Optimal Discrete Systems using Time-Weighted Performance Index with Prescribed Closed-Loop Eigenvalues

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ABSTRACT

An optimization problem minimizing a given time-weighted performance index for discrete-time linear multi-input systems is investigated for the prespecified closed-loop eigenvalues. Necessary conditions for an optimality of the controller that satisfies the specified closed-loop eigen-values are derived. A computational algorithm solving the optimal constant feedback gain is presented and a numerical example is given to show the effect of a time-weighted performance index on the transient responses.

I. Introduction

It is well known that the constant feedback gains giving desired closed-loop eigenvalues in the multi-input system are not unique. So an interesting problem is how to utilize this design freedom. This freedom can be used to minimize a time-weighted performance index, combining the eigenvalue assignment and optimization technique. Such a time-weighted performance index is designed to provide an increasing heavy penality for a sustained error and has more good performance characteristics specified in the time domain such as overshoot, settling time, than conventional quadratic performance index [1-3]. Thus far, the design method of optimal regulator for the prespecified closed-loop eigenvalues has been confined to the study of continuous systems [3-6].

A simple design method of an optimal controller minimizing a given time-weighted performance index for discrete-time linear multi-input systems is presented in this paper when the closed-loop eigenvalues are prespecified. The necessary conditions for the solution are derived by using Lyapunov function and Lagrange multiplier. An algorithm computing the optimal controller is presented and a numerical example is given to show the effect of a time-weighted performance index on the transient responses.

II. Problem Formulation

Consider the linear discrete-time time-invariant system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_o, \qquad (1)$$

where the n-dimensional vector x represents the state, the m-dimensional vector u is the control. The matrices A and B are constant matrices of appropriate dimensions. It is assumed that the above system is completely controllable and the matrix B is full rank. Let the time-weighted performance index be given as

$$J = \sum_{k=o}^{\infty} [k^N x'(k) Q_o x(k) + u(k)'Ru(k)], \qquad (2)$$

where Q_o and R are symmetric positive semidefinite and positive definite matrices, respectively, and N is a non-negative integer. The state feedback gain minimizing the performance index (2) for the specified closed-loop eigenvalues depends on the initial state x_o. To avoid this dependency, the expected value of (2) with respect to the initial state is chosen as follows :

$$J_1 = E(J) . \tag{3}$$

The design problem is to find the constant state feedback control

$$u(k) = Fx(k) \tag{4}$$

which assigns any prespecified closed-loop eigenvalues and at the same time minimizes a time-weighted quadratic performance index (3). The closed-loop eigenvalues are any given symmetric set $\Lambda = \{\lambda_1, \ldots, \lambda_s : s \leq n\}$ such that $\sum_{i=1}^{s} d_i = n$, d_i being the multiplicity degree associated with λ_i .

In the following, the notation A' will denote transpose of the matrix A, tr[A] its trace, $\underset{i=1,k}{\mathrm{diag}}(A_i)$ diagonal matrix with diagonal entries $A_1, A_2, \ldots, A_k$ and $\underset{i=1,k}{\mathrm{col}}(A_i) = [A_1, A_2, \ldots A_k]$.

III. Eigenvalue Assignment

All the class of controllers satisfying pole constraint can be characterized through a set of parameter p_{ijk} ($i=1,\ldots,s$; $j=1,\ldots,d_i$; $k=1,\ldots,m$) as follows [7-9].

1) Compute the maximal rank matrices

$$N_i = \begin{bmatrix} N_{1i} \\ N_{2i} \end{bmatrix}, \quad S_i = \begin{bmatrix} S_{1i} \\ S_{2i} \end{bmatrix}, \quad (i=1,\ldots,s) \tag{5}$$

satisfying

$$[\lambda_i I - A,\ B] N_i = 0, \qquad [\lambda_i I - A,\ B] S_i = I .$$

2) Define a parameter vector p_{ij} ($i=1,\ldots,s; j=1,\ldots,d_i$) as

$$p_{ij} = [p_{ij1}, \ldots, p_{ijm}]' \tag{6}$$

and form the following matrices

$$V = V_0 P, \quad W = W_0 P, \tag{7}$$

where

$$V = \underset{i=1,s}{\mathrm{col}} [N_{1i}, S_{1i} N_{1i}, \ldots, S_{1i}^{d_i - 1} N_{1i}]$$

$$W = \underset{i=1,s}{\mathrm{col}} [N_{2i}, S_{2i} N_{1i}, \ldots, S_{2i} S_{1i}^{d_i - 2} N_{1i}]$$

$$P = \underset{i=1,s}{\mathrm{diag}} \begin{pmatrix} p_{i1} & p_{i2} & \cdots & p_{id_i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & p_{i2} \\ 0 & & & p_{i1} \end{pmatrix} .$$

3) For each value of the parameter vector p_{ij} compute the following feedback matrix

$$F(P) = -WV^{-1} . \tag{8}$$

Thus, the design freedom remaining after eigenvalue assignment is represented as any parameter p_{ijk} selected such that the matrix V is nonsingular.

IV. Selection of the optimal eigenvectors

The evaluation of a general time-weighted cost function for discrete-time time-invarient systems is given in the following lemma.

Lemma 1 [10] : For an asymptotically stable system

$$x(k+1) = Ax(k) , \quad x(0) = x_0 \tag{9}$$

the cost functional

$$J = \sum_{k=0}^{\infty} k^N x'(k) Q\, x(k), \tag{10}$$

where N is a non-negative integer and Q is symmetric, is calculated by

$$J = x_0' Q_{N+1} x_0 , \tag{11}$$

where Q_{N+1} is given by the following equations;

$$A' Q_1 A - Q_1 + Q = 0$$

$$A' Q_{i+1} A - Q_{i+1} + \sum_{r=1}^{i} {}_{N-i+r}C_r\, A' Q_{i+1-r} A = 0, \quad (i=1,\ldots,N). \tag{12}$$

The following lemma is useful in computating the gradient matrix and it is derived easily by using Lagrangian approach.

Lemma 2 [11] : If the function of a matrix X is given by

$$f(X) = tr[AX(BX)^{-1}], \tag{13}$$

then the gradient matrix is found to be

$$\frac{df(X)}{dX} = [I - B'(X'B')^{-1}X']A'(X'B')^{-1}, \tag{14}$$

where a matrix I is the identity matrix.

Substituting (4) into (1) yields

$$x(k+1) = (A+BF)x(k) \triangleq A_c x(k). \tag{15}$$

Applying the Lemma 1 to (3), we obtain

$$J_1 = E[x_o'(Q_{N+1} + G)x_o] = tr[(Q_{N+1} + G)X_o], \tag{16}$$

where Q_{N+1} and G are the solution of the following equations;

$$A_c'Q_1A_c - Q_1 + Q_o = 0$$
$$A_c'Q_{i+1}A_c - Q_{i+1} + \sum_{r=1}^{i} {}_{N-i+r}C_r A_c'Q_{i+1-r}A_c = 0 \quad (i=1,2,\dots,N)$$
$$A_c'GA_c - G + F'RF = 0 \tag{17}$$

and $X_o = E[x_o x_o']$. To determine the optimal parameter p_{ij} which minimize the performance index (3) subject to the constraints in (17), the Hamiltonian for this problem is expressed as

$$H(Q_i, L_i, G, L, p_{ij}) = tr[(Q_{N+1}+P)X_o] + tr[L_1'(A_c'Q_1A_c - Q_1 + Q_o) + \sum_{i=1}^{N} L_{i+1}'(A_c'Q_{i+1}A_c - Q_{i+1} + \sum_{r=1}^{i} {}_{N-i+r}C_r A_c'Q_{i+1-r}A_c) + L'(A_c'GA_c - G + F'RF)], \tag{18}$$

where L_i, i=1,2,...,N+1, and L are the symmetric Lagrange multiplier matrices. Then, the necessary conditions are

$$\partial H/\partial Q_j = 0 \quad (j=1,2,\dots,N+1) \tag{19-a}$$
$$\partial H/\partial G = 0 \tag{19-b}$$
$$\partial H/\partial L_i = 0 \quad (i=1,2,\dots,N+1) \tag{20-a}$$
$$\partial H/\partial L = 0 \tag{20-b}$$
$$\partial H/\partial p_{ijk} = 0 \quad (i=1,\dots,s; j=1,\dots,d_i; k=1,\dots,m). \tag{21}$$

Here, we can obtain the following equation

$$tr[\sum_{i=1}^{N}\sum_{r=1}^{i} {}_{N-i+r}C_r A_c'Q_{i+1-r}A_cL_{i+1}'] = tr[\sum_{j=1}^{N}\sum_{r=o}^{N-j} {}_{N+1-j}C_r A_c'Q_jA_cL_{N+1-r}']. \tag{22}$$

Substituting (22) into (17), the(22-a) leads to the following equations;

$$A_cL_jA_c' - L_j + \sum_{r=o}^{N-j} {}_{N+1-j}C_r A_cL_{N+1-r}A_c' = 0 \quad (j=1,\dots,N)$$
$$A_cL_{N+1}A_c' - L_{N+1} + X_o = 0. \tag{23}$$

(22-b) implies that $L=L_{N+1}$ and (20) yields (17). Finally (21) can be simply determined through the matrix form of $\partial H/\partial P$ which is derived using the Lemma 2.

Necessary conditions

In order that the parameter p_{ijk} be optimal with respect to the performance index (3), it is necessary that

$$\partial H/\partial p_{ijk} = 0, \quad (i=1,\dots,s; j=1,\dots,d_i; k=1,\dots,m).$$

Here $\partial H/\partial p_{ijk}$ can be given from the following equation;

$$\partial H/\partial P = 2(W_o - FV_o)'[B'PA_cL_{N+1} + B'(\sum_{i=1}^{N+1} Q_iA_cL_i) + B'(\sum_{i=1}^{N}\sum_{r=1}^{i} {}_{N-i+r}C_r Q_{i+1-r}A_cL_{i+1}) + RFL_{N+1}](V')^{-1},$$

where Q_i and L_i together with A_c and F satisfy the following equations;

$$A_c'Q_1A_c - Q_1 + Q_o = 0$$
$$A_c'Q_{i+1}A_c - Q_{i+1} + \sum_{r=1}^{i} {}_{N-i+r}C_r A_c'Q_{i+1-r}A_c = 0 \quad (i=1,2,\dots,N)$$
$$A_c'GA_c - G + F'RF = 0$$
$$A_cL_jA_c' - L_j + \sum_{r=o}^{N-j} {}_{N+1-j}C_r A_cL_{N+1-r}A_c' = 0 \quad (j=1,2,\dots,N)$$

$$A_c L_{N+1} A_c' - L_{N+1} + X_0 = 0.$$

Then, the final cost becomes

$$J_1 = tr[(Q_{N+1} + P)X_0].$$

V. A Numerical Example

Consider a two-input system whose dynamics is given by

$$A = \begin{bmatrix} 1.105 & 0 \\ 0.057 & 1.162 \end{bmatrix}, \quad B = \begin{bmatrix} 0.053 & 0.105 \\ 0.055 & 0.057 \end{bmatrix}.$$

The weighting matrices are chosen as $Q_0 = R = I$ and also X_0 is identity matrix. The eigenvalues of the closed-loop system are specified as $\lambda_1 = 0.9$, $\lambda_2 = 0.8$. Then,

$$N_1' = \begin{bmatrix} 0.2585 & 0.5122 \\ 0.1537 & 0.1061 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} -4.8781 & 0 \\ 1.0613 & -3.8168 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$N_2 = \begin{bmatrix} 0.1738 & 0.3443 \\ 0.1246 & 0.1033 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -3.2787 & 0 \\ 0.5163 & -2.7624 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The algotithm converges to the following solution;

$$F = \begin{bmatrix} 1.20 & -6.61 \\ -0.94 & -2.96 \end{bmatrix} \quad \text{for } N=0$$

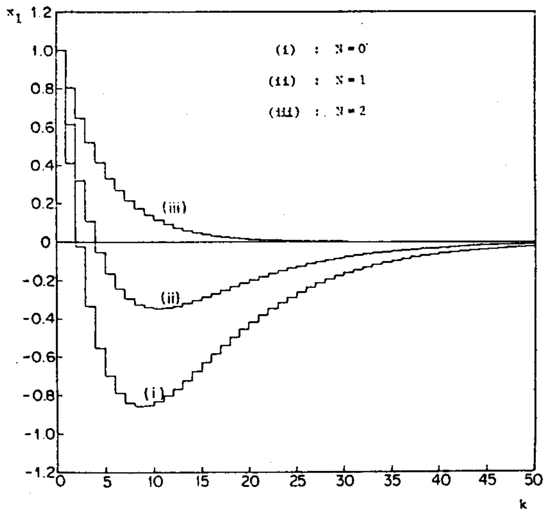
$$F = \begin{bmatrix} 1.88 & -8.84 \\ -2.35 & 1.17 \end{bmatrix} \quad \text{for } N=1$$

$$F = \begin{bmatrix} 1.54 & -11.2 \\ -3.01 & 5.01 \end{bmatrix} \quad \text{for } N=2.$$

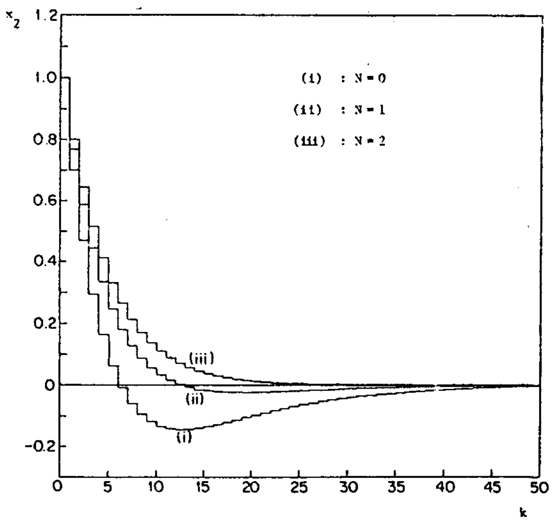
The transient responses of the closed-loop system are shown in Fig.1. It is evident that the responses based on the time-weighted performance index do decay faster and exhibit smaller overshoots than those based on the conventional quadratic performance index.

VI. Conclusioning Remarks

The design freedom, which remains after the prespecified closed-loop eigenvalues for discrete-time linear multi-input systems being assigned, has been used to minimize a given time-weighted quadratic performance index. Necessary conditions for an optimality of the controller have been derived. It has been shown that the design method using the time-weighted performance index presented in this paper provides an additional freedom of analytical design approach for better transient responses.



(a)



(b)

Fig. 1 Response of system with x(0) = [1, 1]'

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Appendix

A computational algorithm

To obtain the optimal solution, one can use the following iterative algorithm.

(1) For each eigenvalue λ_i, calculate N_i and S_i.

(2) Select any initial parameter vector p_{ij} such that the matrix V is nonsingular.

(3) Determine $\partial H/\partial p_{ij}$ from the gradient $\partial H/\partial P$.

(4) If $\partial H/\partial p_{ij}$ satisfies convergence criteria, iteration is completed. Otherewise find a new value of parameter p_{ij} using any gradient-based method.

(5) Return to the step 3.