzzy integral which is generalized fuzzy integral was proposed by Suarez and Gill [10],[11]. In this paper, we investigate some properties of semi-normed fuzzy integral, and for a continuous t-seminorm we discuss how to define a fuzzy measure by using the semi-normed fuzzy integral of a given measurable function with respect to another given fuzzy measure.

II. Preliminaries

In this section, we introduce some notions which will be used in this paper, and investigate elementary properties of Fuzzy measure and seminormed fuzzy integral.

Let X be a nonempty set, $\mathcal{F}$ be a σ-algebra of subsets of X, and $g : \mathcal{F} \to [0,1]$ be a set function.

A set function $g : \mathcal{F} \to [0,1]$ is called a **fuzzy measure** if

(1) $g(\emptyset) = 0$ (vanishing at $\emptyset$);

(2) $A \in \mathcal{F}, B \in \mathcal{F}$, and $A \subset B$ imply $g(A) \subset g(B)$ (monotonicity);

(3) For any $A_n \subset \mathcal{F}$, with $A_1 \subset A_2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ imply

$$\lim_{n\to\infty} g(A_n) = g(\bigcup_{n=1}^{\infty} A_n)$$

(continuity from below);

(4) For any $A_n \subset \mathcal{F}$, with $A_1 \supset A_2 \supset \cdots$, and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$ imply

$$\lim_{n\to\infty} g(A_n) = g(\bigcap_{n=1}^{\infty} A_n)$$

(continuity from above).

We call $(X, \mathcal{F}, g)$ a fuzzy measure space if g is a fuzzy measure on a measurable space $(X, \mathcal{F})$. The main difference between fuzzy measures and classical measures is the lack of additivity of the former. However each classical measure is a fuzzy measure. Since the fuzzy measure loose additivity in general, they appear much looser than the classical measures.

A real-valued function $h: X \to [0,1]$ is said to be **measurable** with respect to $\mathcal{F}$ and Ω (measurable, for short, if there is no confusion) if

$$h^{-1}(B) = \{x \mid h(x) \in B\} \in \mathcal{F} \text{ for any } B \in \Omega,$$

where Ω is the σ-algebra of Borel subsets of $[0,1]$. The definition of measurability of function is the same as in the theory of Lebesgue integrals.

From now on, let us consider the set $L^0(X) = \{h: X \to [0,1] \mid h$ is measurable with respect to $\mathcal{F}$ and $\Omega\}$. For any given $h \in L^0(X)$, we write $H_\alpha = \{x \mid h(x) \geq \alpha\}$, where $\alpha \in [0,1]$. Let $A \in \mathcal{F}$, $h \in L^0(X)$. The **fuzzy integral** of h with respect to g, which is denoted by $\int_A h\,dg$, is defined by

$$\int_A h\,dg = \sup_{\alpha\in[0,1]} [\alpha \wedge g(A \cap H_\alpha)].$$

When $A = X$, the fuzzy integral is denoted by $\int_A h\,dg$. Sometimes the fuzzy integral is also called Sugeno's integral in the literature.

A t**-seminorm** is a function $\top: [0,1]\times[0,1] \to [0,1]$ which satisfies ;

(1) For each $x \in [0, 1]$,

$$\top(x, 1) = \top(1, x) = x;$$

(2) For each $x_1, x_2, x_3, x_4 \in [0,1]$, if $x_1 \leq x_3$, $x_2 \leq x_4$, then $\top(x_1, x_2) \leq \top(x_3, x_4)$.

Example 2.1. The following functions are t - seminorms;

(1) $\top(x, y) = x \wedge y$

(2) $\top(x, y) = xy$

(3) $\top(x, y) = 0 \vee (x+y-1)$.

Let $\top$ be a t-seminorm. For all $h \in L^0(X)$ the **seminormed fuzzy integral** of h over $A \in \mathcal{F}$ with respect to the fuzzy measure g is defined as

$$\int_A h \top g = \sup_{\alpha\in[0,1]} \top[\alpha, g(A \cap H_\alpha)]$$

In what follows, $\int_X h\top g$ will be denote $\int h\top g$ for short. The seminormed fuzzy integral contains as a particular case the fuzzy integral of Sugeno with $\top(x, y) = x\wedge y$.

The following Theorem gives the most elementary properties of the semi-normed fuzzy integral.

Theorem 2.2. [12] Let $(X, \mathcal{F}, g)$ be a fuzzy measure space, $\top$ be a t-seminorm, and $h_1, h_2, h \in L^0(X)$. Then we have the following: for any $A \in \mathcal{F}$,

(1) If $g(A) = 0$, then $\int_A h\top g = 0$.

(2) If $h_1 \le h_2$, then $\int_A h_1\top g \le \int_A h_2 \top g$.

(3) $\int_A h\top g = \int (h\cdot\chi_A)\top g$, where χ_A is the characteristic function of A.

(4) $\int_A a\top g = \top(a\wedge 1, g(A))$ for any constant $a\in[0,\infty)$.

(5) $\int_A (a\vee h)\top g = \int_A a\top g \vee \int_A h\top g$ for any constant $a\in[0,\infty)$.

Proof. We only need to prove (3) and (5); the remaining properties can be obtained directly from the definition of the fuzzy integral.

For (3), Let $H_\alpha = \{x| h(x) \ge \alpha\}$ and

$$E_\alpha = \{x|(h\cdot\chi_A)(x) \ge \alpha\},$$

then it is easy to show that $A\cap H_\alpha = E_\alpha$.

Hence, we have

$$\int_A h\top g = \sup_{\alpha\in[0,1]}\top[\alpha, g(A\cap H_\alpha)]$$

$$= \sup_{\alpha\in[0,1]}\top[\alpha, g(E_\alpha)]$$

$$= \int (h\cdot\chi_A)\top g.$$

For (5), we have directly

$$\int_A (a\vee h)\top g \ge \int_A a\top g \vee \int_A h\top g.$$

Conversely,

$$H_\alpha^{a\vee h} = \{x|(a\vee h)\ge\alpha\} = \begin{cases} X & \text{if } a\ge\alpha \\ H_\alpha^h & \text{if } a<\alpha. \end{cases}$$

Hence

$$\int_A (a\vee h)\top g = \sup_{\alpha\in[0,1]}\top[\alpha, g(A\cap H_\alpha^{a\vee h})]$$

$$= \sup_{\alpha\in[0,a]}\top[\alpha, g(A)] \vee \sup_{\alpha\in(a,1]}\top[\alpha, g(A\cap H_\alpha^h)]$$

$$\le \sup_{\alpha\in[0,a]}\top[\alpha, g(A)] \vee \sup_{\alpha\in[0,1]}\top[\alpha, g(A\cap H_\alpha^h)]$$

$$= \int_A a\top g \vee \int_A h\top g,$$

which yields

$$\int_A (a\vee h)\top g \le \int_A a\top g \vee \int_A h\top g.$$

Hence the conclusion follows.

III. Fuzzy Measures Defined by the Semi - Normed Fuzzy Integrals

In this section, we discuss how to define a fuzzy measure by using the seminormed fuzzy integral of a given measurable function with respect to another given fuzzy measure, For each $\{h_n\}$ in $L^0(X)$ and $h\in L^0(X)$, we write $H_\alpha^n = \{x \mid h_n(x)\ge\alpha\}$, for $\alpha\in[0,1]$.

We also use the symbols " $\downarrow$ " (or " $\uparrow$ ", " $\to$ ") to denote "decreasingly converge to " and " increasingly converge to", "converge to", respectively) for both function sequences and number sequences.

Theorem 3.1. [11] Let $(X, \mathcal{F}, g)$ be a fuzzy measure space, $\top$ be a continuous t-seminorm and let

h and $h_1, h_2, \cdots$ be $\mathcal{F}$-measurable functions on X. Suppose that the relations

(1) $$h_1(x) \leq h_2(x) \leq \cdots$$

and

(2) $$h(x) = \lim_{n\to\infty} h_n(x)$$

hold. Then

$$\lim_{n\to\infty} \int_A h_n \top g = \int_A h \top g.$$

In a similar way, we can prove the following corollary.

Theorem 3.2. [11] Let $(X, \mathcal{F}, g)$ be a fuzzy measure space, $\top$ be a continuous t-seminorm and let h and $h_1, h_2, \cdots$ be $\mathcal{F}$-measurable functions on X. Suppose that the relations

(1) $$h_1(x) \geq h_2(x) \geq \cdots$$

and

(2) $$h(x) = \lim_{n\to\infty} h_n(x)$$

hold. Then

$$\lim_{n\to\infty} \int_A h_n \top g = \int_A h \top g.$$

The following example shows that the conclusion in Theorem 3.1 is not true when the continuous condition is omitted.

Example 3.3. Let $X = (0, 1]$, g be the Lebesgue measure and let for $a < 1$

$$h_n(x) = \begin{cases} a & \text{if } x > \frac{1}{n} \\ nax & \text{if } x \leq \frac{1}{n} \end{cases}$$

and

$$h(x) = a \text{ for all X.}$$

If we take

$$\top(x, y) = \begin{cases} x \wedge y & \text{if } x \vee y \geq 1 \\ 0 & \text{if } x \vee y < 1. \end{cases}$$

Then it is not too difficult to verify that $\top$ is not continuous and $h_n \uparrow h$. Since $a < 1$,

$$\begin{aligned} \int h_n \top g &= \sup_{\alpha\in[0,a]} \top[\alpha, (1 - \frac{\alpha}{n} a)] \\ &\quad \vee \sup_{\alpha\in(a,1]} \top[\alpha, (1 - \frac{\alpha}{n} a)] \\ &= 0 \vee 0 \\ &= 0. \end{aligned}$$

But

$$\int h \top g = \sup_{\alpha\in[0,a]} \top[\alpha, g(X)] = a.$$

Consequently we have

$$\lim_{n\to\infty} \int h_n \top g = 0 \neq a = \int h \top g.$$

Now we discuss how to define a fuzzy measure by using the fuzzy integral of a given measurable function with respect to another given fuzzy measure.

Theorem 3.4. Let $(X, \mathcal{F}, g)$ be a fuzzy measure space, $h \in L^0(X)$, and $\top$ be a continuous t-seminorm. Then the set function ν defined by $\nu(A) = \int_A h \top g$ for any $A \in \mathcal{F}$ is a fuzzy measure on $(X, \mathcal{F})$.

Proof . From Theorem 2.1, we know that $\nu(\phi) = 0$ and ν is monotone. So we only need to prove that ν is continuous. Let B_n be an increasing sequence of sets in $\mathcal{F}$ with $B_n \uparrow B \in \mathcal{F}$.

Then we have $h \cdot \chi_{B_n} \uparrow h \cdot \chi_B$. From the Theorem 3.1, and the continuity of $\top$, we have

$$\begin{aligned} \lim_{n\to\infty} \nu(B_n) &= \lim_{n\to\infty} \int_{B_n} h \top g \\ &= \lim_{n\to\infty} \int (h \cdot \chi_{B_n}) \top g \\ &= \int (h \cdot \chi_B) \top g \\ &= \int_B h \top g \\ &= \nu(B). \end{aligned}$$

Furthermore, for any given decreasing set sequence

$\{B_n\} \in \mathcal{F}$ with $B_n \downarrow B \in \mathcal{F}$, from $h \cdot \chi_{B_n} \downarrow h \cdot \chi_B$ and Theorem 3.2 and the continuity of $\top$ we have also $\lim_{n \to \infty} \nu(B_n) = \nu(B)$. That is, ν is continuous from above. Consequently, ν is a fuzzy measure. □

REFERENCES

[1] N. Batle and E. Trillas. "Entropy and fuzzy integral," J. Math. Anal. Appl, Vol.69, pp.469-474, 1979.

[2] W. G. Fee and K. S. Fu , "A formulations of Fuzzy Automata and its Application as a Model of learning Systems," IEEE transactions on SSC, Vol.1. SSC-5, No.3, pp.215-223, 1969.

[3] R. Kruse, "On the construction of fuzzy measures," Fuzzy Sets and Systems, Vol.8, pp.323-327, 1982.

[4] E. P. Klement and D. Ralescu, "Nonlinearity of the fuzzy integral," Fuzzy Sets and Systems, Vol.11, pp.309-31, 1983.

[5] M. Mizumoto, J. Toyoda, and K. Tanaka : "Some Considerations on Fuzzy Automata," Journal of Computers and Systems Science, Vol.3, No.4, pp.409-422, 1969

[6] D. Ralescu and G. Adams, "The Fuzzy integral," J. Math. Anal. App, Vol.75, pp.562-570, 1980.

[7] E. S. Santos, "General Formulation of Sequential Machines," Information and Control, Vol.12, pp.5-10, 1968.

[8] E. S. Santos, "Fuzzy Algorithms," Information and Control, Vol.17, pp.236-339, 1970.

[9] B. Schweizer and A. Sklar, "Associative functions and abstract semi-groups," Jour Pub. Math. Debrecen, Vol.10, pp.69-81, 1963.

[10] F. Suarez Garcia and P. Gil. Alvarez, "Two families of fuzzy integrals," Fuzzy Sets and Systems, Vol.18, pp.67 - 81, 1986.

[11] F. Suarez Garcia and P.Gil Alvarez, "Measures of fuzziness of fuzzy events," Fuzzy Sets and System, Vol.21, pp.147-157, 1987.

[12] M. Sugeno, "Theory of fuzzy integrals and its applications," Ph.D Dissertation Thesis, Tokyo Institute of Technology, 1974.

[13] S. Tamura and K. Tanaka, "Learning of Fuzzy languages," IEEE transactions on SMC, Vol.1. SMC-3, pp.98-102, 1973.

[14] Z. Wang, G. J. Klir, Fuzzy Measure Theory, Plenum Press, New York, 1992.

[15] L. A. Zadeh, "Fuzzy Sets," Information and Control, Vol.8, pp.338-353, 1965.

[16] L. A. Zadeh, "Quantitative Fuzzy sematics," Information Sciences, Vol.3, pp.159-176, 1971.

김 미 혜(Mi-Hye Kim) 정회원

2001년 2월 : 충북대학교 수학과 (이학박사)
2001년 4월 ~ 현재 : 충북대학교 전기전자 및 컴퓨터 공학부 초빙 교수
<관심분야> : 퍼지 이론, 금융 수학

이 순 석(Soon-Seok Lee) 정회원

1999년 2월 : 단국대학교 대학원 전자공학과(공학박사)
1989년 2월 ~ 1995년 3월 : 국방과학연구소 연구원
2000년 6월 ~ 2001년 3월 : 삼성전자 차장
2001년 4월 ~ 현재 : 충북대학교 전기전자및컴퓨터공학부 초빙교수
<관심분야> : PDP, ELD, LCD backlight unit. Driving circuit for flat panel display, Electrical-optic measurement for flat panel display.