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상태변수에 시간지연이 있는 선형시스템의 안정화 조건및 시간지연여유에 관한 연구

°이 준화, 권 욱현

서울 대학교 제어계측 공학과

On Delay Independent Stability Criteria and Delay Margins of Linear State Delay Systems

Joon Hwa Lee and Wook Hyun Kwon

Dept. of Control and Instr. Engr.
Seoul National Univ.

ABSTRACT

Several sufficient conditions which guarantee the stability of linear state delay systems are derived. And the delay margin which guarantees the stability of the delay systems are presented.

I. *INTRODUCTION*

Throughout this paper we will consider the following system

$$\dot{X}(t)=A_0X(t)+A_1X(t-h) \qquad (1)$$

$$\dot{X}(t)=A_0X(t)+A_1X(t-h)+Bu(t) \qquad (1.1)$$

where

$A_0,A_1\in C^{n\times n},X(t)\in C^n,B\in C^{n\times m},u(t)\in C^m \qquad h\geq 0.$

Stability analysis of time delay systems is the first step to stabilize such systems. Numerous reports have been published for the stabilization of the system (1.1) [1,2,3,4,5,6,7,10]. These stabilization methods are obtained from their stability criteria for the homgenous system (1). Thowsen [3] gives the method of the stability test for the delay systems. This method uses the characteristic equation of the delay systems. For multivariable systems this method are inadequate. Mori [2] derives a stability criteria using Lyapunov equation. His method is relatively simple but very conservative. He suggested another delay dependent stability criteria in [7]. Stability criteria which listed above are conservative or difficult. In this note, the new stability criteria is suggested which differs from the others. These methods are easy to use since they have scalar inequalities and simple matrix calculations. The stability criteria are used to find the delay margins which guarantee the stability of the state delay systems as (1). We extend the Halanay's result [9] and then obtain new stability criteria and the delay margin. By some examples, we illustrate that our results are simple but powerful.

II. *MAIN RESULTS*

First we introduce a lemma in [9] which offers a sufficient conditions of delay independent stability ,and extend it generally.

LEMMA 1 [9]:If $\dot{f}(t)\leq -\alpha f(t)+\beta \sup_{t-h\leq\sigma\leq t} f(\sigma)$ *for* $t>t_0$ *and if* $\alpha>\beta>0$, *then there exists* $\gamma>0$ *and* $k>0$ *such that* $f(t)\leq k\exp(-\gamma(t-t_0))$ *for* $t\geq t_0$.

Above lemma says that if $f(t)\geq 0$ for $t\geq t_0$ and satisfies the assumptions then $f(t)$ is asymptotically stable at 0. As an application of Lemma 1 we think $f(t)$ as a positive definite function of $X(t)$ of system (1) and if $f(t)$ satisfies the lemma 1 then we can conclude that the system (1) is asymptotically stable at 0 independent of $h>0$. For example, define the positive definite function $f(t)$ as

$$f(t)=X(t)^*X(t)=|X(t)|^2 \quad .$$

Then

$$\dot{f}(t)=\dot{X}(t)^*X(t)+X(t)^*\dot{X}(t)$$

$$=X(t)^*(A_0^*+A_0)X(t)+X(t-h)^*A_1^*X(t)+X(t)^*A_1X(t-h)$$

$$\leq -\alpha f(t)+\beta \sup_{t-h\leq\sigma\leq t} f(\sigma)$$

where

$$\alpha=-\lambda_{max}(A_0^*+A_0)$$

$$\beta=2\|A_1\|_2 \quad .$$

So, when $\alpha>\beta>0$ i.e.

$$-1/2\lambda_{max}(A_0^*+A_0)>\|A_1\|_2 \quad ,$$

the state delayed system (1) is asymptotic stable and we get following Corollary.

Corollary1 : System (1) is asymptotic stable if $-1/2\lambda_{max}(A_0^*+A_0) >\|A_1\|_2$.

The same results are contained in [2],[7] with different approaches. Now we prove a lemma which extends lemma 1 for several functions.

LEMMA 2: Consider n functions $f_1(t),\dots,f_n(t)$. *If* $\dot{f}_i(t)\leq -\alpha_i f_i(t) + \beta_i \sup_{t-h\leq\sigma\leq t} [\max_{1\leq k\leq n} \{f_k(\sigma)\}]$ *for* $t\geq t_0, 1\leq i\leq n$ *and if* $\alpha_i>\beta_i>0$, $1\leq i\leq n$, *then there exist* $\gamma>0$ *and* $k>0$ *such that* $f_i(t)\leq k\exp(-\gamma(t-t_0))$ *for* $t\geq t_0$, $1\leq i\leq n$.

Proof) Define $g(t)=\max_{1\leq k\leq n} \{f_k(t)\} \quad ,t\geq t_0$

It is clear that at each $t\geq t_0$ there exist a index i such that

$g(t)=f_i(t)$, $1\le i\le n$.

And for such a index i

$$D\_g(t)=\dot{f}_i(t)\le-\alpha_i f_i(t)+\beta_i \sup_{t-h\le\sigma\le t}[\max_{1\le k\le n}\{f_k(t)\}]$$

$$= -\alpha_i g(t)+\beta_i \sup_{t-h\le\sigma\le t} g(\sigma) \quad ,t>t_0$$

where
$D\_g(t)=\lim_{h\to 0-} \inf\{g(t+h)-g(t)\}/h$ as in [9].

Select $k>0,\Gamma>0$ which satisfies

$$k> g(t_0)$$

$$\Gamma< \min\{\Gamma_i>0 \mid \Gamma_i=\alpha_i-\beta_i \exp(\Gamma_i h)\} \quad .$$

Since $\alpha_i>\beta_i>0$, $1\le i\le n$, $\min\{\Gamma_i>0 \mid \Gamma_i=\alpha_i-\beta_i\exp(\Gamma_i h)\}$ is well defined and we can select a $\Gamma>0$ .Define a function y(t) as

$$y(t)=k\exp(-\Gamma_i(t-t_0)) \quad .$$

Then for $t\ge t_0$

$$\dot{y}(t)=-\Gamma k\exp(-\Gamma_i(t-t_0))>-\Gamma_i k\exp(-\Gamma(t-t_0)) \quad , i=1,\dots,n$$

$$=(-\alpha_i+\beta_i\exp(\Gamma_i h))k\exp(-\Gamma(t-t_0))$$

$$\ge(-\alpha_i+\beta_i\exp(\Gamma h))k\exp(-\Gamma(t-t_0))$$

$$=-\alpha_i k\exp(-\Gamma(t-t_0))+\beta_i k\exp(-\Gamma(t-t_0-h))$$

$$=-\alpha_i y(t)+\beta_i \sup_{t-h\le\sigma\le t} y(\sigma)$$

So, we get

$$g(t_0) < y(t_0)$$

$$D\_g(t) \le -\alpha_i g(t)+\beta_i \sup_{t-h\le\sigma\le t} g(\sigma) \qquad (2)$$

for some i , $1\le i\le n$

$$D\_y(t) > -\alpha_i y(t)+\beta_i \sup_{t-h\le\sigma\le t} y(\sigma) \qquad (3)$$

for all i , $1\le i\le n$

It is remained to show that

$$y(t) > g(t) \ , \ t\ge t_0 \ .$$

Define a set Z as followings.

$$Z=\{t\in[t_0,\infty) \mid g(t)\ge y(t)\} \ .$$

Assume $Z=\phi$ and let $t_1=\inf Z$.

Clearly $t_1>t_0$ and

$$g(t_1) = y(t_1) \qquad (4)$$

$$g(t) < y(t) \ , \ t\in[t_0,t_1) \qquad (5)$$

Hence for sufficiently small $h<0$

$$\{g(t_1+h)-g(t_1)\}/h > \{y(t_1+h)-y(t_1)\}/h$$

So, we get

$$D\_g(t_1) \ge D\_y(t_1) \qquad (6)$$

But

$$D\_g(t_1)=-\alpha_i g(t_1)+\beta_i \sup_{t_1-h\le\sigma\le t_1} g(\sigma)$$

for some i, $1\le i\le n$ by (2).

$$=-\alpha_i y(t_1)+\beta_i \sup_{t_1-h\le\sigma\le t_1} g(\sigma) \quad \text{by (4)}$$

$$\le-\alpha_i y(t_1)+\beta_i \sup_{t_1-h\le\sigma\le t_1} y(\sigma) \quad \text{by (5)}$$

$$< D\_y(t_1) \quad \text{by (3)}$$

This results contradicts (6) and claims that $Z=\phi$ i.e.

$$y(t) > g(t) \ ,\text{for all } t\ge t_0 \ .$$

The lemma follows immediately. <Q.E.D.>

Using the above lemma we can derive row by row stability criteria of state delayed systems.Theorem 1 shows this fact.

THEOREM 1: Denote $a_{ij},\hat{a}_{ij}$ are (i,j) element of A_0,A_1 respectively , where A_0,A_1 are matrices of system (1) and $1\le i,j\le n$.
If A_0,A_1 satisfy

$$Re(a_{ii}) < -\{ \sum_{i=j} |a_{ij}| + \sum_{j} |\hat{a}_{ij}| \}$$

for $i=1,\dots,n$

then the state delayed system (1) is asymptotically stable.

Proof) The i'th row of system (1) is

$$\dot{x}_i(t)=a_{ii}x_i(t)+\sum_{i=j} a_{ij}x_j(t)+\sum_{j}\hat{a}_{ij}x_j(t-h).$$

Define

$$f_i(t)=|x_i(t)|^2 \ , \quad i=1,\dots,n \ .$$

Then we get ($i=1,\dots,n$)

$$\dot{f}_i(t)=\dot{x}_i(t)^* x_i(t)+x_i(t)^* \dot{x}_i(t)$$

$$=2Re(a_{ii})|x(t)|^2 + \sum_{i=j} (a_{ij}^* x_j(t)^* x_i(t)+a_{ij} \quad x_j(t)x_i(t)^*) + \sum_{j}(\hat{a}_{ij}^* \quad x_j(t-h)^* x_i(t)+\hat{a}_{ij} \quad x_j(t-h)x_i(t)^*)$$

$$\le 2\{Re(a_{ii})f_i(t) + \sum_{i=j} |a_{ij}| f_i(t)^{1/2} f_j(t)^{1/2} + \sum_{j} |\hat{a}_{ij}| f_i(t)^{1/2} f_j(t)^{1/2}$$

$$\le 2[Re(a_{ii})f_i(t) +(\sum_{i=j}|a_{ij}|+\sum_{j}|\hat{a}_{ij}|)\sup_{t-h\le\sigma\le t}(\max_{1\le j\le n}\{f_j(\sigma)\})]$$

From the above results and lemma 2, theorem 1 is proved. <Q.E.D.>

Another easy proof of theorem 1 is presented in appendix.Theorem1 offers simple stability criteria of state delayed systems but conservative. To reduce the conservatism we use similar transform which transforms the matrix A_0 into the diagonal form or jordan normal form and then apply the stability criteria in theorem 1 . From the above procedure we can get the following Corollarys.

COROLLARY 2: When A_0 of system (1) is similar to $D=diag(d_1,\dots,d_n)$ with S i.e. $SA_0S^{-1}=D$.
Denote $SA_1S^{-1}=[\hat{a}_{ij}]$. If for all $i=1,\dots,n$,

$$Re(d_i) < -\sum_{j} |\hat{a}_{ij}|$$

then system (1) is asymptotically stable.

COROLLARY 3: The matrix A_0 of system (1) is similar to jordan normal form J with S i.e. $SA_0S^{-1}=J$

Denote $SA_1S^{-1}[\hat{a}_{ij}]$, $1\le i,j\le n$.
If for all $i=1,\dots,n$

$$Re(d_i) < -\sum_j |\hat{a}_{ij}| \quad \text{, when i'th row is the last row of some jordan block}$$

$$Re(d_i) < -\sum_j |\hat{a}_{ij}| - 1 \quad \text{, otherwise}$$

,where d_i is the associated eigenvalue

then system (1) is asymptotically stable.

Example 1) Let us consider the 2-dim state delayed system as follows.

$$dx(t)/dt = \begin{vmatrix} 0 & 1 \\ -3 & -4 \end{vmatrix} x(t) + \begin{vmatrix} 0.5 & 0 \\ 0 & 0.5 \end{vmatrix} x(t-h)$$

$$h \ge 0, x(t) \in R^2$$

Theorem 1 does not guarantee the stability of the given system but we transform the system as follows

$$d\bar{x}(t)/dt = \begin{vmatrix} -1 & 0 \\ 0 & -3 \end{vmatrix} \bar{x}(t) + \begin{vmatrix} 0.5 & 0 \\ 0 & 0.5 \end{vmatrix} \bar{x}(t-h)$$

where

$$\bar{x}(t) = \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} x(t)$$

We can guarantee the stability of the given delay system independent of delay.The derived stability criteria requires that all diagonal elements have negative real parts. But when we determine the delay independent stability of system as (1) ,A_0 must be Hurwitz. By suitable equivalence transformation we can obtain the transformed matrix A_0'whose all diagonal elements have negative real parts. And so, corollary 2 and corollary 3 gives delay independent stability criteria of most state delayed systems. Otherwise, when A_0 is not Hurwitz the given system is unstable as $h \to \infty$,one can not guarantees the delay independent stability. When A_0 is not Huriwitz but (A_0+A_1) is Huriwitz, the magnitude of the delay ,i.e. h is a important stability criteria of the delay system. Think a state feedback system as fig 1. Physically there always a delay to detect states and feedback the signals so we can represent all real system as delay systems. Particular ,the system of fig. 1 can be described as a state delayed system

$$\dot{X}(t) = AX(t) - BKX(t-h) \quad . (7)$$

Even when the system designer select the feedback gain K which stabilize the system with non delay i.e. (A-BK) is Huriwitz, system (7) may be unstable because of the delay introduced in control. So,one needs the delay margin which guarantees the stability of the system with the selected feedback gain K. We will derive the theorem 2 which is useful to determine the delay margin which guarantees the stability of state delayed system as (1), when (A_0+A_1) is Huriwitz where A_0,A_1 are matrices of system (1) . This result also applicable to system (7).

***THEOREM 2:** If (A_0+A_1) is a Huriwitz matrix which have distincteigenvalues then there is a $h^*>0$ such that when $0\le h\le h^*$, the state delayed system (1) is asymptotically stable.Let S be a matrix which diagonalize (A_0+A_1) and*

$S(A_0+A_1)S^{-1}=diag(d_1,\dots,d_n)$
h^* is given by

$$h^* = \min_{1\le i\le n} \{-Re(d_i)/\sum_{j=1}^{n}(|m_{ij}|+|n_{ij}|)\}$$

where m_{ij},n_{ij} are (i,j) component of $SA_1A_0S^{-1}$, $SA_1A_1S^{-1}$ respectively

Comment: When (A_0+A_1) has multiple eigenvalue we can calculate h^* as in corollary 3 using $SA_1A_0S^{-1}$,$SA_1A_1S^{-1}$

Proof)

$$\dot{X}(t) = A_0X(t) + A_1X(t-h)$$

$$= (A_0+A_1)X(t) + A_1\int_{t-h}^{t} \dot{X}(\tau)d\tau$$

$$= (A_0+A_1)X(t) + A_1\int_{t-h}^{t} (A_0X(\tau)+A_1X(\tau-h))d\tau$$

$$= (A_0+A_1)X(t) + A_1A_0\int_{t-h}^{t} X(\tau)d\tau + A_1A_1\int_{t-2h}^{t-h} X(\tau)d\tau$$

Define

$$Y(t) = SX(t)$$

where

$$S(A_0+A_1)S^{-1} = diag(d_1,\dots,d_n) = D$$

Then

$$\dot{Y}(t) = DY(t) + M\int_{t-h}^{t} Y(\tau)d\tau + N\int_{t-2h}^{t-h} Y(\tau)d\tau \quad (8)$$

where

$$M = SA_1A_0S^{-1}, N = SA_1A_1S^{-1}$$

For $y_i(t)$ of $Y(t)$, $1\le i\le n$

$$\dot{y}_i(t) = d_iy_i(t) + \sum_j m_{ij}\int_{t-h}^{t} y_j(\tau)d\tau + \sum_j n_{ij}\int_{t-2h}^{t-h} y_j(\tau)d\tau$$

Define

$$f_i(t) = |y_i(t)|^2 = y_i(t)^*y_i(t) \quad , \quad 1\le i\le n$$

Then

$$\dot{f}_i(t) = \dot{y}_i(t)^*y_i(t) + y_i(t)^*\dot{y}_i(t)$$

$$= 2Re(d_i)y_i(t)^*y_i(t) + \sum_j[m_{ij}^* \int_{t-h}^{t} y_j(\tau)^*d\tau\, y_i(t) + m_{ij}\int_{t-h}^{t} y_j(\tau)d\tau\, y_i(t)^*]$$
$$+ \sum_j[n_{ij}^* \int_{t-2h}^{t-h} y_j(\tau)^*d\tau\, y_i(t) + n_{ij}\int_{t-2h}^{t-h} y_j(\tau)d\tau\, y_i(t)^*]$$

$$\le 2Re(d_i)f_i(t) + 2\{\sum_j[|m_{ij}|\int_{t-h}^{t} |y_j(\tau)|d\tau + |n_{ij}|\int_{t-2h}^{t-h} |y_j(\tau)|d\tau]\}|y_i(t)|$$

$$\le 2[Re(d_i)f_i(t) + h\{\sum_j(|m_{ij}|+|n_{ij}|)\} \sup_{t-2h\le\sigma\le t} \max_{1\le k\le n} \{f_k(\sigma)\}]$$

Since

$$\int_{t-h}^{t} |y_j(\tau)|d\tau \le h \sup_{t-2h\le\sigma\le t} |y_j(\sigma)|$$

$$\int_{t-2h}^{t-h} |y_j(\tau)|d\tau \le h \sup_{t-2h\le\sigma\le t} |y_j(\sigma)|$$

From lemma 2 we get stability condition

$$\mathrm{Re}(d_i) < -h\{\sum_j(|m_{ij}|+|n_{ij}|)\} \quad , i=1,\ldots,n$$

i.e.

$$0 \le h \le \min_{1 \le i \le n} \{-\mathrm{Re}(d_i)/\sum_j(|m_{ij}|+|n_{ij}|)\}$$

<Q.E.D.>

Example 2) Let us consider the scalar time delay system described by

$$x(t)=a_0x(t)+b_0x(t-h)$$

where

$$(a_0+b_0)<0 \quad , a_0>0$$

From the theorem 2 we get stable delay margin

$$h^*=|a_0+b_0|/(|a_0b_0|+|b_0^2|)$$

It is easy to check that this delay margin guarantees the stability.[8]

III. *CONCLUDING REMARKS*

Several delay independent stability criteria and delay margins which guarantee the stability for the linear state delayed systems are derived. The Stability criteria in this paper are simple but more useful for some cases than those developed by other researchers. Delay margins derived in this paper can be applied to real design problems which include uncertain dealy factor.

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APPENDIX

Another proof of theorem 1:
We will prove the theorem by contradiction.
Assume that

(A1) : there exist a $s^* \in C$ such that

$$\mathrm{Re}(s^*) \ge 0 \qquad \text{(A2)}$$

and

$$|s^*I-A_0-A_1\exp(s^*h)|=0 \qquad \text{(A3)}$$

By Gershgorin's theorem and (A3)

$$|s^*-a_{ii}| \le \sum_{j \ne i}|a_{ij}|+\sum_j|\hat{a}_{ij}\exp(s^*h)|$$

$$\le \sum_{j \ne i}|a_{ij}|+\sum_j|\hat{a}_{ij}|\,|\exp(s^*h)|$$

$$\le \sum_{j \ne i}|a_{ij}|+\sum_j|\hat{a}_{ij}| \qquad \text{by (A2)}$$

$$< -\mathrm{Re}(a_{ii})$$

by the assumption in the theorem 1 Hence we get that

$$\mathrm{Re}(s^*) < 0 .$$

This contradicts the assumption (A1) and the theorem is proved.

<Q.E.D.>

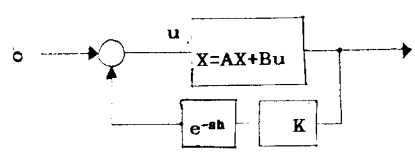

fig . 1