# Degree of 2D Discrete Linear Shift-Invariant System and Reduction of 2D Rational Transfer Function


Shojiro SAKATA

Toyohashi University of Technology
Department of Production Systems Engineering
Tempaku, 440 Toyohashi



Abstract: In this paper we present a method of determining the unknown degree of any 2D discrete linear shift-invariant system which is characterized only by the coefficients of the double power series of a transfer function, i.e. a 2D impulse response array. Our method is based on a 2D extension of Berlekamp-Massey algorithm for synthesis of linear feedback shift registers, and it gives a novel approach to identification and approximation of 2D linear systems, which can be distinguished in its simplicity and potential of applicability from the other 2D Levinson-type algorithms. Furthermore, we can solve problems of 2D Padé approximation and 2D system reduction on a reasonable assumption in the context of 2D linear systems theory.


## 1. Introduction

Signals and systems that depend on two or more independent variables are involved in image processing, control, geophysics and other areas. In extending the one-dimensional (1D) system theory to the multidimensional case, many problems which are important practically as well as theoretically remain to be solved or have not been treated perfectly. Among them, there are problems concerned with two-dimensional (2D) state-space model, notion of controllability, observability, minimality, pole-placement as well as stabilization [1]-[3]. Now we are consider the following three kinds of problems all of which are important in identification and model reduction of 2D discrete linear shift-invariant (DLSI) systems.

First, in the 2D Hankel theory [4] the 2D Hankel matrix is defined via the 2D impulse response array, i.e. the coefficients of the double power series into which the proper 2D rational transfer function is expanded:

$$a(z_1,z_2)=\frac{c(z_1,z_2)}{b(z_1,z_2)}=\frac{\Sigma_{i=0}^{m}\Sigma_{j=0}^{n}c_{m-i,n-j}z_1^{i}z_2^{j}}{\Sigma_{i=0}^{m}\Sigma_{j=0}^{n}b_{m-i,n-j}z_1^{i}z_2^{j}}=\Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}a_{ij}z_1^{-i}z_2^{-j} \quad (b_{00}\neq 0). \qquad (1)$$

On the condition that we are given the 2D array $a=(a_{ij})$ without any knowledge of both system polynomials $b(z_1,z_2)$ and $c(z_1,z_2)$, the system degree $(m,n)$ of the 2D system can be determined through the rank of the 2D Hankel matrix. This is a natural extension of the 1D Hankel theory. However, it is very cumbersome and difficult to obtain the rank of the 2D Hankel matrix since the order of the 2D Hankel matrix in general tends to become much greater as the system degree increases. Furthermore, there has not been given any efficient method of identifying the given system, i.e. of finding the coefficients of system polynomials by means of the 2D Hankel matrix.

Example 1 (Kao and Chen [4]): For the 2D transfer function of a 2D DLSI system over the real number field R:

$$a(z_1,z_2)=\frac{z_1z_2+z_1+2z_2+1}{(z_2+1)(z_2+2)z_1+(z_2+2)z_2}, \qquad (2)$$

we have the 2D impulse response array a shown in Fig. 1. The 2D Hankel theory tells us that the degree of $a(z_1,z_2)$ is $(m,n)=(1,2)$ only on the basis of the rank of the Hankel matrix constructed from the array a.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | -2 | 4 | -8 | 16 | -32 | 64 | -128 | . |
| 0 | 1 | -2 | 4 | -8 | 16 | -32 | 64 | . | |
| 0 | -1 | 3 | -7 | 15 | -31 | 63 | . | | |
| 0 | 1 | -4 | 11 | -26 | 57 | . | | | |
| 0 | -1 | 5 | -16 | 42 | . | | | | |
| 0 | 1 | -6 | 22 | . | | | | | |
| 0 | -1 | 7 | . | | | | | | |
| 0 | 1 | . | | | | | | | |
| 0 | . | | | | | | | | |
| . | | | | | | | | | |

Fig. 1. a 2D impulse response array over the real number field R (Kao and Chen [4]).

Second, a problem of two-variable Padé approximation [5][6] consists of finding the denominator polynomial $\tilde{b}(z_1,z_2)=\Sigma_{i=0}^{m}\Sigma_{j=0}^{m}b_{ij}z_1^{i}z_2^{j}$ $(b_{00}\neq 0)$ and the numerator polynomial $\tilde{c}(z_1,z_2)=\Sigma_{i=0}^{m}\Sigma_{j=0}^{m}c_{ij}z_1^{i}z_2^{j}$ of the rational function s.t.

$$\frac{\tilde{c}(z_1,z_2)}{\tilde{b}(z_1,z_2)}=a(z_1,z_2)+\Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}e_{ij}z_1^{i}z_2^{j}$$

for a given power series

$$a(z_1,z_2):=\Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}a_{ij}z_1^{i}z_2^{j},$$

$$a(z_1,z_2):=\Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}a_{ij}z_1^{i}z_2^{j},$$

where $e_{ij}=0$ for $(i,j)\notin E:=\{(i,j)\mid i+j\geq 4m\}$. This problem is reduced to finding the polynomials

$b(z_1,z_2)$ and $c(z_1,z_2)$ satisfying the identity (1) and having a minimal degree $(m,n)$, by putting $n=m$, $b(z_1,z_2)=z_1^m z_2^m \tilde{b}(z_1^{-1},z_2^{-1})$ and $c(z_1,z_2)=z_1^m z_2^m \tilde{c}(z_1^{-1},z_2^{-1})$.

Example 2 (Buchberger, Krishnamurthy and Winkler [5]): It is required to find the Padé approximant c/b of degree $(m,n)=(1,1)$ which satisfies the following congruence over the field $K=GF(7)$:

$$\frac{\tilde{c}}{\tilde{b}}=\frac{\sum_{i=0}^{1}\sum_{j=0}^{1}c_{ij}z_1^{\,i}z_2^{\,j}}{\sum_{i=0}^{1}\sum_{j=0}^{1}b_{ij}z_1^{\,i}z_2^{\,j}}$$
$$=3z_1^3+z_1^2z_2+2z_2^3+5z_1^2+6z_1z_2+z_2^2+6z_1+4z_2+3 \mod (z_1^4,\ z_1^3z_2,\ z_1^2z_2^2,\ z_1z_2^3, z_2^4). \quad (4)$$

The Groebner basis algorithm [7] applied to the ideal

$$I:=(3z_1^3+z_1^2z_2+2z_2^3+5z_1^2+6z_1z_2+z_2^2+6z_1+4z_2+3,\ z_1^4,\ z_1^3z_2,\ z_1^2z_2^2,\ z_1z_2^3,\ z_2^4)$$

gives $\tilde{c}=5z_2+3\in I$ and $\tilde{b}=5z_1+5z_2+1$. It is known that the Groebner basis algorithm has much complexity, in particular in higher dimensions. By the way, the righthand side of the congruence (4) corresponds to the finite array shown in Fig. 2.

| | | | |
|---|---|---|---|
| 3 | 4 | 1 | 2 |
| 6 | 6 | 0 | |
| 5 | 1 | | |
| 3 | | | |

Fig. 2. a finite 2D array over the Galois field GF(7).

Third, for the minimal implementation of a 2D system, it is required to reduce a given 2D rational function

$$a(z_1,z_2)=\frac{c(z_1,z_2)}{b(z_1,z_2)}=\frac{\sum_{i=0}^{m}\sum_{j=0}^{n}c_{m-i,n-j}z_1^{\,i}z_2^{\,j}}{\sum_{i=0}^{m}\sum_{j=0}^{n}b_{m-i,n-j}z_1^{\,i}z_2^{\,j}} \quad (5)$$

to an irreducible one. A method of solving the problem has been proposed on the basis of resultant matrix of polynomials $b(z_1,z_2)$ and $c(z_1,z_2)$ [8]. Although the method does not have the drawback of coefficient growth during the process of the multivariable Euclidean algorithm, it still has much computational complexity of $O((mn)^3)$ in Gauss elimination of the resultant matrix having order mn, where mn is the size of the given data $b_{ij}$ and $c_{ij}$. Expanding the given rational function (5), we obtain a double power series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}z_1^{-i}z_2^{-j}$ * and then we have again the problem of finding the denominator and numerator polynomials having the possible minimal degree $(m,n)$ and satisfying the identity (1).

---

* If $b_{00}=0$, then it is not always possible to determine the power series consistently and uniquely. In our context, it is enough to consider the proper rational function, for which $b_{00}\neq 0$.

Example 3: The rational function over R:

$$\frac{c}{b}=\frac{z_1^2z_2^2+2z_1z_2^2+2z_1^2z_2+z_2^2+3z_1z_2+z_1^2+z_2+z_1}{z_1^4z_2^3+2z_1^3z_2^3+z_1^4z_2^2+z_1^2z_2^3+z_1^3z_2^2+z_1^2z_2+2z_1z_2+z_1^2+z_2+z_1} \quad (6)$$

can be reduced to the irreducible form:

$$\frac{z_2+1}{z_1^2z_2^2+1} \quad (7)$$

by finding the linearly dependent relations among the rows of the corresponding resultant matrix. By the way, the above rational fuction is expanded into the power series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}z_1^{-i}z_2^{-j},$$

where the coefficient array $a=(a_{ij})$ is shown in Fig. 3.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | . | . |
| 0 | 0 | 0 | 0 | 0 | 0 | . | . |
| 0 | 1 | 0 | 0 | 0 | 0 | . | . |
| 0 | 1 | 0 | 0 | 0 | 0 | . | . |
| 0 | 0 | 0 | -1 | 0 | 0 | . | . |
| 0 | 0 | 0 | -1 | 0 | 0 | . | . |
| 0 | 0 | 0 | 0 | 0 | 1 | . | . |
| 0 | 0 | 0 | 0 | 0 | 1 | . | . |
| . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . |

Fig. 3. a 2D impulse response array over the real number field R.

In the above three kinds of problems, we need to find the polynomials $b(z_1,z_2)$ and $c(z_1,z_2)$ satisfying

$$\sum_{k=0}^{i}\sum_{l=0}^{j}b_{kl}a_{i-k,j-l}=c_{ij},\quad \text{for } (i,j)\in\Gamma_{(m,n)}, \quad (8a)$$

and

$$\sum_{k=0}^{m}\sum_{l=0}^{n}b_{kl}a_{i-k,j-l}=0,\quad \text{for } (i,j)\in\Sigma_{(m+1,0)}\cup\Sigma_{(0,n+1)}, \quad (8b)$$

where

$$\Gamma_{(m,n)}:=\{(i,j)\mid 0\leqq i\leqq m,\ 0\leqq j\leqq n\},$$
$$\Sigma_{(m+1,0)}:=\{(i,j)\mid i\geqq m+1,\ j\geqq 0\},$$
$$\Sigma_{(0,n+1)}:=\{(i,j)\mid i\geqq 0,\ j\geqq n+1\}.$$

If it is possible to find $b(z_1,z_2)$ satisfying (8b), then it is easy to obtain $c(z_1,z_2)$ by using (8a). At any way, we are treating a single quadrant causal AR (or ARMA) model which is defined on $\Sigma_0:=Z_0^2$, i.e. the set of all pairs $p=(p_1,p_2)$ of nonnegative integers $p_1$ and $p_2$, where it has a suitable model degree $(m,n)$, i.e. a window $W=\Gamma_{(m,n)}$. For a while, we assume a priori that $c_{mn}=0$. On this assumption, we have the identity (8b) for any $(i,j)\in\Sigma_w:=\{p\in\Sigma_0\mid w\leqq p\}$, where $w:=(m,n)$ and the partial order $\leqq$ over $\Sigma_0$ is defined as follows:

$$p=(p_1,p_2)\leqq q=(q_1,q_2) \text{ iff } p_1\leqq q_1 \text{ and } p_2\leqq q_2.$$

In this paper, we will consider first how to find the polynomial $b(z_1,z_2)$ which satisfies (8b) for $(i,j)\in\Sigma_w$ and has the 'minimal' degree $w=(m,n)$, where the notion of minimal degree should coinside with that determined based on the rank of the 2D Hankel matrix. Afterwards, we will consider the case where the assumption $c_{mn}=0$ does not hold. Our approach can be regarded as an extension of the method by Conan [9] to two dimensions. To avoid the problem of errors in numerical computation, we sometimes discuss about 2D DLSI systems over a finite field [10]. But, any system over the real or complex number field can be treated similarly.

## 2. Minimal polynomial set for a given 2D array

In the present chapter, we focus our attention to the problem of finding the polynomial $h(z_1,z_2)=\Sigma_{i=0}^{m}\Sigma_{j=0}^{n}b_{m-i,n-j}z_1^{\,i}z_2^{\,j}$ $(b_{00}\neq 0)$ satisfying

$$b_{mn}a_{i-m,j-n}+b_{m-1,n}a_{i-m+1,j-n}+\cdots+b_{00}a_{ij}=0,\quad (i,j)\in\Sigma_w$$

for a given 2D array $(a_{ij})$. As a convention, we put $f(z_1,z_2)=\Sigma_{i=0}^{m}\Sigma_{j=0}^{n}f_{ij}z_1^{\,i}z_2^{\,j}:=b(z_1,z_2)$, where $f_{ij}=b_{m-i,n-j}$, in particular $f_{mn}(=b_{00})\neq 0$. Then, we have a linear recurring (LR) relation which must be satisfied by the given 2D array $a=(a_p)$:

$$\Sigma_{r\in W}f_r a_{r+q-w}=0,\quad q\in\Sigma_w. \tag{9}$$

In our situation, we have nothing but the data $(a_p)$ and we have no assumption or knowledge of the model degree $w=(m,n)$. Thus, we should be led to explore a polynomial $f$ satisfying (9) and having a minimal degree $w=v$, where the term 'minimal' means that there exists no polynomial $f$ satisfying (9) and having some degree $w<v$, i.e. $w\leqq v$ and $w\neq v$. This is almost the same situation as treated before by us [11]. In the following, we review briefly several fundamental concepts about LR relations or bivariate polynomials and then we reformulate our problem exactly.

Let $K$ be any field; $K$ may be the set of real or complex numbers or the Galois field GF(q) of q elements, where q is a power $p^r$ of a prime integer p. Over the 2D lattice $\Sigma_0=Z_0^{\,2}$, we introduce the total ordering $<_T$ as follows:

$$0:=(0,0)<_T(1,0)<_T(0,1)<_T(2,0)<_T(1,1)<_T(0,2)<_T(3,0)<_T\cdots$$

By the way, $p\leqq_T q$ iff $p<_T q$ or $p=q$. By the total degree ordering $<_T$, we have the one-to-one correspondence $|\ \ |:\Sigma_0\to Z_0$. Thus, $|(0,0)|=0$, $|(1,0)|=1$, $|(0,1)|=2$, ... Furthermore, for a 'point' $p=(p_1,p_2)\in\Sigma_0$, we have the 'next' point of $p=(p_1,p_2)$ as follows:

$$p+1:=(p_1-1,p_2+1)\ \text{if}\ p_1\geqq 1;$$
$$:=(p_2+1,0)\ \text{if}\ p_1=0.$$

For $p,q\in\Sigma_0$, the usual vector sum and difference are denoted as $p+q$ and $p-q$, respectively. For $t,p\in\Sigma_0$, let

$$\Sigma_t^{\,p}:=\{q\in\Sigma_0\mid t\leqq q<_T p\}.$$

For $p\in\Sigma_0$, a finite 2D array $a=(a_q)$ of size $k=|p|$ over the field K is a mapping a from $\Sigma_0^{\,p}$ into K. Similarly, a perfect (infinite) 2D array $a=(a_q)$ is a mapping from $\Sigma_0$ into K. For a 2D array a and $p\in\Sigma_0$, $a^p:=(a_q\mid q\in\Sigma_0^{\,p})$ is the restriction of the original array a within $\Sigma_0^{\,p}$.

Let

$$f:=\Sigma_{q\in\Gamma_f}f_q z^q \tag{10}$$

be a polynomial, an element of the bivariate polynomial ring $K[z]:=k[z_1,z_2]$ over K, where $z^q:=z_1^{\,q}\,z_2^{\,q}$ and $\Gamma_f:=\{q\in\Sigma_0\mid f_q\neq 0\}$. The degree Deg(f) of f is defined to be the maximum element q of $\Gamma_f$ w.r.t. the total degree ordering $<_T$. In particular, a polynomial of the form

$$f=\Sigma_{i=0}^{m}\Sigma_{j=0}^{n}f_{ij}z_1^{\,i}z_2^{\,j}\ (f_{mn}\neq 0)$$

has just the degree (m,n) in our definition. Corresponding to a polynomial (10) with Deg(f)=s, the LR relation at a point $p\in\Sigma_0$ for an array a is written as

$$f[a]_p:=\Sigma_{q\in\Gamma_f}f_q a_{q+p-s}=0. \tag{11}$$

For a finite array $a=a^r$, a polynomial f is said to be valid (up to r) iff either $r\leqq_T s$ or the identity (11) holds for any $p\in\Sigma_s^{\,r}$. For a perfect array a, f is said to be valid iff (11) holds for any $p\in\Sigma_s$.

Example 4: For the perfect array over the Galois field GF(2) (Prabhu and Bose [10]), part of which is shown in Fig. 4,

$$f=z_1^{\,2}z_2^{\,2}+z_2^{\,2}+z_1z_2+z_1^{\,2}+z_1$$

is valid at any point $q\geqq(2,2)$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | .. |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | .. |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | .. |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | .. |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | .. |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | .. |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | .. |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | .. |
| . | . | . | . | . | . | . | . | .. |
| . | . | . | . | . | . | . | . | .. |

Fig. 4. a 2D impulse response array over the Galois field GF(2) (Prabhu and Bose [10]).

For any array a, VALPOL(a) is defined to be the set of all valid polynomials for the array a. By the way, for any perfect array a, VALPOL(a) is an ideal of K[z] [12].

We are trying to find a polynomial f having a minimal degree. Since several polynomials in VALPOL(a) possibly have a distinct minimal degree, we introduce the following definition:

Definition 1: A finite subset F of K[z] is said to be a 'minimal polynomial set' for a given 2D array a iff all the following conditions are satisfied:

(1) $F \subseteq VALPOL(a)$;

(2) $S := \{Deg(f) \mid f \in F\}$ is 'nondegenerate', where a finite subset S of $\Sigma_0$ is said to be nondegenerate iff there does not exist any couple $s,t \in S$ s.t. $s \leqq t$ and $s \neq t$;

(3) there does not exists any polynomial g s.t. $g \in VALPOL(a)$ and $Deg(g) \in \Delta(F) := \Sigma_0 / \Sigma_S$, where / is the set difference operator and $\Sigma_S := \bigcup_{s \in S} \Sigma_s$.

We remark that $\Delta(F)$ is unique for the array a by the definition and it can be denoted as $\Delta(a)$, but that the set F is not unique for a. Thus, the class FF(a) of all minimal polynomial sets for a is introduced. Obviously, if $p \leqq_T q$ and $a^p = (a^q)^p$, then $\Delta(a^p) \subseteq \Delta(a^q)$.

## 3. Application of 2D Berlekamp-Massey algorithm

In our paper quoted above [11], we have proposed a 2D extension of Berlekamp-Massey algorithm for finding a minimal polynomial set $F \in FF(a^p)$ of a given finite array $a^p$. During the iterative process, we keep and/or update $F \in FF(a^q)$ and $S = \{s = Deg(f) \mid f \in F\}$ at every point $q \in \Sigma_0^p$. The outline of the algorithm is as follows:

2D Berlekamp-Massey Algorithm (Outline):

(Step 1) $q := 0$; $F := \{1\}$; $S := \{0\}$; $(\Delta := \phi;)$

(Step 2) If $F_N := \{f \in F \mid f[a]_q \neq 0\}$ is nonempty, update F by using $F_N$ and some past information; In particular, if $S_{NN} := \{s = Deg(f) \mid f \in F_N,\ q-s \notin \Delta\}$ is nonempty, then S is updated, i.e. $\Delta$ is replaced by $\Delta \cup \Gamma_{q-S_{NN}}$, where $\Gamma_{q-S_{NN}} := \{t \in \Sigma_0 \mid t \leqq q-s,\ \exists s \in S_{NN}\}$.

(Step 3) $q := q+1$; if $q = p$ then stop, else go to Step 2.

Before applying the algorithm to solve our problem, we remark that, if the given array does not admit the solution polynomial f, i.e. the denominator polynomial of the desired rational function, s.t. $f[a]_w = 0$, in the other words, the LR relation is valid only for (i,j) s.t. i>m or j>n, then we can neglect the first row (i=0) or the first column (j=0) of the original array to have an array w.r.t. which the LR relation is valid for any $(i,j) \in \Sigma_w$. In the following, we show how to solve the three kinds of problems mentioned in Chapter 1 by taking examples.

Example 5 (Example 1 revisited): Applying the algorithm to the array in Fig. 1, we have the unique polynomial $z_1^2 z_2 + z_1^2 + z_1 z_2$ which remains valid after any number of iterations. But, this polynomial does not satisfy the identity (8b) at (i,j) s.t. i=0,1. Next, applying the algorithm to two arrays a' and a" which are obtained by deleting the first row and column, respectively, we have the resulting polynomials:

$z_1 z_2 + z_2 + z_1$ and $z_1 z_2^2 + z_2^2 + 3z_1 z_2 + 2z_2 + 2z_1$,

among which only the latter satisfies the identity (8b) for any (i,j) s.t. i>m or j>n. Thus, we have m=1, n=2, $b(z_1,z_2) = z_1 z_2^2 + z_2^2 + 3z_1 z_2 + 2z_2 + 2z_1$, and, in view of the identity (8a), $c(z_1,z_2) = z_1 z_2 + 2z_2 + z_1 + 1$, which just coinsides with the desired solution and is consistent with the result of the 2D Hankel theory [4].

Example 6 (Example 2 revisited): Applying the algorithm to the array shown in Fig. 2, which corresponds to the righthand side of the congruence (4), we have the polynomial $b(z_1,z_2) = z_1 z_2 + 5z_2 + 5z_1$. Thus, we have the desired denominator $\tilde{b}(z_1,z_2) = z_1 z_2 f(z_1^{-1}, z_2^{-1}) = 5z_1 + 5z_2 + 1$, and, in view of the identity (8a), the numerator $\tilde{c}(z_1,z_2) = 5z_2 + 3$.

Example 7 (Example 3 revisited): Applying the algorithm to the array shown in Fig. 3, we have the desired polynomials $b(z_1,z_2) = z_1^2 z_2^2 + 1$ and $c(z_1,z_2) = z_2 + 1$.

In general, after a proper number of iterations, i.e. at a certain point p, we have a minimal set F of polynomials to which the desired system polynomial f belongs. The other polynomials in F turn out to be not valid at some point away from the point p or fail to satisfy the identity (8b) for some (i,j) s.t. i>m or j>n. The following uniqueness theorem [13] gives a criterion about whether the desired polynomial f has been obtained already at a certain point p.

Theorem 1: Let $F \in FF(a^p)$, $f \in F$ with $s = Deg(f) \leqq p$ and $\Delta = \Delta(a^p)$. If $p - s \notin \Delta$, then the polynomial $f \in VALPOL(a^p)$ having the degree s is unique up to scalar multiplication.

In particular, for the second kind of problem (e.g. Example 2), we have the corollary.

Corollary 1: Let $F \in FF(a^p)$, $p = (4m,0)$ and $f \in F$ with $Deg(f) = (m,m)$. Then, the polynomial $f \in VALPOL(a^p)$ having the degree (m,m) is unique up to scalar multiplication.

It is proven that the computational complexity of the algorithm is of order $O(d^2)$, where d is the size of part of the array which has been examined, i.e. the value $|p|$ for the point p at which the iteration of the algorithm has been stopped. Therefore, in view of Theorem 1, it is of order $O(m^2 n^2)$, provided that the system has degree (m,n).

## 4. Concluding remarks

We have proposed a method for determining the system function and the system degree of a 2D DLSI system over any field which is characterized only by its 2D impulse response array defined on the integral lattice of the first quadrant of the 2D plane. This method is based on an extension of Berlekamp-Massey algorithm to two dimensions [11], where the original (1D) Berlekamp-Massey algorithm is for synthesizing a shortest linear feedback shift register capable of generating a given finite sequence over any field. This approach gives an insight into the concept of 'minimality' in partial realization of 2D discrete linear systems. Our method is also applicable to multidimensional Padé approximation as well as reduction of system degree on the assumption that there exists a proper solution. It is known that the 1D Berlekamp-Massey algorithm is equivalent to Levinson-Trench algorithm. Several authors [14]-[16] have proposed 2D Levinson-type algorithms for identification or approximation of 2D linear systems, where the block Toeplitz structure is scrutinized. Our approach can be distinguished from them by its simplicity and potential of wide applicability. By the way, our method can be applied to not only a 2D impulse response array (first-order data) but also its autocorrelation array (second-order data) [14], since the latter also satisfies a similar form of linear recurring relation. Furthermore, our algorithm has as little computational complexity as the other Levinson-type algorithms, and so our method is effective and efficient in identification of 2D systems having a large degree.

In this paper, we do not deal with the problem on numerical errors incurred during the process of computation and many other important problems in identification and approximation of real 2D systems, e.g. stability, which will be our future problems. In near future, we will extend our approach to the multivariable (i.e. multi-input and multi-output) multidimensional systems.

ACKNOWLEDGEMENT

The author thanks Dr Hiroyoshi Morita of Toyohashi University of Technology for many dicussions.

REFERENCES

[1] R. Eising, 2-D Systems, An Algebraic Approach. Amsterdam: Mathematisch Centrum, 1980.

[2] N. K. Bose, Applied Multidimensional Systems Theory. New York: Van Nostrand Reinhold Comp., 1982.

[3] N. K. Bose (ed.), Multidimensional Systems Theory. Dordrecht: D. Reidel Publishing Comp., 1985.

[4] Y. Kao and C. Chen, "Two-dimensional Hankel theory," Int. J. Control, vol.31, no.2, pp.271-284, 1980.

[5] B. Buchberger, E. V. Krishnamurthy and F. Winkler, "Groebner bases, polynomial remainder sequences and decoding of multivariable codes," in Multidimensional Systems Theory, (Ed. N. K. Bose). Dordrecht: D. Reidel Publishing Comp., 1985, Chapter 8, pp.252-256.

[6] E. V. Krishnamurthy, "Symbolic iterative algorithm for generalised inversion of rational polynomial matrices," J. Symbolic Computation, vol.1, pp.271-281, 1985.

[7] B. Buchberger, "Groebner bases: An algorithmic method in polynomial ideal theory," in Multidimensional Systems Theory, (Ed. N. K. Bose). Dordrecht: D. Reidel Publishing Comp., 1985, Chapter 6, pp.184-232.

[8] Y. Lai and C. Chen, "Reduction of 2-D rational functions," IEEE Trans. Automatic Control, vol.AC-29, no.8, pp.749-752, 1984.

[9] J. Conan, "A recursive procedure for the solution of the minimal partial realization problem for scalar rational sequences," Rev. Roumaine Math. Pures Appl., vol.30, pp.625-645, 1985.

[10] K. A. Prabhu and N. K. Bose, "Impulse response arrays of discrete-space systems over a finite field," IEEE Trans. Acoust., Speech, and Signal Process., vol.ASSP-30, no.1, pp.10-18, 1982.

[11] S. Sakata, "Finding a minimal set of linear recurring relations capable of generating a given finite two-dimensional array," J. Symbolic Computation, vol.5, pp.321-337, 1988.

[12] S. Sakata, "On determining the independent point set for doubly periodic arrays and encoding two-dimensional cyclic codes and their duals," IEEE Trans. Information Theory, vol.IT-27, no.5, pp.556-565, 1981.

[13] S. Sakata, "Synthesis of two-dimensional linear feedback shift registers, "Proceedings of AAECC-5, Menorca, 1987."

[14] L. F. Chaparro and E. I. Jury, "Rational approximation of 2-D linear discrete systems," IEEE Trans. Acoust., Speech, and Signal Process., vol.ASSP-30, no.5, pp.780-787, 1982.

[15] J. H. Justice, "A Levinson-type algorithm for two-dimensional Wiener filtering using bivariate Szego polynomials," Proc. IEEE, vol.65, no.6, pp.882-886, 1977.

[16] T. L. Marzetta, "Two-dimensional linear prediction: Autocorrelation arrays, minimum-phase prediction error filters, and reflection coefficient arrays," IEEE Trans. Acoust., Speech, and Signal Process., vol.ASSP-28, no.6, pp.725-733, 1980.