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A method for deciding weighting matrices in a linear discrete time optimal regulator problems to locate all poles in the specified region

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Abstract: In this paper, a new procedure for selecting weighting matrices in linear discrete time quadratic optimal control problems (LQ-problem) is proposed. In LQ problems, the quadratic weighting matrices are usually decided on trial and error in order to get a good response. But using the proposed method, the quadratic weights are decided in such a way that all poles of the closed loop system are located in a desired area for good responses as well as for stability and values of the quadratic cost functional are kept less tnen a specified value. The closed loop systems constructed by this method have merits of LQ problems as well as those of pole assignment problems. Taking into consideration that little is known about the relationship among the quadratic weights, the poles and the value of cost functional,this procedure is also interesting from the theoretical point of view.

1. Introduction

In many application, it is required that the poles of the closed-loop system lie in a certain restricted region of stability and good responses. Although standard pole placement techniques (see,e.g., Kailath 1980) can be applied to such problem,the exact specification of many poles at once is very difficult. In this respect the LQ-problem has an advantage, since an appropriate stable pole allocation of the closed-loop system is automatically guaranteed.

Several design methods of feedback controllers have been reported; The closed-loop system constructed via LQ-problems has some merits (Safonov and Athans, 1977; Kobayashi and Shimemura, 1981). But, in LQ-problems, weighting matrices of the quadratic cost function must be decide on by trial and error to get the good responses, because only very little is known about the relation between the quadratic weights and dynamical characteristics of the closed-loop system (Harvey and Stein,1978; Stein,1979; Francis,1979). The dynamical characteristics of a linear system are influenced by the location of poles of the system. Therefore to get good responses, it is necessary to locate all poles in some desired positions. But we know that it is sufficient to place all poles in a suitable region instead of placing them in their desired respective position.

In this paper, we give a new method of selecting the quadratic weights in discrete time LQ-problems by which all poles of the closed-loop system can be located in the specified region for good response as well as for stability and values of a given quadratic cost functional are kept less then the specified value. The system constructed by this method has the merits of an LQ-problem as well as a pole-assignment problem and holds down the value of a given quadratic cost functional. Comceptually this decision method may be considered to be derived from the so-called inverse optimal control problems (Thau,1967; Yokoyama and Kinnen,1972; Moylan and Anderson,1973). In continuous time case similar methods of determining quadratic weights have been reported in Kawasaki and Shimemura (1981,1983,1988),in which only pole locations of the closed loop system are considered.

2.Problem formuration

We consider a linear discrete time system described by

$$x(k+1)=Ax(k)+Bu(k) \qquad (1)$$

where $x(k)$ is the state vector of dimension n, $u(k)$ is the input vector of dimension r, and A,B are nxn , nxr constant matrices. The pair (A , B) is assumed to be controllable. Let the performance index be

$$J(u) = \sum_{k=0}^{\infty} (x(k)'Qx(k)+u(k)'Ru(k)) \qquad (2)$$

where Q and R are nxn and rxr positive definite symmetric matrices, respectively. Then it is well known (Kwakernaak and Sivan,1972) that the optimal control which minimizes $J(u)$ subject to (1) is given by the feedback control law

$$u(k) = -Kx(k) \qquad (3)$$

with the optimal feedback gain

$$K = (R+B'PB)^{-1}B'PA \qquad (4)$$

where P is the maximal solution of the algebraic Riccati equation

$$P = A'PA-A'PB(R+B'PB)^{-1}B'PA+Q. \qquad (5)$$

Our problem is to decide quadratic weights Q which give the optimal feedback gain K satisying the condition

1) $\lambda(A-BK) \in \Gamma$

2) $J_o(u) \leq M$

Where Γ is the specified region for a good response as well as for stability (Fig.1), $J_o(u)$ is the quadratic cost function with $(Q,R)=(Q_o,R)$, and M is some realizable positive number, $\lambda(A)$ is a set of eigenvalues of matrix A. In the next section we propose the method of deciding quadratic weights Q in the auxiliary performance index (2), which guarantees the conditions 1) and 2).

3.A method of deciding quadratic weights

3.1 some preliminary lemmas.

Before showing the result,we prepare some preliminary lemmas .

Lemma 1. (Kawasaki and Shimemura,1981)
Let k be an arbitrary real number. Then $(A+kI,B)$ is a controllable pair if and only if (A,B) is a controllable pair. □

Next we consider solutions of Riccati equation (5) and of Liapunov equation

$$P^{\#} = A'P^{\#}A+Q^{\#}. \qquad (6)$$

Lemma 2. (Kodama and Suda,1978)
Let P_1 be a solution of equation (5) with $Q=Q_1$, and P_2 is one with $Q=Q_2$. If $Q_2 \geqq Q_1$ then $P_2 \geqq P_1$. □

Lemma 3. (Kodama and Suda,1978)
Let $Q=Q^{\#} \geqq 0$ in equations (5) and (6). If A is asymptotically stable, then eq. (6) has a real symmetric positive semi-definite solution $P^{\#}$ which satisfies $P^{\#} \geqq P \geqq 0$. □

Lemma 4. (Kodama and Suda,1978)
Let $A_1=A-B(R+B'P_1B)^{-1}B'P_1A$, where P_1 is a solution of the equation(5) with (Q_o,R):

$$P_1 = A'P_1A-A'P_1B(R+B'P_1B)^{-1}B'P_1A+Q_o$$

and $A_2 = A_1-B(R+B'P_2B)^{-1}B'P_2A_1$,where P_2 is a solution of the equation(5) with (Q_1,R):

$$P_2 = A_1'P_2A_1-A_1'P_2B(R+B'P_2B)^{-1}B'P_2A_1 +Q_1.$$

Furthere $A^+=(A-B'(R+B'P^+B)^{-1}B'P^+A,B)$,where P^+ is a solution of the equation (5) with (Q_o+Q_1,R),that is

$$P^+ = A'P^+A-A'P^+B(R+B'P^+B)^{-1}B'P^+A +(Q_o+Q_1).$$

Then $A_2=A^+$.

Lemma 5

Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the eigenvalues of A inside the unit disc, and $\xi_1, \xi_2, \cdots, \xi_m$ be the corresponding eigenvectors. If a symmetric positive semi-definite matrix Q of (5) satisfies the following equation

$$Q\xi_i = 0, \quad i=1,2,\cdots,m \tag{7}$$

the closed loop system matrix $A_c = A-B(R+B'P^+B)^{-1}B'P^+A$ formed with the maximum solution P^+ has the eigenvalue λ_i and the corresponding eigenvector ξ_i.

Proof: Let H be a discrete Hamilton matrix given by

$$H = \begin{bmatrix} A+BR^{-1}B'A'^{-1}Q & -BR^{-1}B'A'^{-1} \\ -A'^{-1}Q & A'^{-1} \end{bmatrix} \tag{8}$$

From (8),(9) and the definition of λ_i and ξ_i, it follows that

$$H\begin{bmatrix} \xi_i \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} \xi_i \\ 0 \end{bmatrix} \tag{9}$$

It is well known (Kimura and Inoue,1978; Pappas et al.,1980) that the optimal closed loop poles are given by the eigenvalues of H inside the unit disc. If the absolute value of λ_i is less then 1, the equation (9) shows that λ_i is an eigenvalue of the closed -loop matrix A_c and ξ_i is the corresponding eigenvector.□

3.2 Fundamental theorems.

In this section, we shall give fundamental theorems which is important to derive the design method. Let λ_i be eigenvalues of the matrix A_c and let ξ_i be the corresponding eigenvector. Then the following relation holds:

$$A-B(R+B'PB)^{-1}B'PA\ \xi_i = \lambda_i\xi_i \tag{11}$$

And we have the following relation to multiply (5) by ξ_i from right:

$$A'P\ A-B(R+B'PB)^{-1}B'PA\ \xi_i = \lambda_i A'P\xi_i = (P-Q)\xi_i \tag{12}$$

From the relations (11) and (12), if λ_i is not eigenvalues of A, λ_i and ξ_i satisfy the following relation:

$$\xi_i = (\lambda_i I-A)^{-1}B(R+B'PB)^{-1}B'(\lambda_i^{-1}I-A')^{-1}Q\xi_i \tag{13}$$

And let Q be given by

$$Q = TQ^*T' \tag{14}$$

where T is some nxr real matrix and Q^* is rxr symmetric positive definite matrix. Substituting (14) into (13) and multiplying it by T' from left, we have the following relation:

$$I_r - T'(\lambda_i I-A)^{-1}B(R+B'PB)^{-1}B'(\lambda_i^{-1}I-A')^{-1}TQ^*T'\xi_i = 0 \tag{15}$$

Before stating the main theorem we show two preliminary results to locate the poles of the closed loop system in the specified region. Let σ_i be real eigenvalues of the matrix A and $\alpha_i \pm j\beta_i$ be complex conjugate pair eigenvalues of A, where $\sigma_i, \alpha_i, \beta_i$ are real numbers. Let ζ_i and $\nu_i \pm j\omega_i$ be the corresponding right eigenvectors of n dimensions , and η'_i and $s'_i \pm jt'_i$ are the corresponding left eigenvectors of n dimension. In order to transform only σ_i , let $T=\eta_i$ and $Q^*=q^*$ in (14), that is,

$$Q = q^*\eta_i\eta'_i \tag{16}$$

Then the matrix Q satisfy the condition (7) of Lemma 5 for the eigenvectors except the eigenvector ζ_j, $j\neq i$.If follows from Lemma 5 that,for the weight Q given by (16), the feedback gain (4) transforms only the real eigenvalue σ_i and the corresponding eigenvector ζ_i. Thus we obtain the following theorem.

Theorem 1

For the quadratic weights Q defined by (16) the feedback gain (4) transforms only the real eigenvalue σ_i. □

Next in order to transform only the complex conjugate pair eigenvalues $\alpha_i \pm j\beta_i$ let

$$T = [s_i \;\; t_i],$$

$$Q^* = \begin{bmatrix} q1^* & q2^* \\ q2^* & q3^* \end{bmatrix} \qquad (17)$$

Then, we define

$$Q = [s_i \;\; t_i] \begin{bmatrix} q1^* & q2^* \\ q2^* & q3^* \end{bmatrix} [s_i \;\; t_i] \qquad (18)$$

From Lemma 5 , we obtain the following theorem.

Theorem 2

For the quadratic weights Q defined by(18), the feedback gain (4) transforms only the complex conjugate pair eigenvalues $\alpha_i \pm \beta_i$. □

We obtained the method of deciding the quadratic weights Q which can transform the closed loop pole via the solution of equation (5). Now we give the formula to evaluate the inrement of the cost functional $J_o(u)$ by a transform of a pair of eigen values. We denote $u_o = -K_o x$ be the optimal control for $J_o(u)$, with $Q=Q_o$, i.e. $K_o = (R+B'P_oB)^{-1}B'P_oA$, and $u = -Kx$ be the optimal control $(Q,R) = (Q_o+Q_1,R)$, where Q_1 is defined by eq. (18). There we can obtain the following main theorem.

Theorem 3

The value of quadratic cost functional $J_o(u)$ satisfies the following inequality

$$J_o(u) \leq J_o(u_o)+\|x_o\|^2 \frac{\lambda\max(Q^*)}{1-\alpha^2} \qquad (19)$$

where x_o is an initial value of the state vector $x(k)$.

Proof: First, we solve the matrix equation

$$P_1 = A_1'P_1A_1-A_1'P_1B(R+B'P_1B)^{-1}B'P_1A_1+Q_1 \qquad (20)$$

where A_1 is a closed-loop matrix $A-BK_o$ and Q_1 is the weight in Theorem 2. We compose the feedback input

$$\begin{aligned} u(k) &= -K_1x(k) \\ &= -(R+B'P_1B)^{-1}B'P_1A_1x(k). \end{aligned} \qquad (21)$$

From Lemma 4, we can obtain the following relation:

$$\begin{aligned} A_1-BK_1 &= A-B(K_o+K_1) \\ &= A-BK \end{aligned} \qquad (22)$$

Let the value of the quadratic cost function (22) be $J_1(u)$. Then we can obtain the following relation:

$$\begin{aligned} J_o(-Kx) &\leq J_1(-Kx) \\ &= J_o(-K_ox)+x_o'P_1x_o \\ &\leq J_o(-K_ox)+\|x_o\|^2 \;\; P1 \end{aligned} \qquad (23)$$

Then from Lemma 2, selecting the norm of left eigenvector $[s_i \;\; t_i] = 1$, and substituting the matrix Q of Theorem 2 into the equation (5),we obtain

$$\lambda\max(P^*) = \frac{\lambda\max(Q^*)}{1-\alpha^2} \qquad (24)$$

Hence,

$$P_1 \leq \frac{\lambda\max(Q^*)}{1-\alpha^2} \qquad (25)$$

This establishes Theorem 3 . □

Here, Q^* of Theorem 3 is a 2x2 symmetric positive definite matrix of eq.(8). Specifically, if $q2^*=0$ then equation (25) is

$$P_1 \leq \frac{\lambda\max(q_1^*\;,\;q_3^*)}{1-\alpha^2} \qquad (26)$$

From Theorem 3, we obtained the value of increment of the quadratic cost functional ΔJ_i. Here, we consider the decision method of the quadratic weights to satisfy the condition 1) and 2). But unfortunately , we can not obtain the determining method q^* and Q^*,Theorem 1 and 2. Then the quadratic weights are decided on trial and error. Concluding the above discussions, we propose a decision method of the quadratic weights as to keep the value of quadratic cost functional less than the specified value as follows.

[Decision method]

Step 1. Solve an LQ-problem for arbitrary quadratic weights (Q_o , R) and solve

$$P_1=A'P_1A-A'P_1B(R+B'P_1B)^{-1}B'P_1A+Q_o \quad (27)$$

to obtain an optimal closed-loop system matrix $A-BK_o=A-B(R+B'P_1B)^{-1}B'P_1A$, and calculate a quadratic cost functional J_o.

Step 2. Choose a real eigenvalue or a pair of complex conjugation eigenvalue which are outside region Γ of Fig.1, and obtain the quadratic weights Q utilizing Theorem 1 or 2 ,and Theorem 3.

Step 3. Utilizing the quadratic weights Q obtained in Step 2, obtain an optimal closed-loop system matrix $A_i-BK_i = A_{i-1}-B(R+B'P_iB)^{-1}B'P_iA_{i-1}$ $(A_o=A)$ and calculate a quadratic cost functional J_i and a value of increment ΔJ_i.

Step 4. Repeat Step 2 and Step 3 so that the all poles enter the region Γ of Fig.1. The least upper bound value of quadratic cost functional is then

$$J_o(-Kx) \leqq J_o(-K_o)+ \Sigma \Delta J_i. \quad (28)$$

Hence, we need to choose the ΔJ_i that is less than M of condition 2).

4. Conslusions.

In this paper,we propose a decision method of determining quadratic weights of an LQ-problem to locate all poles of the closed-loop system in the specified region and to keep the value of the quadratic cost functional less than the specified value.

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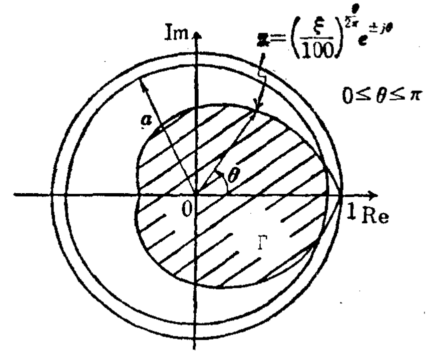

Fig.1 A desired region of closed loop eigenvalue locations.