MASS TRANSPORT IN FINITE AMPLITUDE WAVES

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ABSTRACT

A general scheme is developed which determines the Lagrangian motions of water particles by the Eulerian velocity at their mean positions by use of Taylor's theorem. Utilizing the Stokes finite-amplitude wave theory, the mass transport velocity which includes the effects of higher-order wave components is determined. The fifth-order theory predicts the mass transport velocity less than that given by the existing second-order theory over the whole depth. Limited experimental data for changes in wave celerity in closed wave flumes are compared with the theoretical predictions.
KEYWORDS: Mass Transport, Orbital Motions, Wave Tanks, Stokes Waves

1 INTRODUCTION

In general, the mass transport velocity of a fluid particle under progressive waves is a linear sum of two quantities known as the Stokes drift and the mean Eulerian streaming. (1,5) Stokes drift is a general consequence of the irrotational motion of the fluid while the mean Eulerian streaming arises due to the viscosity in a fluid bounded by a free surface and solid boundaries. For the stokes drift, theoretical expressions have been derived using inviscid wave theories. The magnitude of this drift is first given by Stokes(8) as

$$\bar{u}_L = \frac{H^2 \omega k}{8 \sinh^2 kh} \cdot \cosh 2k(h+y) \qquad (1)$$

in which $\bar{u}_L$ is the mass transport velocity in the Lagrangian reference frame; H, ω, and k are the wave height, the wave frequency, and the wave number, respectively. Skjelbreia (6) investigated the Lagrangian motion of water particles for Stokes' third-order waves. Up to third-order, the Lagrangian mass transport velocity is still given by Eq. (1) with $4a^2$ (a = first-order wave amplitude) instead of H . Dalymple (2) calculated numerically the mass transport velocity in an Eulerian reference frame by using the Stream-function theory presented by Dean (3). These inviscid theories for mass transport may be applied with fair accuracy in deep water during a time not long after the onset of wave motion in which the viscous effects are negligible in core flow of the fluid outside the boundary layer. The existing mass transport theories account only for the first-harmonic linear component of wave motion, and an extention is necessary in order to include the effects of nonlinearities.

2 MATHEMATICAL FORMULATION

Consider a particle initially loacated at point $P(x_o,y_o)$ at $t=0$ in Fig. 1, and examine the particle over one wave period. It is assumed that the coordinate of the mean particle position, $(\bar{x},\bar{y})$, is stationary over one wave period. The coordinate (ξ,ζ) represent the particle displacements with respect to the mean particle position. Let $u_L(x_o,y_o,t')$ and $v_L(x_o,y_o,t')$ represent the horizontal and vertical component of the Lagrangian velocity of the particle at time t'.

Let the Lagrangian and Eulerian velocity vectors of the water particles be denoted by $\vec{u}_L$ and $\vec{u}$, respectively, i.e.,

$$\vec{u}_L = u_L\vec{i} + v_L\vec{j} \; ; \; \vec{u} = u\vec{i} + v\vec{j} \tag{2}$$

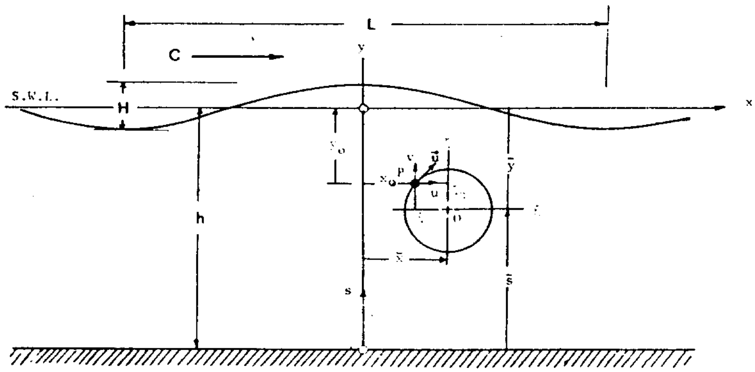

Fig. 1 Definition Sketch: Lagrangian Motion of a Water Particle under Progressive Wave of Finite Amplitude

The Lagrangian velocity of the particle, $\vec{u}_L$, may be related to the Eulerian velocity, $\vec{u}$, by

$$\vec{u}_L(x_o,y_o,t) = \vec{u}\left(x_o + \int_0^t u_L(x_o,y_o,t')dt',\; y_o + \int_0^t v_L(x_o,y_o,t')dt', t\right) \tag{3}$$

Using the relationships, $x_o = \bar{x} + \xi_o$ and $y_o = \bar{y} + \zeta_o$, and noting that

$$\xi = \xi_o + \int_0^t u_L dt' = \int^t u_L dt', \quad \zeta = \zeta_o + \int_0^t v_L dt' = \int^t v_L dt', \tag{4}$$

the Lagrangian velocity of the water particle is related to the Eulerian velocity at the mean poition by

$$\vec{u}_L(x_o,y_o,t) = \vec{u}\left(\bar{x} + \int^t u_L(x_o,y_o,t')dt',\; \bar{y} + \int^t v_L(x_o,y_o,t')dt',\; t\right) \tag{5}$$

in which $\int^t u_L(x_o,y_o,t')dt'$ and $\int^t v_L(x_o,y_o,t')dt'$ represent the integrals evaluated at time t, since the values of the integrals evaluated at time 0 are canceled by ξ_o and ζ_o, respectively, in Eq. (4).

By Taylor's theorem, the right hand side of Eq.(5) may be expanded about the mean position $(\bar{x},\bar{y})$ according to

$$\vec{u}_L(x_o,y_o,t) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \frac{1}{(n-\ell)!\,\ell!} \frac{\partial^n \vec{u}(\bar{x},\bar{y},t)}{\partial x^{n-\ell}\,\partial y^{\ell}} \left[\int^t u_L(x_o,y_o,t')dt'\right]^{n-\ell} \cdot \left[\int^t v_L(x_o,y_o,t')dt'\right]^{\ell} \tag{6}$$

In Eq. (6), the Lagrangian velocity components $u_L(x_o,y_o,t')$ and $v_L(x_o,y_o,t')$ are unknown, and can be related to the Eulerian velocity components at the mean position according to Eq. (6) with t and t' replaced by t' and t", respectively.

Similarly, the Lagrangian velocity components in the right hand side of the resulting equation may be further related to the Eulerian velocity components at the mean position. By substituting the resulting expressions for the Lagrangian velocity components into Eq.(6) successively, the Lagrangian velocity of the water particle may be related solely to the Eulerian velocity at the mean position to the desired accuracy.

The mass transport velocity may be defined by

$$\bar{u}_L = \frac{1}{T} \int_0^T u_L dt, \quad \bar{v}_L = \frac{1}{T} \int_0^T v_L dt \tag{7}$$

in which $T = 2\pi/\omega$ being the wave period.

3 MASS TRANSPORT VELOCITY UNDER STOKES WAVES

For Stokes waves, the Eulerian horizontal and vertical velocity, u and v, are given by

$$u = C \sum_{n=1}^{K} {}_n u \; ; \; v = C \sum_{n=1}^{K} {}_n v \tag{8}$$

in which K is the order of Stokes wave theory being considered, and C = celerity of the wave represented by the Kth-order wave theory. The horizontal velocity components take the form

$$\begin{aligned} {}_1u &= F_1 \cosh ks \cos\theta \;, \quad {}_2u = F_2 \cosh 2ks \cos 2\theta \\ {}_3u &= F_{13} \cosh ks \cos\theta + F_3 \cosh 3ks \cos 3\theta \\ {}_4u &= F_{24} \cosh 2ks \cos 2\theta + F_4 \cosh 4ks \cos 4\theta \\ {}_5u &= F_{15} \cosh ks \cos\theta + F_{35} \cosh 3ks \cos 3\theta + F_5 \cosh 5ks \cos 5\theta \end{aligned} \tag{9}$$

correct to fifth-order. The vertical velocity components take the same form with cosh( ) and cos( ) replaced by sinh( ) and sin( ), respectively. In Eqs.(9), $\theta = kx - \omega t$, $s = y + h$, and k is the wave number.

The Lagrangian horizontal and vertical velocity under Kth-order wave, u_L and v_L, are also assumed to be given by

$$u_L = C \sum_{n=1}^{K} {}_n u_L \; ; \; v_L = C \sum_{n=1}^{K} {}_n v_L \tag{10}$$

After the Lagrangian velocity components in Eq.(6) are expressed by the Eulerian velocity components by successive expansions, the Eulerian velocity components given by Eq. (8) are substituted into Eq. (6). By retaining the terms up to K^{th}-order in Eq. (6), the Lagrangian velocity under a K^{th}-order Stokes wave may be expressed entirely by the Eulerian quantities. When this procedure is carried out for Stokes wave theory at the different orders, the number of terms in the Eulerian quantity increase rapidly with order. The equations for Lagrangian velocity contain 4 terms for a Stokes second-order wave, and successively 16, 64, and finally 292 terms for a Stokes fifth-order wave.

Second-Order Theory

In the Stokes second-order wave theory, the wave height $H = 2a$, and the wave celerity is given by $C = C_o \tanh kh$, $C_o = g/\omega$ being the deep water linear wave celerity. The Lagrangian velocity, $\vec{u}_L$, of the water particle is approximated by the Eulerian velocity, $\vec{u}$, according to

$$\vec{u}_L = C({}_1\vec{u}_L + {}_2\vec{u}_L) = C({}_1\vec{u} + {}_2\vec{u} + \vec{u} \int^t {}_1u dt' + {}_1\vec{u} \int^t {}_1v dt') \tag{11}$$

Substitution of Eqs. (9) for ${}_1\vec{u}$ and ${}_2\vec{u}$ into Eq.(11) and integration according to Eq.(4) yields,

$$\xi = -\frac{H}{2\sinh kh}\cosh k\bar{s}\sin\bar{\theta} - \frac{H^2k}{16\sinh^2 kh}\left(\frac{3}{2\sinh^2 kh}\cosh 2k\bar{s}-1\right)\sin 2\bar{\theta} + (\omega t)\frac{H^2k}{8\sinh^2 kh}\cosh 2k\bar{s} \tag{12a}$$

and

$$\zeta = \frac{H}{2\sinh kh}\sinh k\bar{s}\cos\bar{\theta} + \frac{3H^2k}{32\sinh^4 kh}\sinh 2k\bar{s}\cos 2\bar{\theta} \tag{12b}$$

The components of the mass transport velocity for the Stokes second-order wave given by Eqs.(7) is

$$\bar{u}_L/C_o = \frac{(Hk)^2}{4\sinh 2kh}\cosh 2k\bar{s} \tag{14}$$

It can be readily shown that Eq.(14) is identical with Eq.(1) which was initially reported by Stokes(8).

Third-Order Theory

After collecting the terms up to third-order in Eq.(6), integration yields the horizontal and vertical displacements of the water particles from their mean positions, ξ and ζ, as

$$\begin{aligned}\xi = &-\frac{1}{k}\left[\left(F_1 + F_{13} - \frac{F_1^3}{8}\right)\cosh k\bar{s} + \frac{F_1}{8}(3F_1^2 + 10F_2)\cosh 3k\bar{s}\right]\sin\bar{\theta}\\ &-\frac{1}{2k}\left[F_2\cosh 2k\bar{s} - \frac{F_1^2}{2}\right]\sin 2\bar{\theta}\\ &-\frac{1}{3k}\left[\frac{F_1}{4}(F_1^2 - 5F_2)\cosh k\bar{s} + F_3\cosh 3k\bar{s}\right]\sin 3\bar{\theta}\\ &+\frac{Ct}{2}F_1^2\left(\cosh 2k\bar{s} - F_2\cosh k\bar{s}\,.\,\cosh 2k\bar{s}\,.\,\cos\bar{\theta}\right)\end{aligned} \tag{15a}$$

and

$$\begin{aligned}\zeta = &\frac{1}{k}\left[\left(F_1 + F_{13} - \frac{3}{8}F_1^3\right)\sinh k\bar{s} + \frac{F_1}{8}(F_1^2 + 6F_2)\sinh 3k\bar{s}\right]\cos\bar{\theta}\\ &+\frac{1}{2k}F_2\sinh 2k\bar{s}\cos 2\bar{\theta}\\ &+\frac{1}{3k}\left[F_3\sinh 3k\bar{s} - \frac{3}{4}F_1F_2\sin k\bar{s}\right]\cos 3\bar{\theta}\\ &-\frac{Ct}{2}F_1^3\sinh k\bar{s}\,.\,\cosh 2k\bar{s}\,.\,\sin\bar{\theta}\end{aligned} \tag{15b}$$

The third-order Lagrangian velocity has been reported by Skjelbreia (6) by use of the Taylor series expansion scheme in a different manner. He substituted $\bar{x} + \xi$ for z and $\bar{s} + \zeta$ for s in Eqs.(9) and expanded. The expressions for horizontal and vertical particle positions, ξ and ζ, are found by successive approximations. Equations(15) are identical to the expressions reported by Skjelbreia(6) provided that his expression $3F_1^3$ in $\sin 2\bar{\theta}$ term in Eq.(15a) is changed into $3F_1^2$. This seems to be a typogrpahical error, for the term in question should be a quantity of $O(\varepsilon^2)$. In addition, Skjelbreia(6) included F_{13} into his expression for F_1, which is mathematically inconsistent, because F_{13} is a quantity of $O(\varepsilon^3)$ while F_1 should be of $O(\varepsilon)$.

The components of the mass transport velocity for a Stokes wave at third-order according to the definition given by Eqs.(7) is determined to be

$$\bar{u}_L/C_o = (1 + \varepsilon^2 C_1)\tanh kh\left[\frac{F_1^2}{2}\cosh 2k\bar{s} - \frac{F_1^3}{4}(\cosh k\bar{s} + \cosh 3k\bar{s})\cos k\bar{x}\right] \tag{16a}$$

and

$$\bar{v}_L/C_o = (1 + \varepsilon^2 C_1)\tanh kh[\frac{F_1}{4}(\sinh k\bar{s} - \sinh 3k\bar{s})\sin k\bar{x}] \quad (16b)$$

For the present study, the fifth-order Stokes wave theory presented by Skjelbreia and Hendrickson (7) is utilized. The coefficients in Eqs.(9) are related with those given by Skjelbreia and Herdirickson(7) as $F_1 = \varepsilon A_{11}$, $F_2 = 2\varepsilon^2 A_{22}$, $F_3 = 3\varepsilon^3 A_{33}$, $F_{13} = \varepsilon^3 A_{13}$, $F_4 = 4\varepsilon^4 A_{44}$, $F_{24} = 2\varepsilon^4 A_{24}$, $F_5 = 5\varepsilon^5 A_{55}$, $F_{15} = \varepsilon^5 A_{15}$, $F_{35} = 3\varepsilon^5 A_{35}$. Nishimura et. al. (9) pointed out a minor mistake' in the expression for the fourth-order celerity in the solution given by Skjelbreia and Hendrickson (7) [the sign of + 2592 C^8 in the expression of C_2 should be changed into -2592 C^8]. This correction was accounted for in calculating the wave number, k, and the perturbation parameter, ε, in the present study. In Eqs.(16), $C_1 = (8\cosh^4 kh - 8\cosh^2 kh + 9)/8\sinh^4 kh$ as given by Skjelbreia and Hendrickson (7).

Equations(16) indicate that the mass transport velocity has a component of $O(\varepsilon^2)$ which is dependent on $\bar{x}$ and that the vertical mass transport velocity is nonzero at third-or higher-orders. However, the second term in Eq. (16a) as well as the vertical mass transport velocity tend to average to zero as a water particle travels over a distance equal to one wavelength, so that the long-term mass transport velocity reduces to

$$\bar{u}_L/C_o = (1 + \varepsilon^2 C_1)\tanh kh \frac{F_1^2}{2} \cosh 2k\bar{s} \quad (17)$$

Fourth-Order Theory

According to Eqs. (6)-(8), the components of the mass transport velocity for the fourth-order Stokes wave are determined as follows:

$$\begin{aligned}\bar{u}_L/C_o = (1 + \varepsilon^2 C_1)\tanh kh\{&\frac{1}{32}[F_1^2(3F_1^2 - F_2) - F_1(5F_1^3 + 10F_1F_2 - 32F_{13} \\ &- 16F_1)\cosh 2k\bar{s} + (4F_1^4 + 31F_1^2F_2 + 16F_2^2)\cosh 4k\bar{s}] \\ &- \frac{F_1^3}{4}(\cosh k\bar{s} + \cosh 3k\bar{s})\cos k\bar{x} - \frac{1}{32}[F_1^2(F_1^2 + 8F_2) - 6F_1^4\cosh 2k\bar{s} \\ &- F_1^2(F_1^2 + 8F_2)\cosh 4k\bar{s}]\cos 2k\bar{x}\}\end{aligned} \quad (18a)$$

and

$$\begin{aligned}\bar{v}_L/C_o = (1 + \varepsilon^2 C_1)\tanh kh\{&\frac{F_1^3}{4}(\sinh k\bar{s} - \sinh 3k\bar{s}) \cdot \sin k\bar{x} \\ &- \frac{F_1^2}{32}(F_1^2 + 8F_2)\sinh 4k\bar{s} \cdot \sin 2k\bar{x}\end{aligned} \quad (18b)$$

The terms dependent on $\bar{x}$ in Eqs. (18) tend to average to zero as a water particle travels over a distance equal to one wavelength, so that the long-term mass transport velocity may be given by

$$\begin{aligned}\bar{u}_L/C_o = (1 + \varepsilon^2 C_1)\tanh kh\{&\frac{1}{32}[F_1^2(3F_1^2 - F_2) - F_1(5F_1^3 + 10F_1F_2 - 32F_{13} \\ &-16F_1)\cosh 2k\bar{s} + (4F_1^4 + 31F_1^2F_2 + 16F_2^2)\cosh 4k\bar{s}]\}\end{aligned} \quad (19)$$

Fifth-Order Theory

Due to space limitations, the expressions for displacements of water particles from their mean positions, ξ and ζ, as well as the full expressions for the mass transport velocity, $\bar{u}_L$ and $\bar{v}_L$, are not given here. After the terms dependent on $\bar{x}$ in the mass transport velocity components are eliminated, the long-term mass transport velocity has only the horizontal component given by

$$\bar{u}_L/C_o = (1 + \varepsilon^2 C_1 + \varepsilon^4 C_2)\tanh kh \{\frac{1}{32}[F_1^2(3F_1^2 - F_2) - F_1(5F_1^3 + 10F_1F_2 - 32F_{13} - 16F_1)\cosh 2k\bar{s} + (4F_1^4 + 31F_1^2F_2 + 16F_2^2)\cosh 4k\bar{s}]\} \qquad (20)$$

in which $C_2 = (3840 \cosh^{12} kh - 4076 \cosh^{10} kh - 2592 \cosh^8 kh - 1008 \cosh^6 kh + 5944 \cosh^4 kh - 1830 \cosh^{12} kh + 147)/\{512 \sinh^{10} kh(6\cosh^2 kh - 1)\}$

as given by Skjelbrea and Hendrickson(7).

The orbital motion of the fluid particles whose mean position is located under the wave crest($k\bar{x}$ = 0) at time t = 0 is shown in Figure 2 at the mean depths $\bar{s}/h$=1.0(near the free surface for waves with H/L_o = .0625 and h/L_o = .20 [Case7-B in Dean (3)]. The solid lines represent the orbital motion computed by the Stokes wave theory utilizing the fourth-order Runge-Kutta method of stepwise integration. The dotted lines represent the orbital motions predicted by the expressions for water particle displacements (ξ and ζ)obtained in the study.

The orbital motion predicted by Eqs.(12) has the horizontal mass transport velocity much greater than the computed values for second-order theory. For third- and fifth-order theory, the predicted orbital motions are close to the computed motions. The agreement between the predicted and computed motions improves with increasing vertical distance from the free-surface. The disagreement near the free-surface is due to the large displacement of the water particles from their mean positions which yields a relatively poor approximation by Taylor's theorem.

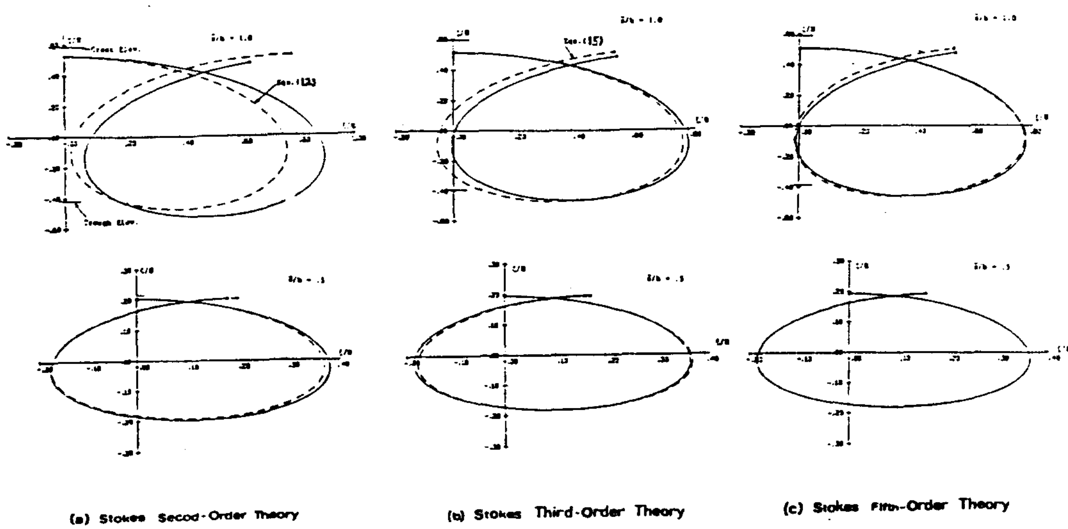

Figure 2. Comparison Between the Predicted and the Computed Orbital Motion of a Water Particle About Its Mean Position at Two Depths ; H/L_o = 0.625, h/L_o = .20, $k\bar{x}$ = 0.

The average (or long-term) horizontal mass transport velocity profiles over depth are plotted in Figure 3 for the three wave conditions.

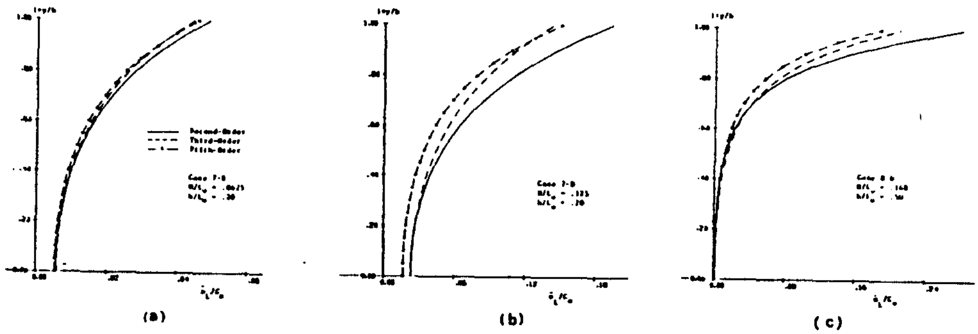

Figure 3. Langrangian Mass Transport Velocity Profiles Over Depth for Stokes Waves.

4 MASS TRANSPORT EFFECTS IN CLOSED WAVE FLUMES

In closed wave flumes such as laboratory channels, the mass flux transported by the waves must be balanced by the mass flux carried by the return flow when a steady-state condition is established. In an inviscid fluid, it may be assumed that this return flow is uniform over depth so that its magnitude, R_c, is determined by

$$R_c = -\frac{1}{h} \int_{-h}^{0} \bar{u}_L(y)dy \qquad (21)$$

in which $\bar{u}_L(y)$ is the Lagrangian mass transport velocity as determined in the previous section. As a result, the wave celerity and angular frequency measured by a fixed observer differ from their values in still water, while the wave number (wavelength) is unchanged. The observed (or apparent) wave frequency, σ, is related to the wavemaker (intrinsic) frequency, ω, by

$$\sigma = \omega + kR_c \qquad (22)$$

Accordingly, the observed (or apparent) wave celerity of the main wave in a closed wave flume, C_c, is given by

$$C_c = C + R_c \qquad (23)$$

in which $C = \omega/k$ is the wave celerity of the Stokes wave in still water.

Iwagaki and Yamaguchi (4) obtained some experimental results of wave celerity in a closed wave flume. Figure 4 shows the comparison between the theoretical values and their experimental results. In Figure 4, the notations S5-O and S5-C represent the Stokes fifth-order theory of Skjelbreia and Hendrickson (7) for

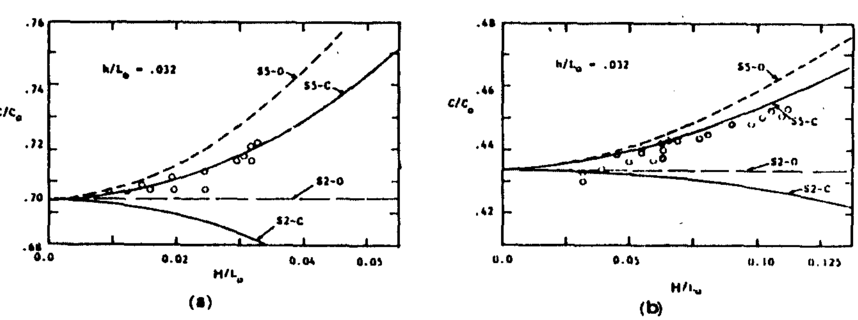

Figure 4. Comparison Between the Theoretical and the Measured Wave Celerity; Data from Iwagaki and Yamuguchi(4).

open and closed wave flume conditions, respectively. The notations S2-O and S2-C represent the Stokes second-order theory applied in the open and closed wave flume condition, respectively, in which wave celerity in the open flume condition is independent of wave height. The measured data show good agreement with the present fifth-order theory which includes the effect of the second-harmonic Stokes wave component in closed laboratory channels.

5 CONCLUSIONS

The existing inviscid theories for mass transport have been extended by use of Taylor's theorem to include higher-harmonic wave components. According to this new mass transport theory, the effects of higher-harmonic wave components in the Stokes waves on the mass transport for both closed and open flume conditions have been examined. In the third-or higher-order approximation for the Lagrangian motion of water particles, the mass transport velocity over one wave cycle is dependent on the initial position of water particles. The decrease in the surface drift velocity ($\bar{u}_L$ at y = 0) due to the existence of the return flow in a closed wave flume is as large as 20% of the magnitude in an open flume for waves with H/L_o=0.14 at $0.3<h/L_o<0.5$. Experimental data for changes in wave celerity show good agreement with the theoretical predictions in the study.

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REFERENCES

1. Craik, A. D. D. (1982) "The drift velocity of water waves," Journal of Fluid Mechanics, 116:187-205.
2. Dalrymple, R. A. (1976) "Wave-induced mass transport in water waves," Journal of Waterways, Harbors and Coastal Engineering Division, ASCE, 102(WW2): 255-264.
3. Dean, R. G. (1974) "Evaluation and development of water wave theories for engineering application," Vol I. Special Report No. 1, U.S. Army Corps of Engineers, Coastal Engineering Research Center. Fort Belvoir, Virginia.
4. Iwagaki, Y., and M. Yamaguchi. (1968) "Studies on cnoidal waves (Sixth report)-Limiting condition for application of cnoidal wave theory," Annuals, D. P. R. I. No. 11B, Kyoto University, Japan (in Japaneses), 477-502.
5. Mei, C. C. (1983) The applied dynamics of ocean surface waves. John Wiley & Sons, Inc., New York, New York, Chapters 8 and 9.
6. Skjelbreia, L. (1959) Gravity Waves: Stokes' Third-order Approximation, Tables of Functions, the Engineering Foundation Council on Wave Research, Berkeley, CA, U.S.A., pp. 231-251.
7. Skjelbreia, L., and J. H. Hendrickson. (1960) "Fifth-order gravity wave theory," Proceedings, 7th Conference on Coastal Engineering, Hague, The Netherlands, 184-196.
8. Stokes, G. G. (1847) "On the theory of oscillatory waves," Transactions, Cambridge Philosophical Society, London, England, 8:441-455.
9. Nishimura, H., M. Isobe, and K. Horikawa (1977) "Higher-order solutions of the Stokes and the Cnoidal waves", Journal of the Faculty of Engineering, The University of Tokyo, Vol. 34, No. 2, pp. 267-293.