표 준 분 해 시 스 템

하 인 중
서울대학교 공과대학 제어계측공학과

Standard Decomposed System

In Joong Ha
Dept. of Control and Instrumentation Eng., Seoul National Univ.

Abstract

Conditions for achieving noninteraction in nonlinear multivariable systems via the decomposition of state space are well established. The main contribution of this paper is to present a Standard Decomposed System (SDS). The SDS is similar to the decomposed system of Isidorl, Krener, Gori-Giorgi, and Monaco but has a finer structure. The finer structure parallels the one used by Gilbert for linear systems. A weaker form of noninteraction, based on input-output behaviour, is decoupling. Some connections between decomposition and decoupling are also established.

1. Introduction

Consider a nonlinear system of the form :

$$(1.1)\quad \dot{x}(t)=F(x,u)\overset{\Delta}{=}X_o(x)+\sum_{i=1}^{m} X_i(x)u_i,\quad y(t)=H(x),$$

where X_i, $i=1,\dots,m$ are vector fields on an n-dimensional manifold χ ; $H:\chi\to R^m$ is the output map of the system ; and $u_i(t)\in R$, $y_i(t)\in R$ are the ith components of $u(t)\in R^m$, $y(t)\in R^m$, respectively. Decomposition (which is called noninteracting control in [8]) concerns the dynamic structure of systems in state space. Roughly speaking, the system (1.1) is decomposed if in an appropriate system of coordinates, it appears as a system having m independent subsystems such that for the ith subsystem, the input and output are u_i, y_i, respectively. consider the following class of control laws :

$$(1.2)\quad u=K(x,\hat{u})\overset{\Delta}{=}\alpha(x)+\beta(x)\hat{u},$$

where $\alpha:\chi\to R^m$ and $\beta:\chi\to R^{m\times m}$. The system (1.1) is decomposable if there is a control law (1.2) such that the feedback system :

$$(1.3)\quad \dot{x}=F(x,\alpha(x)+\beta(x)\hat{u}),\quad y=H(x).$$

is decomposed. The corresponding control law is called a decomposing control law.

Conditions for (feedback) decomposition of nonlinear systems have been studied by many authors. See, e.g., [8-11,13,17]. The most common theoretical framework is some generalization of the geometric approach introduced by Wonham and Morse [12,22] for linear systems. Here we emphasize instead decomposition structure and describe a nonlinear system of a special structure, which we will call a Standard Decomposed System (SDS). We present conditions under which a nonlinear system can have the special structure of SDS. The SDS is similar to the decomposed system of Isidori, Krener, Gori-Giorgi, and Monaco [8] but has a finer structure. The structure parallels the one used by Gilbert [4] for linear systems. The SDS is useful for some engineering applications, which will be presented later.

A weaker form of noninteracting than decomposition is decoupling [2.3,5,9,14-16,18]. Essentially, the system (1.1) is decoupled if for each $i=1,\dots,m$, u_i affects only y_i. Note that decoupling concerns only the input-output map of systems, whereas decomposition concerns both the input-output map and the dynamic structure of systems in state space. Thus, decomposition may be more interesting from engineering viewpoint. As part of our presentation we will establish some connections between the two concepts of noninteraction.

The paper is organized as follows. In this section, we introduce notation and some basic differential geometric tools used in later sections. Precise definitions of decomposition and decomposability are given. In Section 2, necessary and sufficient conditions for a system to be decomposed and for a system to be decomposable are discussed. Section 3 contains the definition of the SDS and conditions under which nonlinear system can have the form of SDS. We also examine the relationship between the class of decomposing control laws and the class of the closed-loop decomposed systems. Section 4 contains concluding remarks.

Let $M_{i,j}$ denote the set of integers $\{i,i+1,\dots,j\}$. We denote by $\{F,H,\chi\}$ the abstact system (1.1) defined on an manifold . At each $p\in\chi$, there exists a chart or coordinate neighborhood (U,ϕ) such that in the coordinates $z=\phi(x)$, the system (1.1) is described by

$$(1.4)\quad \dot{z}(t)=f(z(t),u(t))\overset{\Delta}{=}f_o(z(t))+\sum_{j=1}^{m} f_i(z(t))u_i(t)$$
$$y(t)=h(z(t)),$$

where the functions $f_i:\phi(U)\to R^n$, $i\in M_{0,m}$ are determined by, respectively, the local representation of the vector fields X_i in the chart. We denote this local representation by $\{f,h,U\}$. we denote by h_i, H_i the ith components of h, H, respectively. To simplify our definitions and proofs, all systems and control laws considered in this paper are assumed to be at least smooth (C^∞). See [5] for the precise definitions of smoothness and real analyticity of systems and control laws. All control laws considered in this paper are assumed to be nonsingular ; $\beta(x)$ is nonsingular, $x \in \chi$.

Let T be a C^∞-mapping from an n-dimensional smooth manifold χ into an m-dimensional smooth manifold $\hat{\chi}$. A smooth function $\hat{\phi}$ from $\hat{\chi}$ into R is T-related on χ to a smooth function ϕ from χ into R if $\phi(p)=\hat{\phi}\circ T(p)$, $p \in \chi$, where $\circ$ denotes the function decomposition. For a smooth vector field Y on χ, Y_p denotes the tangent vector at $p \in \chi$ assigned by Y. A smooth vector field $\hat{Y}$ on $\hat{\chi}$ is T-related on χ to a smooth vector field Y on χ if $\hat{Y}_{T(p)}\hat{\phi}=Y_p(\hat{\phi}\circ T)$, $p \in \chi$ for all C^∞-functions $\hat{\phi}$ from $\hat{\chi}$ into R. The Lie bracket of two vector fields Y,Z on χ is denoted by $[Y,Z]\overset{\Delta}{=}YZ-ZY$.

The following definition concerns state transformations between systems.

<u>Definition 1.1.</u> Suppose for two systems $\{F,H,\chi\}$, $\{\hat{F},\hat{H},\hat{\chi}\}$, there exists a C^∞-diffeomorphism $T:\chi \to \hat{\chi}$ such that (i) $\hat{X}_i$ is T-related on χ to X_i, $i \in M_{0,m}$, and (ii) $\hat{H}_i$ is T-related on χ to H_i, $i \in M_{1,m}$. Then, $\{\hat{F},\hat{H},\hat{\chi}\}$ is T-related on χ to $\{F,H,\chi\}$. □

Definitions similar to Definition 1.1 are found in the prior literature including [20].

Next, we introduce a general relation between systems, which takes into account both state and input-feedback transformations. Let T, α, β be mappings from χ into $\hat{\chi}$, R^m, and $R^{m\times m}$, respectively, such that $\beta(x)$ is nonsingular, $x \in \chi$. Define a mapping $J:\chi \times R^m \to \hat{\chi}\times R^m$ by

$$(1.5)\quad J(x,u)\overset{\Delta}{=}\begin{pmatrix} T(x) \\ [\beta(x)]^{-1}(u-\alpha(x)) \end{pmatrix},\quad (x,u)\in\chi\times R^m.$$

We often write $J=\{T,\alpha,\beta\}$. Let $\{F,H,\chi\}^{\alpha,\beta}$ denote the feedback system of $\{F,H,\chi\}$ corresponding to a control law $u=\alpha(x)+\beta(x)\hat{u}$. In other words, $\{F,H,\chi\}^{\alpha,\beta}$ stands for the feedback system $\{\hat{F},H,\chi\}$, where $\hat{F}(x,u)\overset{\Delta}{=}F(x,\alpha(x)+\beta(x)\hat{u})$.

<u>Definition 1.2.</u> Suppose there exists a C^∞-diffeomorphism $J:\chi\times R^m \to \hat{\chi}\times R^m$ defined by (1.5) such that $\{\hat{F},\hat{H},\hat{\chi}\}$ is T-related on χ to the system $\{F,H,\chi\}^{\alpha,\beta}$. Then, $\{\hat{F},\hat{H},\hat{\chi}\}$ is J-feedback related on χ to $\{F,H,\chi\}$. □

Similar definitions appeared in the prior literature including [7,10].

Now, we define decomposition and decomposability. the definitions are similar to those in [8].

<u>Definition 1.3.</u> $\{F,H,\chi\}$ is decomposed at $x_o \in \chi$ if there exist : (a) an open neighborhood E of x_o ; (b) an open subset $\bar{\chi}$ of R^n ; (c) a C^∞-diffeomorphism $T:E \to \bar{\chi}$; (d) integers $\bar{s}_i \geq 1$, $i \in M_{1,m}$ and $\bar{s}_{m+1} \geq 0$ satisfying $n=\sum_{i=1}^{m+1}\bar{s}_i$; and (e) a system $\{\bar{F},\bar{H},\bar{\chi}\}$ which is T-related on E to $\{F,H,E\}$ such that its coordinate representation $\{\bar{f},\bar{h},\bar{\chi}\}$ has the form :

$$(1.6)\quad \dot{\bar{x}}_i=\bar{f}_i(\bar{x}_i)+\bar{g}_i(\bar{x}_i)\bar{u}_i,\quad \bar{y}_i=\bar{h}_i(\bar{x}_i),\quad i \in M_{1,m},$$

$$\dot{\bar{x}}_{m+1}=\bar{f}_{m+1}(\bar{x})+\sum_{j=1}^{m}\bar{b}_j(\bar{x})\bar{u}_j,$$

where $\bar{x}_i(t) \in R^{\bar{s}_i}$, $i \in M_{1,m+1}$, and $\bar{x}\overset{\Delta}{=}(\bar{x}_1,\ldots,\bar{x}_m,\bar{x}_{m+1})$. If E= χ in the above statement, $\{F,H,\chi\}$ is decomposed on χ. □

<u>Definition 1.4.</u> $\{F,H,\chi\}$ is decomposable at $x_o \in \chi$ if there exists a control law $u=\alpha(x)+\beta(x)\hat{u}$ such that $\{F,H,\chi\}^{\alpha,\beta}$ is decomposed at x_o. $\{F,H,\chi\}$ is decomposable on χ if there exists a control law $u=\alpha(x)+\beta(x)\hat{u}$ such that $\{F,H,\chi\}^{\alpha,\beta}$ is dcomposed on χ. □

Similar terminology arises in the precise definition of decoupling [5,9]. It should be clear from definitions of decomposition and decoupling that a decomposed system is always decoupled. As will be discussed later in Section 2, the converse statement is not necessarily true.

Let Z be a smooth vector field on χ. The codistribution $\Delta^\perp$ of a distribution Δ on χ is Z-invariant on χ if for any smooth function ϕ from χ into R, $d\phi \in \Delta^\perp$ on χ always implies $dZ\phi \in \Delta^\perp$ on χ. Let ψ_i, $i \in M_{1,k}$ be smooth functions from χ into R. The differentials $d\psi_i$, $i \in M_{1,k}$ are linearly independent on χ if $d\psi_i(p)$, $i \in M_{1,k}$ are linearly independent, $p \in \chi$. The first derivative of a smooth function $:R^n \to R$ at x is denoted by $D\psi(x) \in R^{1\times n}$.

For each $i \in M_{1,m}$, $\hat{M}_i$ denotes the set $\{j:j \in M_{1,m}, j\neq i\}$. For each $i \in M_{1,m}$, let

$$(1.7)\quad \overset{\circ}{\Delta}_i(\{F,H,\chi\}) \overset{\Delta}{=} [X_{i_1},[X_{i_2},[\ldots[X_{i_k},X_j]\ldots]]] \;:\; i_r \in \{0,i\},\ r \in M_{1,k},\ k \in M_{0,\infty},\ \text{and}\ j \in \hat{M}_i\},$$

where $[X_{i_1},[X_{i_2},[\ldots[X_{i_k},X_j]\ldots]]]\overset{\Delta}{=}X_j$ if k=0. Define $\Delta_i(\{F,H,\chi\})$ as the smallest subalgebra containing $\overset{\circ}{\Delta}_i(\{F,H,\chi\})$. Note that $\Delta_i^\perp$ is X_o-invariant and X_i-invariant on χ.

2. Conditions for Decomposition and Decomposability

To state our results a variety of assumptions beyond smoothness and the nonsingularity of control laws are needed. To simplify the presentation we list them together here.

(A.1) The system $\{F,H,\chi\}$ satisfies the controllability rank condition on χ ([19]).

(A.2) $\Delta_i^\perp(\{F,H,\chi\})$ has a dimension $p_i \geq 1$ on χ, $i \in M_{1,m}$,

(A.3) There exist nonnegative integers d_i, $i \in M_{1,m}$ such that the following m-row vector conditions are satisfied :

$$(2.1)\quad [X_1X_0^kH_i(x),\ldots,X_mX_0^kH_i(x)] = 0,\quad x \in \chi,\quad k \in M_{0,(d_i-1)},\ \text{applies when}\ d_i > 0,$$

$$(2.2)\quad D_i^*(x) \overset{\Delta}{=} [X_1X_0^{d_i}H_i(x),\ldots,X_mX_0^{d_i}H_i(x)] \neq 0,\quad x \in \chi.$$

The following Theorem concerns necessary and sufficient conditions for local decomposition.

<u>Theorem 2.1.</u> The system $\{F,H,\chi\}$ is decomposed at $x_o \in \chi$ if and only if there exist an open neighborhood E of x_o and m involutive distributions Δ_i^* on E which has dimension $r_i < n$ such that

on E.

(i) $dH_i \in (\Delta_i^*)^{\perp} \subset \Delta_i^{\perp}$, $i \in M_{1,m}$,

(ii) $(\Delta_i^*)^{\perp}$ is X_o-invariant and X_i-invariant, $i \in M_{1,m}$,

(iii) $(\Delta_i^*)^{\perp}$, $i \in M_{1,m}$ are mutually disjoint. □

Theorem 2.1 is implied by Theorem 5.1 in [8]. Note that it is not easy to check for the existence of Δ_i^*, $i \in M_{1,m}$ satisfying conditions specified in Theorem 2.1. This motivates the following result.

Theorem 2.2. Suppose that $\{F,H,\chi\}$ satisfies (A.1) and (A.2). Then, $\{F,H,\chi\}$ is decomposed at each $x_o \in \chi$ if and only if

$$(2.3)\quad dH_i \in \Delta_i(\{F,H,\chi\}) \text{ on } \chi,\ i \in M_{1,m}. \qquad □$$

Apart from giving an easily verified condition for local decomposition, this result has other important implications. In [2,5,9], it was shown that (2.3) is a necessary and sufficient condition for decoupling of real analytic systems. From this and Theorem 2.1, we see that the conditions for decomposition are more complex that those for decoupling. Moreover, for real analytic systems satisfying the hypotheses of Theorem 2.2, the concepts of decomposition and decoupling are (at least locally) equivalent. It appears that this observation has not been made before. Proofs of Theorem 2.2 were obtained independently by Ha [6] and Nijmeijer [15].

We now turn to the question of when a system is decomposable by a control law. When (A.3) is satisfied, let $D^*(x)$ and $A^*(x)$ denote, respectively the (m x m) and (m x 1) matrices of functions defined by

$$(2.4)\quad D^*(x) \triangleq \begin{bmatrix} D_1^*(x) \\ \cdots \\ D_m^*(x) \end{bmatrix}, \quad A^*(x) \triangleq \begin{bmatrix} X_0^{(d_1+1)} H_1(x) \\ \cdots \\ X_0^{(d_m+1)} H_m(x) \end{bmatrix}$$

Theorem 2.3. Suppose $\{F,H,\chi\}$ satisfies (A.3). then, $\{F,H,\chi\}$ is decomposable at each $x_o \in \chi$ if and only if

$$(2.5)\quad D^*(x) \text{ is nonsingular at each } x \in \chi.$$

Furthermore, $u=[D^*(x)]^{-1}(\hat{u}-A^*(x))$ decomposes $\{F, H,\chi\}$ at each $x \in \chi$. That is, for $\alpha(x) \triangleq -[D^*(x)]^{-1} A^*(x)$ and $\beta(x) \triangleq [D^*(x)]^{-1}$, the system $\{F,H,\chi\}^{\alpha,\beta}$ is decomposed at each $x \in \chi$. □

In [5,6,9,18], it was shown that (2.5) is a necessary and sufficient condition for a smooth system to be decouplable on χ. Hence, Theorem 2.3 has the important implication that under the assumption (A.3), decouplability and decomposability are locally equivalent.

the sufficiency of Theorem 2.3 was shown in [2,6,9]. We believe that the necessity of Theorem 2.3 is new. Since some details of the proofs are essential in the development of Section 3, we give the brief proofs.

Lemma 2.1. Suppose that $\{F,H,\chi\}$ satisfies (A.3). Let $\{\hat{F},\hat{H},\hat{\chi}\}$ be J-feedback related on χ to $\{F,H,\chi\}$ by $J=\{T,\alpha,\beta\}$. Let $\hat{X}_i$, $i \in M_{0,m}$ be vector fields corresponding to $\{\hat{F},\hat{H},\hat{\chi}\}$. Similarly, let $\hat{d}_i$, $\hat{D}^*$, $\hat{A}^*$ be determined by (2.1),(2.2), and (2.5) by replacing the X_i by the $\hat{X}_i$. Then :

(i) $\{\hat{F},\hat{H},\hat{\chi}\}$ satisfies (A.3) with $\hat{d}_i=d_i$, $i \in M_{1,m}$,

(ii) $\hat{D}^*(T(x)) = D^*(x)\,\beta(x)$, $\hat{A}^*(T(x)) = A^*(x) + D^*(x)\,\alpha(x)$, $x \in \chi$,

(iii) $\hat{X}_o^{k}\hat{H}_i(T(x)) = X_o^{k}H_i(x)$, $x \in \chi$, $k \in M_{0,di}$.

Lemma 2.2. Suppose that a system $\{F,H,\chi\}$ satisfies (A.3) and (2.5). Then, $dX_o^{k}H_i$, $k \in M_{o,di}$, $i \in M_{1,m}$ are linearly independent on χ. □

In [16] Lemma 2.1 was shown for the case when T is the identity map and $\chi=R^n$. Lemma 2.2 appeared in [6,9].

Proof of Theorem 2.3. suppose there exists α,β such that $\{F,H,\chi\}^{\alpha,\beta}$ is decomposed at each $x_o \in \chi$. Let $\{\hat{F},H,\chi\} \triangleq \{F,H,\chi\}^{\alpha,\beta}$. Then, by Theorem 2.1,

$$(2.6)\quad dH_i \in \Delta_1^{\perp}(\{\hat{F},H,\chi\}) \text{ on } \chi,\ i \in M_{1,m}.$$

By the definition of the Lie bracket and Δ_i, this implies

$$(2.7)\quad \hat{X}_j\hat{X}_o^{d_i} H_i(x) = 0, \text{ on } \chi,\ j \in M_i,\ i \in M_{1,m}.$$

By Lemma 2.1-(i) and the definition of $\{d_i,\ i \in M_{1,m}\}$, (2.7) implies

$$(2.8)\quad \lambda_i(x) \triangleq \hat{X}_i\hat{X}_o^{d_i} H_i(x) \neq 0,\ x \in \chi,\ i \in M_{1,m}.$$

On the other hand, by Lemma 2.1-(ii), (2.7), and (2.8),

$$(2.9)\quad D^*(x)\,\beta(x) = \operatorname{diag} \lambda_i(x),\ x \in \chi.$$

Then, (2.5) is a direct consequence of (2.8),(2.9), and the nonsingularity assumption of control laws.

Next, assume (2.5). Let $\{\hat{F},H,\chi\} \triangleq \{F,H,\chi\}^{\alpha,\beta}$ with $\alpha(x) \triangleq -[D^*(x)]^{-1}A^*(x)$ and $\beta(x) \triangleq [D^*(x)]^{-1}$. By Lemma 2.1, direct computation shows

$$(2.10)\quad \hat{X}_o^{k}\hat{H}_i(x) = X_o^{k}H_i(x),\quad k \in M_{0,d_i},$$

$$\hat{X}_o^{(d_i+1)}H_i(x) = 0,$$

$$(2.11)\quad \hat{X}_j\hat{X}_o^{k}H_i(x) = \begin{cases} 1 & \text{if } j = i \text{ and } k = d_i, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by Lemma 2.1-(iii) and Lemma 2.2, $d\hat{X}_o^{k}H_i$, $k \in M_{0,d_i}$, $i \in M_{1,m}$ are linearly independent on χ. Let $T_{i,j} \triangleq \hat{X}_o^{(j-1)}H_i$, $j \in M_{1,(d_i+1)}$, $i \in M_{1,m}$. Let $T_i \triangleq (T_{i,1},\ldots,T_{i,(d_i+1)})$, $i \in M_{1,m}$. Let $p \triangleq \sum_{i=1}^{m}(d_i+1)$ and $p_{m+1} \triangleq n-p$. Fix $x_o \in \chi$. Because $d\hat{X}_o^{k}H_i$, $k \in M_{o,d_i}$, $i \in M_{1,m}$ are linearly independent on χ, it is possible to choose a C^∞-mapping $T_{m+1}:\chi \to R^{p_{m+1}}$ such that $T \triangleq (T_1,\ldots,T_m,T_{m+1})$ has rank n at x_o. Then, it can be shown that there exist an open neighborhood E of x_o and a system $\{\bar{F},\bar{H},\bar{\chi}\}$ with $\bar{\chi}=T(E)$, $\bar{s}_i=d_i+1$, $i \in M_{1,m}$. $\bar{s}_{m+1}=p_{m+1}$ meet the requirement of Definition 1.3. In particular, its coordinate representation $\{\bar{f},\bar{h},\bar{\chi}\}$ has the form (1.6) such that for each $i \in M_{1,m}$,

$$(2.12) \quad \bar{f}_i(\bar{x}_i) = \bar{A}_i\bar{x}_i,\ \bar{g}_i(\bar{x}_i) = \bar{B}_i,\ \bar{h}_i(\bar{x}_i) = \bar{C}_i\bar{x}_i,$$

where

$$\bar{A}_i \triangleq \begin{bmatrix} 0 & I_{d_i} \\ \vdots & \\ 0 & \dots 0 \end{bmatrix},\ \bar{B}_i \triangleq \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix},\ \bar{C}_i \triangleq [1,0,\dots,0],$$

and I_d is the d by d identity matrix. Since $\{\hat{F}, H,\chi\}$ is decomposed at each $x_o \in \chi$, $\{F,H,\chi\}$ is decomposable at each $x_o \in \chi$ by the control law $u = [D^*(x)]^{-1}(\hat{u}-A^*(x))$.

Because of its importance in our subsequent developments, we henceforth reserve the notation $\{F^*,H,\chi\}$ for the system $\{F,H,\chi\}^{\alpha,\beta}$ with $\alpha(x) \triangleq -[D^*(x)]^{-1}A^*(x)$ and $\beta(x) = [D^*(x)]^{-1}$. It is well known [2,3,5,9,16,18] that $\{F^*,H,\chi\}$ is decoupled on χ. In other words, the control law $u=[D^*(x)]^{-1}(\hat{u}-A^*(x))$ is a "global" decoupling control law. In [2,8,9] and here, it is shown to be also a "local" decomposing control law. Note, however, that it is not necessarily a global decomposing control law.

3. Standard Decomposed System (SDS)

We begin this section by defining SDS which ha a more detailed structure for $\{\bar{f},\bar{h},\bar{\chi}\}$ than the one in (1.6) and (2.12).

Definition 3.1. Let $\bar{\chi}$ be an open connected subset of R^n. A system $\{\bar{F},\bar{H},\bar{\chi}\}$is a standard decomposed system (SDS) if its coordinate representation $\{\bar{f},\bar{h},\bar{\chi}\}$ has the following properties :

(1) There exist nonnegative integers $\bar{d}_i$, $i \in M_{1,m}$ and $\bar{p}_i$, $i \in M_{1,m+1}$, satisfying $n=\sum_{i=1}^{m+1}\bar{p}_i$, $\bar{p}_{m+1} \geq 0$, and $\bar{p}_i \geq \bar{d}_i + 1$, $i \in M_{1,m}$ so that $\{\bar{f},\bar{h},\bar{\chi}\}$ has a form :

$$(3.1) \quad \dot{\bar{x}}_i = \bar{f}_i(\bar{x}_i,\bar{u}_i) \triangleq \begin{bmatrix} \bar{A}_i\bar{x}_i \\ \bar{\theta}_i(\bar{x}_i) \end{bmatrix} + \begin{bmatrix} \bar{B}_i \\ \bar{\gamma}_i(\bar{x}_i) \end{bmatrix} \bar{u}_i,$$

$$\bar{y}_i=\bar{h}_i(\bar{x}_i)\triangleq\bar{C}_i\bar{x}_i,\quad i \in M_{1,m},$$

$$(3.2) \quad \dot{\bar{x}}_{m+1} = \bar{f}_{m+1}(\bar{x}) + \sum_{i=1}^{m}\bar{b}_i(\bar{x})\bar{u}_i,$$

where : $\bar{x}_i(t) \in R^{\bar{p}_i}$, $i \in M_{1,m+1}$; $\bar{x}\triangleq(\bar{x}_1,\dots,\bar{x}_{m+1}) \in R^n$; $\bar{A}_i,\bar{B}_i,\bar{C}_i$ are respectively $(\bar{d}_i+1)\times\bar{p}_i$, $(\bar{d}_i+1)\times 1$, $1\times\bar{p}_i$ matrices such that

$$\bar{A}_i \triangleq \begin{bmatrix} 0 & I_{di} & \vdots \\ \vdots & & \vdots\ \mathbf{0} \\ 0\dots 0 & & \vdots \end{bmatrix}\quad \bar{B}_i \triangleq \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix},\ \bar{C}_i \triangleq [1\ 0,\dots,0];$$

(2) Let $\bar{\chi}\triangleq\{\bar{x}_i : \bar{x}\triangleq(\bar{x}_1,\dots,\bar{x}_{m+1}) \in \bar{\chi}\}$. Each subsystems $\{\bar{f}_i,\bar{h}_i,\bar{\chi}_i\}$, $i \in M_{1,m}$, in (3.6) satisfies the controllability rank condition on $\bar{\chi}_i$,

(3) dim. $\Delta_i^{\perp}(\{\bar{F},\bar{H},\bar{\chi}\})=\bar{p}_i$ on $\bar{\chi}$, $i \in M_{1,m}$. □

Remark 3.1. The SDS in Definition 3.1 is a nonlinear version of the system introduced by Gilbert [4]. When $\{F,H,\chi\}$ is a linear system, it can be shown that property (3) is equivalent to condition (iv) in Definition 6 of [4]. It is worth noting that properties (2),(3) together imply the SDS $\{\bar{F},\bar{H},\bar{\chi}\}$satisfies (A.1). □

Now, we are ready to state the following result.

Theorem 3.1. suppose that a system $\{F,H,\chi\}$) satisfies (A.1), (A.3), and (2.5). Further, assume that $\{F^*,H,\chi\}$ satisfies (A.2). then, at each $x_o \in \chi$, there exist : (a) an open neighborhood E of x_o ; (b) an open connected subset $\bar{\chi}$ of R^n ; (c) a C^∞-diffeomorphism $T : E \to \bar{\chi}$; and (d) a system $\{\bar{F},\bar{H},\bar{\chi}\}$, which is T-related on E to $\{F^*, H,E\}$ and is a SDS with $\bar{d}_i=d_i$, $\bar{p}_i=p_i$, $i \in M_{1,m}$, and $\bar{p}_{m+1}=p_{m+1}\triangleq n-\sum_{i=1}^{m}p_i$, where the p_i and d_i appear in (A.2) and (A.3). □

For the proof of Theorem 3.1, we need the following Lemmas.

Lemma 3.1. Suppose that $\{\hat{F},\hat{H},\hat{\chi}\}$ is J-feedback related on χ to $\{F,H,\chi\}$ by $J=\{\alpha,\beta,T\}$. Then, if $\{F,H,\chi\}$ satisfies (A.1) on χ, $\{\hat{F},\hat{H},\hat{\chi}\}$ satisfies (A.1) on $T(\chi)$. □

Lemma 3.2. suppose that $\{F,H,\chi\}$ satisfies (A.1) and (A.2). Then, at each point $x_o \in \chi$, there exist $(\sum_{i=1}^{m}p_i)$ C^∞-functions $\xi_{i,j}$, $j \in M_{1,p_i}$, $i \in M_{1,m}$ from an open neighborhood υ of x_o into R such that

(i) $d\xi_{i,j}$, $j \in M_{1,p_i}$ $i \in M_{1,m}$ are linearly independent on υ,

(ii) $d\xi_{i,j} \in \Delta_i^{\perp}(\{F,H,\chi\})$ on υ, $j \in M_{1,p_i}$, $i \in M_{1,m}$. □

Lemma 3.1 seems to be well known. The proof of Lemma 3.2 is omitted because of limited space. It can be found in [6].

Proof of Theorem 3.1. Let $\{\hat{F},H,\chi\}\triangleq\{F^*,H,\chi\}$. By given hypotheses and Lemma 3.1, $\{\hat{F},H,\chi\}$ satisfies (A.1). Fix $x_o \in \chi$. Then, by Lemma 3.2, there exist an open neighborhood υ of x_o and $(\sum_{i=1}^{m}p_i)$ C^∞-functions $\phi_{i,j} : \upsilon \to R$, $j \in M_{1,pi}$, $i \in M_{1,pi}$, such that on υ,

(3.3) $d\phi_{i,j}$, $j \in M_{1,pi}$, $i \in M_{1,m}$ are linearly independent,

(3.4) $d\phi_{i,j} \in \Delta_i^{\perp}(\{\hat{F},H,\chi\})$, $j \in M_{1,pi}$, $i \in M_{1,m}$.

As was shown in the proof of Theorem 2.3, $\{\hat{F},H,\chi\}$ is decomposed at x_o. therefore by (i) of Theorem 2.1, there exists an open neighborhood $\hat{\upsilon} \subset \upsilon$ of x_o such that

(3.5) $dH_i \in \Delta_i^{\perp}(\{\hat{F},H,\chi\})$ on $\hat{\upsilon}$, $i \in M_{1,m}$.

Since $\Delta_i^{\perp}$ is $\hat{X}_o$-invariant on χ, this implies

(3.6) $d\hat{X}_o^{K}H_i \in \Delta_i(\{\hat{F},H,\chi\})$ on $\hat{\upsilon}$, $k \in M_{o,di}$, $i \in M_{1,m}$.

This, (3.6), and Lemma 2.2 show

(3.7) $p_i \geq d_i + 1, \ i \in M_{1,m}$.

Next, we show that there exists an open neighborhood $W \subset \hat{U}$ of x_o and a basis of $\Delta_i^{\perp}(\{\hat{F},H,\chi\})$ on W which contains $d\hat{X}_o{}^k H_i$, $k \in M_{0,d_i}$. By (3.3), (3.4), and (3.6), for each $i \in M_{1,m}$, there exist an open neighborhood $V_i \subset \hat{U}$ of x_o and C^∞-functions $\psi_{i,j}$ from an appropriate subset of R^{p_i} into R, $j \in M_{1,(d_i+1)}$ such that

(3.8) $T_{i,j}(x) \triangleq \hat{X}_o{}^{(i-1)}H_i(x) = \psi_{i,j}(\phi_{i,1}(x),\ldots,\phi_{i,p_i}(x))$, $x \in V_i$, $j \in M_{1,(d_i+1)}$.

By Lemma 2.1-(iii), Lemma 2.2, and (3.3), $D\psi_{i,j}(\phi_{i,1}(x_o),\ldots,\phi_{i,p_i}(x_o))$, $j \in M_{1,(d_i+1)}$ are linearly independent $(1 \times p_i)$ row vectors. Now, for each $i \in M_{1,m}$, let $r_i \triangleq p_i - d_i - 1$ and choose r_i $(1 \times p_i)$ row vectors $\eta_{i,j}$ such that $D\psi_{i,j}(\phi_{i,1}(x_o),\ldots,\phi_{i,p_i}(x_o))$, $j \in M_{1,(d_i+1)}$ and $\eta_{i,j}$, $j \in M_i$, are linearly indecpendent. Let

(3.9) $E_i \triangleq (\phi_{i,1},\ldots,\phi_{i,pi})$, $T_{i,(d_i+i+j)} \triangleq \eta_{i,j}E_i$, $j \in M_{1,ri}$, $i \in M_{1,m}$.

Then, by the construction of $T_{i,j}$, $j \in M_{i,pi}$, $i \in M_{1,m}$,

(3.10) $dT_{i,j}(x_o)$, $j \in M_{1,pi}$, $i \in M_{1,m}$ are linearly independent,

(3.11) $dT_{i,j}(x_o) \in \Delta_i^{\perp}(\{\hat{F},H,\chi\})$ on V_1, $j \in M_{1,pi}$, $i \in M_{1,m}$.

Let $V \triangleq V_1 \cap \ldots \cap V_m$ and $p_{m+1} \triangleq n - \sum_{i=1}^{m} p_i$. If $p_{m+1} \geq 1$, choose a C^∞-mapping T_{m+1} from V into $R^{p_{m+1}}$ such that T has rank n at x_o, where

(3.12) $T \triangleq (T_1,\ldots,T_m,T_{m+1})$,

$T_i \triangleq (T_{i,1},\ldots,T_{i,p_i})$, $i \in M_{1,m}$.

Then, by the local inverse function theorem, there exists an open neighborhood W V of x_o such that

(3.13) T is a C^∞-diffeomorphism on W,

(3.14) $\{dT_{i,j}(p),\ j \in M_{1,pi}\}$ is a basis of

Now, using (3.13) and (3.14), we show property (1) of Definition (3.1). Since $_i$ is X_o-invariant and X_i-ubvaruabt ib m (3,14)

(3.15) $d\hat{X}_oT_{i,j}$, $d\hat{X}_1T_{i,j} \in \Delta_i^{\perp}(\{\hat{F},H,\chi\})$ on W, $j \in M_{1,pi}$, $i \in M_{1,m}$.

Then, (3.14) and (3.15) imply that there exist an open connected neighborhood $E \subset W$ of x_o and C^∞-functions $\bar{\theta}_{i,j}$, $\gamma_{i,j}$ from appropriate subsets of R^{pi} into R, $j \in M_{1,ri}$, $i \in M_{1,m}$ such that

(3.16) $\hat{X}_oT_{i,(di+1+j)}(x) = \bar{\theta}_{i,j}(T_i(x))$, $\hat{X}_iT_{i(di+1+j)}(x) = \bar{\gamma}_{i,j}(T_i(x))$, $x \in E$

On the other hand, by (3.13), there exist C^∞-functions $\bar{f}_{m+1,j}$, $\bar{b}_{i,j}$, $i \in M_{1,m}$, $j \in M_{1,pm+1}$ defined on appropriate subsets of R^n such that

(3.17) $\hat{X}_oT_{m+1,j}(x) = \bar{f}_{m+1,j}(T(x))$, $\hat{X}_iT_{m+1}(x) = \bar{b}_{i,j}(T(X))$, $x \in E$.

Let $\bar{\chi} \triangleq T(E)$. Let $\bar{x} \triangleq (\bar{x}_1,\ldots,\bar{x}_m,\bar{x}_{m+1}) \triangleq (T_1(x),\ldots,T_{m+1}(x))$. Let $\bar{\theta}_i \triangleq (\bar{\theta}_{i,1},\ldots,\bar{\theta}_{i,ri})$, $\bar{\gamma}_i \triangleq (\bar{\gamma}_{i,1},\ldots,\bar{\gamma}_{i,ri})$, $i \in M_{1,m}$. Let $\bar{f}_{m+1} \triangleq (\bar{f}_{m+1,1},\ldots,\bar{f}_{m+1,pm+1})$, $\bar{b}_i \triangleq (\bar{b}_{i,1},\ldots,\bar{b}_{i,pm+1})$, $i \in M_{1,m}$. Define vector fields $\bar{X}_i$, $i \in M_{0,m}$ by

(3.18) $$\bar{X}_o(\bar{x}) \triangleq \sum_{i=1}^{m}\Big(\sum_{j=1}^{di}\bar{x}_{i,(i+1)}\,\partial/\partial\bar{x}_{i,j} + \sum_{j=di+2}^{pi}\bar{\theta}_{i,(j-di-1)}(\bar{x}_i)\,\partial/\partial\bar{x}_{i,j}\Big) + \sum_{j=1}^{pm+1}\bar{f}_{m+1,j}(\bar{x})\,\partial/\partial\bar{x}_{m+1,j}.$$

(3.19) $$\bar{X}_i(\bar{x}) \triangleq \partial/\partial\bar{x}_{i,(di+1)} + \sum_{j=d_i+2}^{p_i}\gamma_{i,(j-d_i-1)}(\bar{x}_i)\,\partial/\partial\bar{x}_{i,j}) + \sum_{j=1}^{p_{m+1}}b_{i,j}(\bar{x})\,\partial/\partial\bar{x}_{m+1,j},\ i \in M_{1,m}.$$

(3.20) $\bar{H}_i(\bar{x}) \triangleq \bar{x}_{i,1}$, $i \in M_{1,m}$.

where. $x_{i,j}$ is the jth component of $\bar{x}_i$. Let $\{\bar{F},\bar{H},\bar{\chi}\}$ be the system constructed as above. By (2.10),(2.11) and (3.16)-(3.20), the coordinate representation $\{\bar{f},\bar{h},\bar{\chi}\}$of $\{\bar{F},\bar{H},\bar{\chi}\}$ has the form indicated in (1) of Definition 3.1, where $\bar{d}_i = d_i$, $i \in M_{1,m}$ and $\bar{p}_i = p_i$, $i \in M_{1,m+1}$.

Let $\bar{Y}_i$ be a C^∞-vector field in $\Delta_i(\{\bar{F},\bar{H},\bar{\chi}\})$. Then, using (3.18) and (3.19), we can show that if $\bar{Y}_i \triangleq \sum_{j=1}^{m+1}\sum_{k=1}^{pi}\gamma_{j,k}(.)\,\partial/\partial\bar{x}_{j,k}$ is a local representation of $\bar{Y}_i$ on $\bar{\chi}$,

(3.21) $\gamma_{i,k}(\bar{x})=0$, $\bar{x} \in \bar{\chi}$, $k \in M_{1,pi}$

By Lemma 3.1, $\{\bar{F},\bar{H},\bar{\chi}\}$must satisfy (A.1). Thus, (3.26) implies property (2) of Definition 3.1. Property (3) follows from the fact that by (3.13), $(\Delta_i)_p(\{F,H,E\})$ and $(\Delta_i)_{T(p)}(\{\bar{F},\bar{H},\bar{\chi}\})$ are isomorphic at each $p \in E$. □

Next, we state a converse result.

Theorem 3.2. suppose that at each $x_o \in \chi$, there exist : (a) an open neighborhood E of x_o ; (b) an open connected subset $\bar{\chi}$ of R^n ; and (c) a SDS $\{\bar{F},\bar{H},\bar{\chi}\}$ which is J-feedback related on E to

$\{F,H,E\}$ by $J=\{T,\alpha,\beta\}$. Then, the following properties hold :

(i) $\{\bar{F},\bar{H},\bar{\chi}\}$ satisfies (2.5), (A.1), and (A.3) $d_i=\bar{d}_i$, $i \in M_{1,m}$,

(ii) $\{F^*,H,\chi\}$ satisfies (A.2) with $p_i=\bar{p}_i$, $i \in M_{1,m}$,

(iii) $\alpha(x)=-[D^*(x)]^{-1}A^*(x)$ and $\beta(x)=[D^*(x)]^{-1}$.

Proof. By Remark 3.1, $\{\bar{F},\bar{H},\bar{\chi}\}$ satisfies (A.1). By Lemma 3.1, this implies that $\{F,H,\chi\}$ satisfies (A.1). Direct computation shows that $\{\bar{F},\bar{H},\bar{\chi}\}$ satisfies (A.3) with $d_i=\bar{d}_i$, $i \in M_{1,m}$, $\bar{D}^*(x)=I_m$, and $\bar{A}^*(x)=0$. By this, Lemma 2.1, and the nonsingularity assumption of control laws, we see that $\{F,H,\chi\}$ satisfies (2.5), (A.3) with $d_i=\bar{d}_i$, $i \in M_{1,m}$, and, furthermore (iii). Since $\{\bar{F},\bar{H},\bar{\chi}\}$ is J-feedback related on E to $\{F,H,E\}$ by $J=\{T,\alpha,\beta\}$, (iii) implies that $\{\bar{F},\bar{H},\bar{\chi}\}$ is T-related on E to $\{F^*,H,\chi\}$. Consequently, $(\Delta_i)_q(\{F^*,H,\chi\})$ and $(\Delta_i)_{T(q)}(\{\bar{F},\bar{H},\bar{\chi}\})$ are isomorphic at each $q \in E$. This implies (ii). □

5. CONCLUSION

In this paper, we have characterized a class of nonlinear systems which can be transformed intc a SDS by the J-feedback relation. The two concepts, decoupling and decomposition are not globally equivalent for nonlinear systems. Some conditions under which the two concepts are at least locally equivalent are found.

The results in this paper can be extended to more general class of nonlinear systems : (i) systems in which F(x,u) is not affine in u and/or (ii) systems in which the $y_i(t)$ are vectors instead of scalars. Decoupling conditions for the above more general cases, obtained by Flies [3] and Nijmeijer [15], are useful for the extension because under their assumptions, the two concepts of decoupling and decomposition are locally equivalent.

REFERENCES

[1] W.M.Boothby, An Introduction to Differential Manifolds and Riemanian Geometry, New York : Academic Press, 1975.

[2] D.Claude, Decouplage des systemes nonlineaires, series generatrices non commutatives et algebres de Lie, this journal to appear.

[3] M.Fliess, A new approach to the noninteracting control problem in nonlinear systems theory, Proc. Allerton Conf. Commun. Contr. Comput, 1985, pp. 123-129.

[4] E.G.Gilbert, The decoupling of multivariable systems by state variable feedback, this journal, 7(1969), pp. 50-64.

[5] I.J.Ha and E.G.Gilbert, A complete characterization of decoupling control laws for a general class of nonlinear systems, presented in Conf. Infor. Scien. Sys., Johns Hopkins Univ., 1985. To appear in IEEE Trans. Automat. Control , 31(1986).

[6] I.J.Ha, Nonlinear decoupling theory with applications to robotics, Ph.D. Dissertation, CRIM Report No. RSD-TR-8-85, Univ. of Michigan, Ann Arbor, 1985.

[7] L.R.Hunt, H.Luksic, and R.Su, Exact linearization of input-output system, Int. J. Control, 43(1986), pp. 247-255.

[8] A Isidori, A.J.Krener, C.Gori-Giorgi, and S. Monaco., Nonlinear decoupling via feedback : a differential geometric approach, IEEE Trans. Automat. Control, 26(1871), pp. 331-345.

[9] A Isidori, Nonlinear Control Systems : An Introduction, Lecture Notes in Control and Information Sciences 72, Springer-Veriag, 1985.

[10] A Isidori and A.J.Krener, On Feedback equivalence of nonlinear systems, System. Control Lett., 2(1982), pp. 118-121.

[11] S.H.Mikhail and W.H.Wonham, Local decomposability and the disturbance decoupling in nonlinear autonomous systems, Proc. Allerton Conf. Commun. Contr. Comput, 1978, pp. 664-669.

[12] A.S.Morse and W.M.Wonham, Status of noninteracting control, IEEE Trans. Automat. Control, 16(1971), pp. 568-581.

[13] H.Nijmeijer, Feedback decomposition of nonlinear control systems, IEEE Trans. Automat. Control, 28(1983), pp. 861-862.

[14] H.Nijmeijer, Zeros at infinity for affine nonlinear control systems, IEEE Trans. Automat. Control, 30(1985), pp. 566-573.

[15] H.Nijmeijer, " The regular local noninteracting control problem for nonlinear system," this journal, to appear.

[16] W.A.Porter, Diagonalization and inverses for nonlinear systems, Int. J. Control, 10(1970), pp. 252-264.

[17] W.Respondek, On decomposition of nonlinear control systems, System Control Lett, 1(1982), pp. 301-308.

[18] S.N. Singh and W.J.Rugh, Decoupling in a class of nonlinear systems by state variable feedback, ASME J.Dynamic System. Measurement. Control, 94(1972), pp. 323-329.

[19] H.J.Sussmann and V.Jurdjevic, Controllability of nonlinear systems, J. Differential Equations, 12(1972), pp. 95-116.

[20] H.J.Sussmann, Existence and uniqueness of minimal realizations of nonlinear systems, Math. System Theory, 10(1977), pp. 263-284.

[21] F.W.Warner, Foundations of Differentiable Manifolds and Lie Groups, Scott, Foresman and Co., Glenview IL, 1971.

[22] W.M.Wonham, Linear Multivariable Control : a Geometric Approach, 2nd ed., Springer-Verlag, New York, 1979.