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Cheap Control for a Class of Nonlinear Systems

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A quadratic regulator problem for a class of nonlinear system, in which a small parameter multiplies the control cost, is considered. In the analysis of the problem, we utilize the method of multiple time-scale decomposition which has been devised for analyzing complex linear cheap control problems. In so doing, we extend the class of nonlinear systems, considerably, for which the minimum cost becomes zero as the small parameter goes to zero.

1. Introduction

A quardratic regulator problem in which a small parameter multiplies the control cost is known as cheap control. The problem is important in analyzing the limiting possibility of the quardratic regulator[4] and it enriches insights on the system, such as invertibility of systems[9]. When the system dynamics is linear, the problem is extensively studied[4,6,8,9] and still more its unique area of practical application is reported[7]. But earlier researches are limited to linear systems.

Among works about the linear cheap control, the multiple time-scale decomposition by Sannuti[7] makes the analysis of the problem be possible in great details. In this note, we apply the multiple time-scale decomposition to the problem of cheap control for a class of nonlinear systems too and, in so doing, extend the results of previous paper[2]. The multiple time-scale decomposition makes the problem easier to facilitate a direct application of readily available perturbation literatures. The transformed problem is solved by the power series method[3] and behaviors of each term of power series solution are investigated, as the small parameter multiplying the control cost goes to zero.

2. Problem Statement and Multiple Time-Scale Decomposition.

Consider a single-input single-output system described by the vector differential equation of the form

$$\dot{\tilde{x}}_i = \tilde{x}_{i+1}\ , \quad i=1,2,\cdots,(q-1) \tag{1}$$

$$\dot{\tilde{x}}_q = \sum_{i=1}^{q} a_{qi}\tilde{x}_i + a_{q\nu}\tilde{x}_\nu + \tilde{f}_q(\tilde{x}) + b_q\tilde{u} + \tilde{g}_q(\tilde{x})\tilde{u}$$

$$\dot{\tilde{x}}_\nu = a_{\nu 1}\tilde{x}_1 + a_{\nu\nu}\tilde{x}_\nu + \tilde{f}_\nu(\tilde{x}) + \tilde{g}_\nu(\tilde{x})\tilde{u}$$

$$y = \tilde{x}_1 \tag{2}$$

The the quadratic performance index to be minimized is given as

$$\tilde{V} = \int_0^\infty (y^2(\tau) + \mu^2\tilde{u}^2(\tau))d\tau \tag{3}$$

Here $\tilde{x}=(\tilde{x}_1\ ,\ \tilde{x}_2\ ,\ \dots\ \tilde{x}_q\ ,\ \ \tilde{x}_\nu\ )^T$ is the (q+1)-dimensional system state vector, and u and y are the scalar input and output variables, respectively. The scalar-valued functions $\tilde{f}_q(\tilde{x})$ and $\tilde{f}_\nu(\tilde{x})$ are analytic functions beginning with second order terms about $\tilde{x}$. The scalar-valued functions $\tilde{g}_q(\tilde{x})$ and $\tilde{g}_\nu(\tilde{x})$ are analytic functions beginning with first order terms about $\tilde{x}$.

The problem is to investigate the asymptotic behaviors of the optimal cost

$$\tilde{\Phi}(\tilde{x}) = \inf_{\tilde{u}(\cdot)} \tilde{V}(\tilde{x}, \tilde{u}(\cdot)) \tag{4}$$

for an arbitrary initial state $\hat{x}(0)$ in a neighborhood of the origin as the weighting, η, multiplying the control cost goes to zero. To be specific, conditions under which the optimal cost becomes to zero (perfect regulation) as η goes to zero are isolated.

For this we shall make the following assumptions:

(A1) $b_g \neq 0$

(A2) The linearized system is controllable, observable and of minimum phase.

In order to utilize previous results about perturbation, we apply a multiple magnitude scaling to the state variables as

$$x_i = \varepsilon^{i-1} \tilde{x}_i, \quad i = 1, 2, \cdots, g \tag{5}$$
$$u = \varepsilon^{g} \tilde{u}$$

where
$$\varepsilon = \eta^{1/g}$$

Then Eq.(1) can be rewritten as

$$\varepsilon \dot{x}_f = A_{ff} x_f + A_{fs} x_s + b_m \varepsilon^g f_f(x) + b_f u + b_m g_f(x) u$$
$$\dot{x}_s = A_{sf} x_f + A_{ss} x_s + f_s(x) + \varepsilon^{-g} g_s(x) u \tag{6}$$

where

$$x_f = (x_1 \ x_2 \ \cdots \ x_g)^T$$
$$x_s = x_\nu$$
$$A_{ff} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ \varepsilon^g a_{g1} & \varepsilon^{g-1} a_{g2} & \varepsilon^{g-2} a_{g3} & \cdots & \varepsilon a_{gg} \end{pmatrix}$$
$$A_{fs} = (0 \quad 0 \quad 0 \quad \cdots \quad \varepsilon^g a_{g\nu})$$
$$b_m = (0 \quad 0 \quad 0 \quad \cdots \quad 1)^T$$
$$b_f = (0 \quad 0 \quad 0 \quad \cdots \quad b_g)^T$$
$$f_f(x) = \tilde{f}_g(\tilde{x})$$
$$g_f(x) = \tilde{g}_g(\tilde{x})$$
$$A_{sf} = (a_{\nu 1} \quad 0 \quad 0 \quad \cdots \quad 0)$$
$$A_{ss} = a_{\nu\nu}$$
$$f_s(x) = \tilde{f}_\nu(\tilde{x})$$
$$g_s(x) = \tilde{g}_\nu(\tilde{x})$$

The quadratic performance index (3) becomes

$$V = \int_0^\infty (y^2(\tau) + u^2(\tau))\, d\tau \tag{7}$$

We solve the Hamilton-Jacobi equation representing the problem of Eqs. (6) and (7) by the power series method and investigate behaviors of each term of the power series solution as the weighting, η, multiplying the control cost. That is, for a fixed η, the optimal cost

$$\phi(x) = \inf_{u(\cdot)} V(x, u(\cdot)) \tag{8}$$

is the solution of the Hamilton-Jacobi equation :

$$\nabla\phi(x) \cdot (Ax + f(x)) - \tfrac{1}{4} (\nabla\phi(x) \cdot (b + g(x)))^2 + y^2 = 0,$$
$$\phi(0) = 0 \tag{9}$$

where

$$Ax = \begin{pmatrix} \frac{1}{\varepsilon} A_{ff} & \frac{1}{\varepsilon} A_{fs} \\ A_{sf} & A_{ss} \end{pmatrix} \begin{pmatrix} x_f \\ x_s \end{pmatrix} \begin{matrix} g \\ 1 \end{matrix}$$
$$\qquad\qquad g \qquad 1$$

$$f(x) = \begin{pmatrix} b_m \varepsilon^g f_g(x) \\ f_s(x) \end{pmatrix} \begin{matrix} g \\ 1 \end{matrix}, \quad g(x) = \begin{pmatrix} b g_g(x) \\ g_s(x) \end{pmatrix} \begin{matrix} g \\ 1 \end{matrix}$$

$$b = \begin{pmatrix} \frac{1}{\varepsilon} b_f \\ 0 \end{pmatrix} \begin{matrix} g \\ 1 \end{matrix}$$

The partial differential equation (9) can be solved by the power series method as described in [3]. In the power series method $\phi(x)$ is first expanded in series in the form:

$$\phi(x) = \phi^{(2)}(x) + \phi^{(3)}(x) + \cdots \tag{10}$$

Then they are substituted to Eq.(9) and coefficients of the same order terms about x in both side are compared.

The leading term is just the solution of the linearized LQ problem. Say,

$$\phi^{(2)}(x) = x \cdot Px$$

where P is the unique positive-definite solution of the matrix Riccati equation

$$PA + A^T P - Pbb^T P + cc^T = 0 \tag{11}$$

Higher order terms are calculated from:

$$\nabla\phi^{(m)}(x) \cdot A^* x = -\sum_{i=2}^{m-1} \nabla\phi^{(i)}(x) \cdot f^{(m-i+1)}(x) + \frac{1}{4} \sum_{i=1}^{m-1} \left[\sum_{j=2}^{m-1} \nabla\phi^{(j)}(x) \cdot g^{(i-j+1)}(x) \right] \left[\sum_{k=2}^{m-1} \nabla\phi^{(k)}(x) \cdot g^{(m-i-k+1)}(x) \right]$$

where $\tag{12}$

$$A^* = A - bb^T P$$

The coefficients of $\phi^{(m)}(x)$ is determined by equating coefficients of the same order terms about x in both sides of Eq.(12) and solving the resulting system of linear equations.

That is, higher order terms can be computed from Eq.(12) by solving successively higher order systems of linear equations.

In the following sections, we investigate the asymptotic behiviors of $\phi(x)$ as ε goes to zero via the power series method.

3. Main Result

For the problem described in section 2, we may obtain the following results.

Lemma 1 : When the linearized system of Eq. (6) satisfies the Assumptions (A1) and (A2), then the solution of the matrix Riccati equation (11) is

$$P(\varepsilon) = \left(\begin{array}{c|c} L & M \\ \hline M^T & N \end{array} \right) \begin{matrix} q \\ 1 \end{matrix} \qquad (13)$$

where

$$L = \varepsilon L_0 + O(\varepsilon^2) \qquad (14)$$
$$M = \varepsilon^{q+1} M_0 + O(\varepsilon^{q+2})$$
$$N = \varepsilon^{2q} N_0 + O(\varepsilon^{2q+1})$$

Proof : From the matrix Riccati equation (11), we can obtain

$$M^T\left(\frac{1}{\varepsilon}A_{ff} - \frac{1}{\varepsilon^2}b_f b_f L\right) + A_{ss}^T M^T + N A_{sf} + \frac{1}{\varepsilon}A_{fs}^T L = 0 \qquad (15)$$
$$\frac{1}{\varepsilon}M^T A_{fs} + \frac{1}{\varepsilon}A_{fs}^T M + N A_{ss} + A_{ss}^T N - \frac{1}{\varepsilon^2}M^T b_f b_f^T M = 0 \qquad (16)$$

In [8], it was shown that

$$L = \varepsilon L_0 + O(\varepsilon^2),$$
$$M = O(\varepsilon),$$

and $N = O(\varepsilon)$

To extract more close approximation for M and N, we solve Eq.(15) about $(m_1, m_2, \ldots, m_q, N)$ in terms of L and m_q (m_i's are elements of M) and substitute them to Eq.(16). Then Eq.(16) becomes

$$-\frac{1}{\varepsilon^2} b_q^2 m_q^2 + \alpha m_q + \beta = 0 \qquad (17)$$

where $\alpha = O(\varepsilon)$ and $\beta = O(\varepsilon^{q+2})$

Among two solutions of Eq.(17), the solution of $O(\varepsilon^{q+1})$ is the valid one. This lemma follows by substituting this solution to Eqs.(15) and (16).

Before characterizing higher order terms, we analyze equations of the form of Eq.(12) as

$$\nabla\phi^{(j)}(x) \cdot \varepsilon A^* x = \xi(x) \qquad (18)$$

Lemma2 : When $\xi(x) = h_a(x_f) + h_b(x_f, x_s) + h_c(x_s)$ and

$$h_a(x_f) = O(\varepsilon^m),\ h_b(x_f, x_s) = O(\varepsilon^n), \qquad (19)$$
$$h_c(x_s) = O(\varepsilon^{n+1}),\quad q \ge n \ge m \ge 1$$

then the solution of Eq.(18) satisfies that

$$\phi^{(j)}(x) = O(\varepsilon^m) \qquad (20)$$
$$\nabla_{x_s}\phi^{(j)}(x) = O(\varepsilon^n)$$

Proof : When the eigenvalues of A^* are distinct, Eq.(18) can be solved effectively by the spectral resolution of A^* [1]. The eigenvalues of A^* for our problem are all distinct and so we prove this lemma via solving method of the spectral resolution.

Since

$$\varepsilon A^* = \left(\begin{array}{c|c} A_{ff} - \frac{1}{\varepsilon} b_f b_f^T L & A_{fs} - \frac{1}{\varepsilon} b_f b_f^T M \\ \hline \varepsilon A_{sf} & \varepsilon A_{ss} \end{array} \right) \qquad (21)$$

eigenvalues of εA^* are

$$\lambda_i = O(1),\quad i = 1, 2, \cdots, q \qquad (22)$$

and $\lambda_s = O(\varepsilon)$

Let v^i be right eigenvectors. Then it is of the form

$$v^i = (1\ \ \lambda_i\ \ \lambda_i^2\ \cdots\ \lambda_i^{q-1}\ \ O(1))^T,\ i = 1, 2, \cdots, q \qquad (23)$$

and

$$v^s = (\varepsilon^q\ \ \varepsilon^q \lambda_s\ \cdots\ O(1))^T$$

The spectral transformation,

$$x = T\bar{x} \qquad (24)$$

where $T = (v^1\ \ v^2\ \cdots\ v^q\ \ v^s)$ and its inverse transformation,

$$\bar{x} = T^{-1}x = \left(\begin{array}{c|c} O(1) & O(\varepsilon^q) \\ \hline O(\varepsilon) & O(1) \end{array} \right) \begin{matrix} q \\ 1 \end{matrix}\ x \qquad (25)$$

will proves the relation of this lemma.

With above two lemmas, we can extract specific but useful results.

Theorem 1 : When the linearized system of Eq.(6) satisfies the Assumptions (A1) and (A2) and

$$f_f(x) = O(\varepsilon^{-q-1+n}) \qquad (26)$$
$$g_f(x) = O(\varepsilon^{-1+n})$$
$$f_s(x) = O(\varepsilon^0)$$
$$g_s(x) = O(\varepsilon^{-1}),\quad q \ge n \ge 1$$

then

$$\phi^{(j)}(x) = O(\varepsilon^n),\quad j = 3, 4, \cdots \qquad (27)$$

Proof is straightforward, if considering above two lemmas, but lengthy. So we ommit here.

Corrolary 1: When the linearized system of Eq.(6) satisfies the Assumptions (A1) and (A2) and

$$f_f(x) = O(\epsilon^{-q-1+n}) \qquad (28)$$
$$g_f(x) = O(\epsilon^{-1+n})$$
$$f_f(x) \equiv g_f(x) \equiv 0, \quad q \ge n \ge 1$$

then

$$\phi^{(j)}(x) = O(\epsilon^n), \quad j = 3, 4, \cdots \qquad (29)$$

4. Example

Example 1 : To test the lemma 1, following simple LQ problem [8] is composed. A system of minimun phase is given by

$$\dot{\tilde{x}}_1 = \tilde{x}_2, \quad \dot{\tilde{x}}_2 = \tilde{x}_3 + \tilde{u}, \quad \dot{\tilde{x}}_3 = \tilde{x}_1 - \tilde{x}_3 \qquad (30)$$

and $y = \tilde{x}_1$

It is required to find $\tilde{u}$ such that $\tilde{V}$ is minimized.

$$\tilde{V} = \int_0^\infty (y^2 + \epsilon^4 \tilde{u}^2)\, dt \qquad (31)$$

Using the transformation,

$$x_1 = \tilde{x}_1, \quad x_2 = \epsilon \tilde{x}_2, \quad x_3 = \tilde{x}_3, \quad u = \epsilon^2 \tilde{u}$$

one has the equivalent optimization problem of finding u for the system

$$\epsilon \dot{x}_1 = x_2 \qquad (32)$$
$$\epsilon \dot{x}_2 = \epsilon^2 x_3 + u$$
$$\dot{x}_3 = x_1 - x_3$$

to minimize

$$V = \int_0^\infty (y^2 + u^2)\, dt \qquad (33)$$

The Riccati equation for the problem satisfies the equations

$$P = \left(\begin{array}{cc|c} l_{11} & l_{12} & m_1 \\ l_{12} & l_{22} & m_2 \\ \hline m_1 & m_2 & n \end{array} \right) \qquad (34)$$

$$2m_1 - \frac{1}{\epsilon^2} l_{12}^2 + 1 = 0$$
$$m_2 + \frac{1}{\epsilon} l_{11} - \frac{1}{\epsilon^2} l_{12} l_{22} = 0$$
$$\frac{2}{\epsilon} l_{12} - \frac{1}{\epsilon^2} l_{22}^2 = 0$$
$$n + \epsilon l_{12} - m_1 - \frac{1}{\epsilon^2} l_{12} m_2 = 0$$
$$\frac{1}{\epsilon} m_1 + \epsilon l_{22} - m_2 - \frac{1}{\epsilon^2} l_{22} m_2 = 0$$
$$2\epsilon m_2 - 2n - \frac{1}{\epsilon^2} m_2^2 = 0$$

For a sufficiently small ϵ, m_2 is given by

$$m_2^2 + (2\epsilon l_{22} + l_{12}) m_2 - 2\epsilon^3 (l_{12} + \epsilon l_{22}) = 0 \qquad (35)$$

Between two solutions, $m_2 = O(\epsilon)$ and $m_2 = O(\epsilon^3)$, $m_2 = O(\epsilon^3)$ is the stabilizing solution. Subsequently

$$n = \epsilon m_2 - \frac{1}{2\epsilon^2} m_2^2 = O(\epsilon^4) \qquad (36)$$
$$m_1 = -\epsilon^2 l_{22} + \epsilon m_2 + \frac{1}{\epsilon} l_{22} m_2 = O(\epsilon^3)$$

Example 2(Van der Pol oscillator): we consider a system

$$\dot{\tilde{x}}_1 = \tilde{x}_2 \qquad (37)$$
$$\dot{\tilde{x}}_2 = -\tilde{x}_1 + \tilde{x}_2 - \tilde{x}_1^2 \tilde{x}_2 + \tilde{u}$$

Cost functional to be minimized is given by

$$\tilde{V} = \int_0^\infty (\tilde{x}_1^2 + \epsilon^4 \tilde{u}^2)\, dt \qquad (38)$$

The optimal cost, $\tilde{\phi}(\tilde{x})$, is calculated as

$$\tilde{\phi}(\tilde{x}) = l_{11}\tilde{x}_1^2 + 2l_{12}\tilde{x}_1\tilde{x}_2 + l_{22}\tilde{x}_2^2 + r_1\tilde{x}_1^4 + r_2\tilde{x}_1^3\tilde{x}_2 + r_3\tilde{x}_1^2\tilde{x}_2^2 + r_4\tilde{x}_1\tilde{x}_2^3 + r_5\tilde{x}_2^4 + \cdots$$

where

$$r_1 = \frac{\epsilon^4}{2}\left[\alpha - 1 - \alpha(\beta+1)(9\alpha-3)/(\beta(10\alpha-3))\right] \qquad (39)$$
$$r_2 = 0$$
$$r_3 = -\epsilon^4 (\beta+1)(9\alpha-3)/(\beta(10\alpha-3))$$
$$r_4 = -4\epsilon^4 (\beta+1)/(10\alpha-3)$$
$$r_5 = -4\epsilon^4(\beta+1)/(\beta(10\alpha-3)), \quad \alpha = \sqrt{1-\epsilon^{-4}}, \quad \beta = \sqrt{2\alpha-1}$$

The transformations,

$$x_1 = \tilde{x}_1, \quad x_2 = \epsilon\tilde{x}_2, \quad u = \epsilon^2 \tilde{u} \qquad (40)$$

and some calculations will prove Corrollary 1.

5. Conclusion

Utilizing the multiple time-scale transformation, we can extend class of nonlinear systems for which perfect regulation (the optimal cost of quadratic regulator becomes zero as the small parameter multiplying the control cost goes to zero) is possible.

This cheap controller will be applied to nonlinear systems which can not be handled by the celebrating method of external linearization [5] for nonlinear systems.

6. Reference

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