# Hilbert Space에서 대수 Riccati 방정식으로 얻어지는 교란된 Co-Semigroup의 상한에 대한 연구

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# A Study on Upper Bounds of the Perturbed Co-Semigroups via the Algebraic Riccati Equation in Hilbert Space

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Abstract

Upper bounds of the perturbed Co-semigroups of the infinite dimensional systems are investigated by using the algebraic Riccati equation (ARE). In the case that the solution P of the ARE is strictly positive, the pertrubed semigroups are uniformly bounded. A sufficient condition for the solution P to be strictly positive is provided. The uniform boundedness plays an important role in extending approximately weak stability to weak stability on the whole space. Exponential Stability of the perturbed semigroups is studied by using the Young's inequlity. Some further discussions on the uniform boundedness of the perturbed semigroups are given.

## 1. Introduction

Let H be a complex Hilbert space with an inner product $[\cdot, \cdot]$ and a norm $\|\cdot\|$ and let (A, B) denote the abstract differential equation :

$$\dot{x} = Ax + Bu,$$

where A is taken to be the infinitesimal generator of a Co -- strongly continuous at the origin -- semigroup $[T(t),\ t \geq 0]$ of bounded linear operators over H, and B is bounded linear from another Hilbert space U to H.

The pair (A, B) is said to be "stable" if for each x in H : $T(t)x \to 0,\ t \to \infty$ -- in a prescribed sense. If (A, B) is not stable and if there exists a bounded linear state feedback operator $F : H \to U$, $u = Fx$, such that $A + BF$ generates a stable semigroup $[S(t),\ t \geq 0]$, say, then (A, B) is said to be "state feedback stabilizable".

In this paper we consider upper bounds and stability of a Co semigroup $[S(t),\ t \geq 0]$ whose generator is $A - BB^* P$, where P is bounded, linear, self-adjoint and nonnegative, $P \geq 0$. P satisfies the algebraic Riccati Equation (ARE) :

$$[Px, Ax] + [Ax, Px] + [Rx, x] - [PBB^* Px, x] = 0$$

for x in the domain $\mathcal{D}(A)$ of A. Here R is also linear bounded, self-adjoint and nonnegative, $R \geq 0$. Throughout this paper the existence of the bounded solution of the ARE is always assumed.

Stabilizability of infinite dimensional linear dynamic systems has been studied by many researchers [1], [2], [3]. In infinite dimensional systems the relationships between controllability, observability and stabilizability are much more complicated. We cannot have exact controllability when the semigroup $[T(t),\ t \geq 0]$ or the operator B is compact [4]. Also exponential stability cannot be achieved by a compact feedback in the case of a linear oscillatory system [5].

Since exponential stability is somewhat restricted in many applications, the notions of weak and strong stability have been investigated by several authors [6], [7]. A concept of approximate stability is introduced to explain the behavior of the semigroup pertrubed by a feedback operator $-B^* P$ via the ARE when the solution of the ARE is just positive [8]. If the pertrubed semigroup is approximately stable and uniformly bound, the approximate stability can be extended to the whole space H. Considering the upper bounds of the pertrubed semigroup, we can investigate its stability.

## 2. Basic Concepts of the Infinite Dimensional Systems

We begin with basic concepts of the infinite dimensional systems. A Co semigroup $[T(t),\ t \geq 0]$ is said to be exponentially stable if there exist constants $M \geq 1$, $\alpha \geq 0$ such that $\|T(t)\| \leq M e^{-\alpha t}$, $t \geq 0$. Also if $\lim_{t \to \infty} \|T(t)x\| = 0$, for each x in H, then $[T(t),\ t \geq 0]$ is said to be strongly stable. Suppose $[T(t),\ t \geq 0]$ satisfies the following : $\lim_{t \to \infty} [T(t)x, y] = 0$, for x, y in H, then it is said to be weakly stable. It is easy to note that the following relationships hold :

exponential stability ⇒ strong stability ⇒
weak stability ⇒ uniform boundedness.

If the pertrubed semigroup $\{S(t),\ t \geqq 0\}$ is stable on a dense subset of H, then it is said to be approximatdly stable.

The notion of the controllability also can be extended to the non-finite demensions. The system (A, B) is said to be exactly controllable, if there exists $t > 0$ and $\alpha > 0$ such that

$$\int_0^t \| B^* T^*(\sigma)x \|^2 d\sigma \geqq \alpha \| x \|^2 \text{, for x in H.}$$

The system (A, B) is exactly controllable, if and only if the reachable set $\bigcup_{t\geq 0} K(t)$ is the whole space, where $\bigcup_{t\geq 0} K(t)$ is the range space of the solution of the pair (A, B), i.e.

$$x(t) = L(t)U = \int_0^t T(t-\sigma)Bu(\sigma)d\sigma,\ K(t) = \mathcal{R}(L(t)).$$

Now if the reachable set is dense in H, then we say the system (A, B) is approximately controllable. It is easy to prove that a necessary and sufficient condition for the system (A, B) to be approximately controllable is that for any $t > 0$, $\int_0^t \| B^* T^*(\sigma)x \| d\sigma = 0$ implies that $x = \theta$, i.e. $\bigcap_{t\geq 0} \text{Ker. } B^* T^*(t) = \{\theta\}$.

## 3. Lower Bounds of the Solution P of the ARE

We now show that the semigroup $\{S(t),\ t \geqq 0\}$ generated from the ARE have the following property :

$$[PS(t)x,\ S(t)x] \leqq [Px,\ x],\quad t \geqq 0.$$

From the ARE, we have

$$2\ \text{Re}\ [P(A-BB^*P)x,x] = -[Rx,\ x] - \| B^* Px \|^2 \leqq 0.$$

Since $BB^* P$ is bounded linear, $A - BB^*P$ generates a Co semigroup $\{S(t),\ t \geqq 0\}$ on H :

$$[PS(t)x,\ S(t)x] - [Px,\ x] = -\int_0^t \{[RS(\sigma)x,\ S(\sigma)x] + \| B^* PS(\sigma)x \|^2\}d\sigma,\quad t \geqq 0.$$

We thus have $[PS(t)x,\ S(t)x] \leqq [Px,\ x]$ on H.

Suppose that P is strictly positive, $P \geqq \alpha I$, $\alpha > 0$, then $\alpha \| S(t)x \|^2 \leqq [PS(t)x,\ S(t)x] \leqq [Px,\ x] \leqq k \| x \|^2$, for some k and x in H. Hence $\| S(t)x \| \leqq \sqrt{k/\alpha} \| x \|$ for each x in H. $\{S(t),\ t \geqq 0\}$ is uniformly bounded. This leads us to finding conditions for P to be strictly positive.

Fact I

If there exists $\alpha > 0$ such that for some $t \geqq 0$ and for x in H

$$\int_0^t \{[RS(\sigma)x,\ S(\sigma)x] + \| B^* PS(\sigma)x \|^2\}d\sigma \geqq \alpha \| x \|^2$$

then $[Px,\ x] \geqq \alpha \| x \|^2$.

Since $[PS(t)x,\ S(t)x] - [Px,x] = -\int_0^t \{[RS(\sigma)x,\ S(\sigma)x] + \| B^* PS(\sigma)x \|\}\ d\sigma \leqq -\alpha \| x \|^2$, for x in H, $t \geqq 0$. It follows that

$$-[Px,\ x] \leqq [PS(t)x,\ S(t)x] - [Px,\ x] \leqq -\alpha \| x \|^2$$

for x in H. Hence $[Px,\ x] \geqq \alpha \| x \|^2$, $\alpha > 0$, for x in H. If P is strictly positive, then $[Px,\ x] \triangleq \| x \|_P$ defines a new norm.

It is natural to ask whether exact controllability of $(A^*,\ R)$ implies strict positivity of P. This is not generally the case. However we can prove :

Theorem I.

If a Co semigroup $\{T(t)x,\ t \geqq 0\}$ with a generator A is exponentially stable and the pair $(A^*,\ R)$ is exactly controllable, then the bounded linear, self-adjoint and nonnegative solution of the ARE is strictly positive.

Proof

Since $\{T(t),\ t \geqq 0\}$ is exponentially stable, there exists a bounded linear, self-adjoint and nonnegative solution P of the ARE. Suppose that the pair $(A^*,\ R)$ is exactly controllable and there exists an arbitrary small $\delta$ such that for some x in H,

$$\int_0^\epsilon \{[RS(t)x,\ S(t)x] + \| B^* PS(t)x \|^2\}\ dt < \delta \text{ for all } \epsilon > 0 \tag{3.1}$$

Then

$$\int_0^\epsilon [RS(t)x,\ S(t)x]dt < \delta \tag{3.2}$$

and

$$\int_0^\epsilon \| B^* PS(t)x \|^2\ dt < \delta \tag{3.3}$$

for all $\epsilon$ and for some $x$ in $H$.

Since $S(t)x = T(t)x - \int_0^t T(t-\sigma)BB^*PS(\sigma)x\,d\sigma$,

$t \geqq 0$,

let $W(t, x) = \int_0^t T(t-\sigma)BB^*PS(\sigma)x\,d\sigma$, then from (3.2)

we have

$$\int_0^\epsilon \{[RT(t)x, T(t)x] + [RW(t,x), W(t, x)]\}dt \leqq \delta +$$

$$\int_0^\epsilon \{[RT(t)x, W(t, x)] + [RW(t, x), T(t)x]\}dt \qquad (3.4)$$

Now, for some $x$ in $H$

$$\int_0^\epsilon [RT(t)x, W(t, x)]dt$$

$$= \int_0^\epsilon [RT(t)x, \int_0^t T(t-\sigma)BB^*PS(\sigma)x\,d\sigma]dt$$

$$= \int_0^\epsilon\int_0^t [B^*T^*(t-\sigma)RT(t)x, B^*PS(\sigma)x]d\sigma\,dt$$

$$\leqq \int_0^\epsilon \{\int_0^t \|B^*T^*(t-\sigma)RT(t)x\|^2\,d\sigma$$

$$\int_0^t \|B^*PS(\sigma)x\|^2\,d\sigma\}^{1/2}dt. \qquad (3.5)$$

From (3.3), $\int_0^t \|B^*PS(\sigma)x\|^2 d\sigma < \delta,\ 0 \leqq t \leqq \epsilon$.

Hence we have from (3.4) and (3.5)

$$\int_0^\epsilon [RT(t)x, T(t)x]dt < \delta + 2\sqrt{\delta}\int_0^\epsilon [\int_0^t$$

$$\|B^*T^*(t-\sigma)RT(t)x\|^2\,d\sigma]^{1/2}\,dt. \qquad (3.6)$$

Since $\{T(t), t \geqq 0\}$ is exponentially stable, there exist $M \geqq 1$ and $\alpha$ such that

$$\|T(t)\| < Me^{-\alpha t},\ t \geqq 0.$$

Thus

$$\int_0^\epsilon [\int_0^t \|B^*T^*(t-\sigma)RT(t)x\|^2\,d\sigma]^{1/2}\,dt$$

$$\leqq \int_0^\infty [\int_0^t \|B^*T^*(t-\sigma)RT(t)x\|^2\,d\sigma]^{1/2}$$

dt

$$\leqq K\int_0^\infty [\int_\sigma^t e^{-2\alpha(2t-\sigma)}\|x\|^2\,d\sigma]^{1/2}\,dt$$

$$= K\int_0^\infty \frac{1}{\sqrt{2\alpha}}e^{-\alpha t}[1 - e^{-2\alpha t}]\ \|x\|dt,$$

where $K = M^2\|B^*\|\ \|R\|$.

Now $\int_0^\infty K\frac{1}{\sqrt{2\alpha}}e^{-\alpha t}[1 - e^{-2\alpha t}]^{1/2}\|x\|dt = \frac{k\pi}{4\sqrt{2}\,\alpha^{3/2}}\|x\|$

$\triangleq$ N, hence, from (3.6), for some $x$ in $H$,

$$\int_0^\epsilon [RT(t)x, T(t)x]dt < \delta + 2\sqrt{\delta}N, \text{ for all } \epsilon > 0.$$

This contradicts the assumption of $(A^*, R)$ - exact controllability. Therefore there exists $\gamma > 0$ such that for some $\epsilon$.

$$\int_0^\epsilon \{[RS(t)x, S(t)x] + \|B^*PS(t)x\|^2\}dt \geqq \gamma\|x\|^2,$$

for each $x$ in $H$. Consequently by Fact I $P$ is strictly positive. This completes the proof.

Now we recall Lyapunov's theorem : A semigroup $\{T(t), t \geqq 0\}$ with a generator $A$ is exponentially stable if and only if there exists $P \geqq 0$ such that

$$2\,\mathrm{Re}[PAx, x] = -\|Wx\|^2, \text{ for } x \text{ in } \mathcal{D}(A),$$

where $W \geqq \omega I,\ \omega > 0$. Therefore

$$[PT(t)x, T(t)x] - [Px, x] = -\int_0^t \|WT(\sigma)x\|^2\,d\sigma,$$

for $x$ in $H$.

Since $[Px, x] = \int_0^\infty \|WT(\sigma)x\|^2\,d\sigma$, if for $\epsilon > 0$,

$$\int_0^\infty \|WT(t)x\|^2\,dt \geqq \alpha\|x\|,\ \alpha > 0, \text{ for } x \text{ in } H, \qquad (3.7)$$

then $P \geqq \alpha I,\ \alpha > 0$. Hence $[Px, x]$ defines an equivalent norm on $H$. It is evident that if we set $W = R^{1/2}$, $R = R^*$,

then (3.7) implies that for $\epsilon > 0$, $\int_0^\epsilon [RT(t)x, T(t)x]dt \geqq \alpha\|x\|^2,\ \alpha > 0$, for $x$ in $H$. If thus follows that the pair $(A^*, R)$ is exactly controllable. This is the special case of theorem I in which $B = 0$

## 4. Upper Bounds of the Perturbed Semigroups.

The lower bound of the solution $P$ of the ARE : $P \geqq \alpha I,\ \alpha > 0$ in the previous section implies uniform boundedness of the perturbed semigroup $\{S(t), t \geqq 0\}$. We now study upper bounds for the pertrubed semigroup $\{S(t), t \geqq 0\}$, since uniform boundedness plays an important role in extending stability of $\{S(t), t \geqq 0\}$ to the whole space.

We consider, for $x$ in $H$,

$$S(t)x = T(t)x - \int_0^t T(t-\sigma)BB^*PS(\sigma)x\,d\sigma,\ t \geqq 0.$$

It then follows that

$$\|S(t)x\| \leqq \|T(t)x\| + \int_0^t \|T(t-\sigma)\|\ \|BB^*P\|$$

$\|S(\sigma)x\|\,d\sigma$.

Let $\|T(t)\| \leqq Me^{\omega_0 t}\|x\|$, $M \geqq 1$, $\omega_0 > 0$, $t \geqq 0$, then

for $x$ in $H$,

$$\|S(t)x\| \leqq Me^{\omega_0 t}\|x\| + \int_0^t Me^{\omega_0(t-\sigma)}\|BB^*P\|\ \|S(\sigma)\|\,d\sigma$$

By dividing $e^{\omega_0 t}$ on both sides, we have, for x in H,

$$e^{-\omega_0 t}\| S(t)x \| \leq M\| x \| + M\| BB^* P \| \int_0^t e^{-\omega_0 \sigma}$$

$$\| S(\sigma)x \| d\sigma$$

Now set $e^{-\omega_0 t}\| s(t)x \| = g(t)$, then

$$g(t) \leq M\| x \| + M\| BB^* P \| \int_0^t g(\sigma) d\sigma,$$

by Gronwall's Inequality, we have

$$g(t) \leq Me^{M\| BB^* P \| t}\| x \|, \; t \geq 0, \quad \text{for } x \text{ in } H.$$

Therefore

$$\| S(t)x \| \leq Me^{(\omega_0 + M\| BB^* P\|)t}\| x \|, \; t \geq 0, \; x \text{ in } H. \tag{4.1}$$

From (4.1) we cannot get any new information about on upper bound of {S(t), t ≧ 0}, since the upper bound of {S(t), t ≧ 0} is greater than that of {T(t), t≧ 0}. Even if {T(t), t ≧ 0} is exponentially stable, we cannot tell whether {S(t), t ≧ 0} is exponentially stable or not by (4.1).

Now using Young's Inequality [9], we prove that if {T(t), t ≧} is exponentially stable, then so is {S(t), t≧ 0}.

Consider : for x in H, t ≧ 0,

$$\| S(t)x \| \leq \| T(t)x \| + \| B \| \int_0^\infty \| T(t-\sigma) \|$$

$$\| B^* PS(\sigma)x \| d\sigma. \tag{4.2}$$

Since {T(t), t≧ 0} is exponentially stable and the solution P of the ARE exists, we have

$$\int_0^\infty \| T(t) \| dt < \infty$$

$$\int_0^\infty \| T(t)x \|^2 dt < \infty$$

and $\int_0^\infty \| B^* PS(t)x \|^2 dt < \infty$ , for x in H.

By Young's Inequality, we have from (4.2)

$$\left(\int_0^\infty \| S(t)x \|^2 dt\right)^{1/2} \leq \left(\int_0^\infty \| T(t)x \|^2 dt\right)^{1/2} + \| B \| \int_0^\infty$$

$$\| T(t) \| dt \quad \left(\int_0^\infty \| B^* PS(\sigma)x \|^2 d\sigma\right)^{1/2} < \infty$$

Hence {S(t), t≧ 0} is exponentially stable [3].

From (4.2) if {T(t), t ≧ 0}, is uniformly bounded, $\| T(t) \| \leq M$, t ≧ 0, then

$$\| S(t)x \| \leq M\| x \| + \int_0^t M\| B \| \| B^* PS(\sigma)x \| d\sigma,$$

$$t \geq 0, \quad \text{for } x \text{ in } H. \tag{4.3}$$

Since the exisitenc of P of the ARE implies that

$$\int_0^t \| B^* PS(\sigma)x \|^2 d\sigma < \infty, \; t \geq 0, \text{ for } x \text{ in } H,$$

$Uop(t) = -B^* PS(t)x$ belongs to $L_2$ {0, ∞ }. Hence Uop(t) does not necessarily belong to $L_1$ {0, ∞ }. If Uop belongs to $L_1$ {0, ∞ }, then

$$\int_0^\infty \| B^* PS(\sigma)x \| d\sigma < \infty, \text{ for } x \text{ in } H.$$

Therefore from (4.3), we have, for some N > 0

$$\| S(t)x \| \leq N\| x \|, \; t\geq 0, \text{ for } x \text{ in } H.$$

If $U = L_1 \{0, \infty\} \cap L_2 \{0, \infty\}$, then the semigroup {S(t), t ≧ 0} is uniformly bounded as soon as {T(t), t≧ 0} is.

## 5. Conclusions

Upper bounds of the perturbed semigroup {S(t), t ≧ 0} generated from the ARE are studied. If the bounded, linear, self-adjoint and nonnegative solution P is strictly positive, then the perturbed semigroup is uniformly bounded. By uniform boundedness, the approximate stability can be extended to the whole space H. A necessary and sufficient condition for the solution P to be strictly positive is given.

The result obtained from Granwall's Inequality does not provide any new information on the upper bounds of {S(t), t ≧ 0} in terms of {T(t), t≧ 0}. Using Young's Inequality we can prove that if {T(t), t ≧ 0} is exponentially stable, then so is {S(t), t ≧ 0}.

And another condition for the uniform boundedness is also investigated.

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