OPTIMAL CONTROL OF STOCHASTIC CONTINUOUS DISCRETE SYSTEMS APPLIED TO FMS *

E.K. Boukas[1]

Ecole Polytechnique de Montréal
C.P. 6079, Succursale "A"
Montréal, Québec H3C 3A7 CANADA

Keywords: Piecewise Deterministic Systems, Jump Markov Process

This paper deals with the control of system with controlled jump Markov disturbances. A such formulation was used by Boukas to model the planning production and maintenance of a FMS with failure machines. The optimal control problem of systems with controlled jump Markov process is addressed. This problem describes the planning production and preventive maintenance of production systems. The optimality conditions in both cases finite and infinite horizon, are derived. A numerical example is presented to validate the proposed results.

1. Introduction

The primary aim of control theory is to develop mathematical models and algorithms for the design of complex dynamic systems. The goal of design is to realize a desired system function within the constraints imposed by nature, economics, and the current state of technology. In general, the function of control system is to maintain a given set of variables prescribed bounds. The necessity for control arises from the fact that in operation a physical system is usually subject to perturbations which cannot be exactly predicted, and thus corrected for, in advance. For this reason, the presence of chance phenomena imposes a basic constraint on system performance, and it is thus appropriate to investigate control processes with the aid of stochastic models.

This paper deals with a general class of piecewise deterministic control systems that encompasses the flexible manufacturing systems (FMS) flow control as well as other related models. This class of systems is also known in the litterature as system with jump disturbances.

Krasovskii and Lidski [1961] where the first to study the optimal control problem of systems with jump Markov disturbances. Rishel [1975] has developed a rigorous continuous time dynamic programming approach.

The used Markov process by Krasovskii and Lidski or Rishel to model the disturbances, was not controlled and the rates where supposed to be constant. This formalism has been used to model the problem of control production of the flexible manufacturing systems (see. Olsder and Suri [1980], Kimemia and Gershwin [1983], and Akella and Kumar [1986]).

The aim of this paper consists to extend the formalism used by Rishel or Krasovskii and Lidski. The extension stems the facts that the jump Markov disturbances are controlled, and also from the discontinuities in the system trajectory (of the continuous state).

This extension was motivated by the optimal control problem of the planning production and preventive maintenance of a FMS [see. Boukas [1987] or Boukas and Haurie [1988]).

The paper is organized as follows: In section 2, we give the formulation of the optimal control problem. In section 3, we develop the optimality conditions under some appropriate assumptions. In section 4, we use these conditions to present some numerical results with a sample systems producing one product.

2. Optimal control problem

Consider a system described by the state equations:

$$\dot{z}(t) = f^{\zeta(t)}(z(t), u(t)), \quad \forall t \in [t_n, t_{n+1}) \qquad (2.1)$$

$$z(t_n) = g^{\zeta(t_n)}(z(t_n^-)), \quad n = 0, 1, 2, \ldots \qquad (2.2)$$

In (2.1), $\zeta = (\zeta(t) : t \geq 0)$ is a finite state controlled Markov process and the derivatives changes from $f^{\beta}(z, u, t)$ to $f^{\beta'}(z, u, t)$ as $\zeta(t)$ jumps from β to β'. This process is defined by the jump rates $q(\beta, z, u)$, $\beta \in \mathbf{E}$, $z \in \mathbb{R}^p$, $u(.) \in U(\beta)$, and the transition probabilities $\pi(\beta'|\beta, z, u), \beta, \beta' \in \mathbf{E}$, $z \in \mathbb{R}^p$, $u(.) \in U(\beta)$. The set $\mathbf{E}$ is assumed to be finite, t_n (random variable) is the time of the n^{th} jump of the process ζ which takes its values in $\mathbf{E}$.

At a jump time t_n, the state z is reset at a value $z(t_n)$ defined by Eq. (2.2) where $g^{\beta}(.) : \mathbb{R}^p \mapsto \mathbb{R}^p$ is, for any value $\beta \in \mathbf{E}$, a given function.

This description of the system dynamics generalizes the control framework already studied in depth by Rishel

* This research has been supported by NSERC-Canada, Grants # OGP003 6444, A4952.

[1] K. Boukas is with the mechanical Engineering department, Ecole Polytechnique de Montréal and the GERAD, Montréal, Que., Canada. Tel. (514) 340-4007

[1975]. The generalisation stems from the fact that the jump Markov distrubances are controlled, and also from the discontinuities in the z-trajectory generated by Eq. (2.2).

Remark 3.1: The reader will easily check that this class of system encompass the FMS model described in Boukas and Haurie [1988].

Remark 3.2: The ζ process could be replaced by a more general controlled jump process, with a larger memory, as in Rishel [1977]. We have retained this description for the state of simplicity in the exposition.

The control constraint set $U(\beta)$ is a closed subset of $\mathbb{R}^q$, and for each $\beta \in \mathbf{E}$, $f^\beta(.,.) : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^p$ is bounded continuously differentiable function with bounded partial derivatives in z.

Let $\mathcal{U}$ be a class of control functions $u_\beta(z,t)$, with values in $U(\beta)$ defined on $\mathbf{E} \times \mathbb{R}^p \times \mathbb{R}$, called the class of admissible policies. The control function $u_\beta(z,t)$ is supposed to piecewise continuous in t and continuously differentiable with bounded partial derivatives in z. The continuous differentiability assumption is a severe restriction on the class of optimization problems considered, but it is the assumption which allows th simpler exposition that will be given. We seek a control law u in $\mathcal{U}$ which minimizes the conditionnal expectation:

$$\mathbf{E_u}\{\int_0^T e^{-\rho t} c^\beta(z,u)dt + \phi(z(T))|z(0) = z, \zeta(0) = \beta\} \tag{2.3}$$

where $\rho > 0$ is a continuous discount rate, and $c^\beta(.,.) : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^+$, $\beta \in E$ is a family of cost rate functions, satisfying the same assumptions as f^β.

We now proceed to give a more precise definition of the controlled stochastic process. Let $(\Omega, \mathcal{F})$ be a measure space. We consider a function $X(t,\omega)$ defined as:

$$X : D \times \Omega \mapsto \mathbf{E} \times \mathbb{R}^n, \qquad D \in \mathbb{R}^+$$
$$X(t,\omega) = (z(t,\omega), \zeta(t,\omega)) \tag{2.4}$$

which is measurable with respect to $\mathcal{B_D} \times \mathcal{F}$.

Let $\mathcal{F}_t = \sigma\{X(s,.) : s \leq t\}$ be the σ-field generated by the past observations of X up to time t. we now assume the following:

Assumption 1: The behavior of the system under an admissible policy $u \in \mathcal{U}$ is completely described by a probability measure P_u on the $(\Omega, \mathcal{F}_\infty)$. Thus the process

$$X_u = (X(t,.), \mathcal{F}_t, P_u) \quad t \in D$$

is well defined.

For a given $\omega \in \Omega$ with $z(0,\omega) = z^0$ and $\zeta(0,\omega) = \beta_0$, given, we define:

$$t_1(\omega) = inf\{t > 0 : \zeta(t,\omega) \neq \beta_0\} \tag{2.5}$$
$$\beta_1(\omega) = \zeta(t_1(\omega),\omega) \tag{2.6}$$
$$\vdots$$
$$t_{n+1}(\omega) = inf\{t > t_n(\omega) : \zeta(t,\omega) \neq \zeta(t_n,\omega)\} \tag{2.7}$$
$$\beta_{n+1}(\omega) = \zeta(t_{n+1}(\omega),\omega) \tag{2.8}$$
$$\vdots$$

Assumption 2: For any admissible policy $u \in \mathcal{U}$, and almost any $\omega \in \Omega$, there exists a finite number of jump times $t_n(\omega)$ on any bounded interval $[0,T], t > 0$. Thus the function $X_u(t,\omega) = (\zeta_u(t,\omega), z_u(t,\omega))$ satisfy:

$$\zeta_u(0,\omega) = \beta_0 \tag{2.9}$$
$$z_u(t,\omega) = z^0 + \int_0^t f^{\beta_0}(z_u(s,\omega), u^{\beta_0}(z(s,\omega)))ds \quad \forall t \in [0, t_1(\omega)[, \tag{2.10}$$
$$\vdots$$
$$\zeta_u(t,\omega) = \beta_n(\omega) \tag{2.11}$$
$$z_u(t,\omega) = g^{\beta_n(\omega)}(z_u(t_n^-(\omega),\omega)) + \int_0^t f^{\beta_n(\omega)}(z_u(s,\omega), u^{\beta_0}(z(s,\omega)))ds, \ \forall t \in [0, t_1(\omega)[, \tag{2.12}$$
$$\vdots$$

Assumption 3: For any admissible policy $u \in \mathcal{U}$, we have:

$$P_u[t_{n+1} \in [t, t+dt]|t_{n+1} \geq t_n, \zeta(t) = \beta_n, z(t) = z] = q(\beta_n, z, u^{\beta^n}(z))dt + o(dt) \tag{2.13}$$
$$P_u[\zeta(t) = \beta_{n+1}|t_{n+1} = t, \zeta(t^-) = \beta_n, z(t^-) = z] = \pi(\beta_{n+1}|\beta_n, x, u) \tag{2.14}$$

Optimality conditions

Given the assumption of the section 2., the problem consists to establish the opptimality conditions for the the optimal control problem in finite and infinite horizon. For a control $u = (u_\beta(z,t) \in \mathcal{U}$, let $z^\beta(s;t,z)$ denote the value of the solution of the system (2.1)-(2.2) at time s.

3.1 Finite horizon

In this section, we study the optimal control problem in finite horizon with a terminal cost. Then we use these results to establish the optimality conditions to the optimization problem with infinite horizon.

For any $(t,z) \in [0,T] \times \mathbb{R}^n$ define the value function $v^\beta(z,t)$ associated to the control law $u_\beta(z,t)$ taking value in $U(\beta)$ by:

$$v^\beta(z,t) = \mathbf{E_u}\{\int_t^T e^{-\rho t} c^\beta(z,u)dt + \phi(z(T))|z(0) = z, \zeta(0) = \beta\}$$

3.1.1 Terminal cost

Let us now consider the case where $c^\beta(.) = 0$ for all $\beta \in \mathbf{E}$. This class of optimization problem is always encountered in control applications. It consists to penalize only the final state of the dynamic system.

Theorem 1. For each admissible control law $u_\beta(z,t)$ withvalue in $U(\beta)$, the value function $v^\beta(z,t)$ satisfies the integral equation:

$$v^\beta(z,t) = \phi(z^\beta(T;t,z)) - \int_t^T q(\beta, z^\beta(s;t,z), u_\beta(z^\beta(s),s))v^\beta(z^\beta(s),s)ds + \sum_{\beta'\neq\beta,\beta\in\mathbf{E}} [\int_t^T q(\beta, z^\beta(s;t,z), u_\beta(z^\beta(s),s))v^{\beta'}(z^\beta(s),s))\pi(\beta,z,u)ds] \quad (3.1)$$

Proof: see Boukas [1987]

Theorem 2. For the optimal control law $u_\beta(z,t)$ with value in $U(\beta)$, the value function $v^\beta(z,t)$ satisfy:

1. the system of partial differential equations:

$$v_t^\beta(z,t) + \sum_{i=1}^n v_{z_i}^\beta(z,t)f_i^\beta(z(t),u_\beta(z,t)) - q(\beta,z,u_\beta(z,t))v^\beta(z,t) + \sum_{\beta'\in\mathbf{E}-\{\beta\}} q(\beta,z,u) v^{\beta'}(g^{\beta'}(z(t),\beta'))\pi(\beta'|\beta,z,u) = 0 \forall\beta\in\mathbf{E} \quad (3.2)$$

2. the boundary conditions:

$$v^\beta(z,T) = \phi(z(T)) \quad (3.3)$$

Proof: see Boukas [1987].

Remark To establish the Rishel's optimality conditions, recall that we have:

$$\lambda_{\beta\beta} = -\sum_{\beta\in\mathbf{E}-\{\beta\}} \lambda_{\beta\beta'} \quad (3.4)$$

$$\pi(.) = \frac{\lambda_{\beta\beta'}}{\sum_{\beta\in\mathbf{E}-\{\beta\}}\lambda_{\beta\beta'}} \quad (3.5)$$

By replacing in our optimality conditions we obtain the Rishel's conditions.

Theorem 3. Let $h^\beta(z,t)$, $\beta\in\mathbf{E}$ be bounded continuous functions defined on $[T_0,T]\times\mathbb{R}^n$ such that $h^\beta(z,t)$ is continuously differentiable in z and piecewise continuously differentiable in t.

$$h_t^\beta(z,t) + \sum_{i=1}^n h_{z_i}^\beta(z,t)f_i^\beta(z(t),u_\beta(z,t)) - q(\beta,z,u_\beta(z,t))h^\beta(z,t) + \sum_{\beta'\in\mathbf{E}-\{\beta\}} q(\beta,z,u) h^{\beta'}(g^{\beta'}(z(t),\beta'))\pi(\beta'|\beta,z,u) \geq 0, \forall\beta\in\mathbf{E} \quad (3.6)$$

and $h^\beta(z,T)\leq 0$

Then

$h^\beta(z,T)\leq 0$ on $[T_0,T]\times\mathbb{R}^n$

Proof. see Boukas [1987]

The previous severe assumptions on the class control policy will produce a necessary and sufficient optimality conditions. Without these assumptions the optimiality conditions will be become only sufficient.

Theorem 4. A necessary and sufficient condition that a control $u_\beta(z,t)\in U(\beta)$ be optimum is that for each $\beta\in\mathbf{E}$, its performance function $v^\beta(z,t) = E_u\{\phi(z(T))|z(t) = z, \zeta(t)=\beta\}$ satisfy the partial differential equation:

$$\min_{u(.)\in U(\beta)}\{v_t^\beta(z,t) + \sum_{i=1}^n v_{z_i}^\beta(z,t)f_i^\beta(z(t),u_\beta(z,t)) - q(\beta,z,u_\beta(z,t))v^\beta(z,t) + \sum_{\beta'\in\mathbf{E}\{\beta\}} q(\beta,z,u) v^{\beta'}(g^{\beta'}(z(t),\beta'))\pi(\beta'|\beta,z,u)\} = v_t^\beta(z,t) + \sum_{i=1}^n v_{z_i}^\beta(z,t)f_i^\beta(z(t),u_\beta(z,t)) - q(\beta,z,u_\beta(z,t)) v^\beta(z,t) + \sum_{\beta'\in\mathbf{E}-\{\beta\}} q(\beta,z,u)v^{\beta'}(g^{\beta'}(z(t),\beta')) \pi(\beta'|\beta,z,u) = 0, \forall\beta\in\mathbf{E} \quad (3.7)$$

2. the boundary conditions:

$$v^\beta(z,T) = \phi(z(T)) \quad (3.8)$$

Proof. see Boukas [1987].

3.1.2 Cost defined by an integrale

This class of optimization problem can formulated in the optimization problem of the sub-section 3.1.1.

Theorem 5. A necessary and sufficient condition that a control $u_\beta(z,t)\in U(\beta)$ be optimum is that for each $\beta\in\mathbf{E}$, its performance function $v^\beta(z,t) = E_u\{\int_0^T e^{\rho t}c^\beta(z(t), u(t))dt|z(t) = z, \zeta(t)=\beta\}$ satisfy the partial differential equation:

$$\rho v^\beta(z,t) = \min_{u(.)\in U(\beta)}\{c^\beta(z,u) + v_t^\beta(z,t) + \sum_{i=1}^n v_{z_i}^\beta(z,t)f_i^\beta(z(t),u_\beta(z,t)) - q(\beta,z,u_\beta(z,t)) v^\beta(z,t) + \sum_{\beta'\in\mathbf{E}-\{\beta\}} q(\beta,z,u)v^{\beta'}(g^{\beta'}(z(t),\beta')) \pi(\beta'|\beta,z,u)\}, \qquad \forall\beta\in\mathbf{E} \quad (3.9)$$

2. the boundary conditions:

$$v^\beta(z,T) = \phi(z(T)) \quad (3.10)$$

Proof. see Boukas [1987].

3.2 Optimization problem over infinite horizon

Let now $t \to \infty$, and suppose that the final cost converges to 0 as $T \to \infty$. Let $\mathcal{U}$ be the set of admissible laws with value in $U(\beta)$ with the same assumption in previous section. This class $\mathcal{U}$ is such that for each β, the mapping $u_\beta(.) : z \to U(\beta)$ is sufficiently smooth. Thus for each control law $u \in \mathcal{U}$ is associated a probability measure P_u on $(\Omega, \mathcal{F})$ such that the process (z, ζ) is well defined and the cost (3.3) if finite.

Theorem 6. A necessary and sufficient condition that a control $u_\beta(z) \in U(\beta)$ be optimum is that for each $\beta \in \mathbf{E}$ its performance function $v^\beta(z) = E_u\{\int_0^\infty e^{-\rho t} c^\beta(z(t), u(t))dt | z(t) = z, \zeta(t) = \beta\}$ satisfy the partial differential equation:

$$\rho v^\beta(z) = \min_{u(.)\in U(\beta)} \{c^\beta(z,u) + \sum_{i=1}^{n} v^\beta_{z_i}(z) f^\beta_i(z(t), u_\beta(z)) - q(\beta, z, u_\beta(z))v^\beta(z) + \sum_{\beta' \in \mathbf{E}-\{\beta\}} q(\beta, z, u) v^{\beta'}(g^{\beta'}(z(t), \beta'))\pi(\beta'|\beta, z, u)\} \forall \beta \in \mathbf{E} \tag{3.11}$$

Proof. see Boukas [1987].

4. A numerical results

To illustrate the application of the previous results, let us consider a sample system which produce a single product. Such problem have been considered by many authors (see Akella and Kumar [1986], Sharifnia [1988], Bielecki and Kumar [1987] Malhamé and Boukas [1989]).

4.1 A model

Let $u(t)$ be the production rate of the work station at time t. Let $\lambda_{\beta\beta'}(a, v)$ be the transition rate from state β to state β' for the work station at time t. Let d be a given demand rate for the considered part.

We consider in this example that the aging of the work station is proportional to the production of this work station at time t. Thus the cumulative age is the solution of the following differential equation

$$\frac{da(t)}{dt} = u(t), \quad \forall t > T \tag{4.1}$$

$$a(T) = 0, \tag{4.2}$$

where T is the last restart time of the work station.

We assume in this example, that the intervention restores the age at a zero value.

The state equation of the inventory level is given by

$$\frac{dx(t)}{dt} = u(t) - d, \tag{4.3}$$

$$x(0) = x^0, \tag{4.4}$$

where x^0 is a given initial inventory

The operational state of the system is described by a controlled jump process with 3 possible states, indexed over the set $E = \{1, 2, 3\}$.

The cost rate $\psi^\beta(a, x, \pi)$ is given by the following expression

$$\psi^\beta(a, x, \pi) = c^+x^+ + c^-x^- + c^\beta, \quad \forall \beta \in E \tag{4.5}$$

where:
c^+, c^- are positive constants, and x^+, x^-, and c^β are defined by:

$$x^+ = max(0, x);$$
$$x^- = max(0, -x);$$
$$c^\beta = [c_1 Ind\{\zeta(t) = 2\} + c_2 Ind\{\zeta(t) = 3\}],$$

with

c_1 : cost rate (positive constant) applying to machine's repair activity;

c_2 : cost rate (positive constant) applying to machine's preventive maintenance activity.

Applying the results of Boukas [1987] or Boukas and Haurie [1988] concerning the approximation technique for the numerical solution on the DP equation, we obtain the corresponding discrete Dynamic Programming equation.

$$V^\beta_h(z) = \min_{\pi\in\Pi(\beta)} \{\frac{\psi^\beta(z,\pi)}{Q^\beta_h(z,\pi)[1 + \frac{\rho}{Q^\beta_h(z,\pi)}]} + \frac{1}{[1 + \frac{\rho}{Q^\beta_h(z,\pi)}]} [\sum_{z'\in G} p^\beta_h(z, z', \pi)V^\beta_h(z') + \sum_{\beta'\in E-\{\beta\}} p^\beta_h(z, \beta, \beta', \pi)V^{\beta'}_h(\varphi(a, \beta'), x)]\}, \quad \forall \beta \in E, \quad \forall z \in G \tag{4.6}$$

where:

G : is a finite grid;

$\Pi(\beta)$: control constraint set;

$$Q^\beta_h(z,\pi) = -q_{\beta\beta}(z,\pi) + \frac{\sum_{i=1}^n |F_i(z,\pi)|}{h_i}$$

$$p^\beta_h(z, z \pm e_i h, \pi) = \frac{F^\pm_i(z,\pi)}{h_i Q^\beta_h(z,\pi)}$$

$$p^\beta_h(z, \beta, \beta', \pi) = \frac{q_{\beta\beta}(z,\pi)}{Q^\beta_h(z,\pi)}$$

with

$$F^+_i(z,\pi) = max(0, F_i(z,\pi))$$
$$F^-_i(z,\pi) = max(0, -F_i(z,\pi))$$

For a given policy π, we introduce the operators T_π and T^* acting on $V_h = (V^\beta_h)_{\beta\in E}$, and defined by

$$T_\pi(V_h)(\beta, z) = \frac{\psi^\beta(z,\pi)}{Q_h^\beta(z,\pi)[1+\frac{\rho}{Q_h^\beta(z,\pi)}]} + \frac{1}{[1+\frac{\rho}{Q_h^\beta(z,\pi)}]}$$
$$[\sum_{z'\in G} p_h^\beta(z,z',\pi)V_h^\beta(z') + \sum_{\beta'\in E-\{\beta\}} p_h^\beta(z,\beta,$$
$$\beta',\pi)V_h^{\beta'}(\varphi(a,\beta'),x)], \quad (4.7)$$

$$T^*(V_h)(\beta, z) = \min_{\pi\in\Pi(\beta)}\{T_\pi(V_h^\beta(z))\} \quad (4.8)$$

where the constraint set $\Pi(\beta)$ is defined for each β by:

$\Pi(1) = \{\pi = (u,v) : 0 \le u \le u^{max}, 0 \le v \le v^{max},\}$
$\Pi(2) = \Pi(3) = \{(0,0)\}$.

The discrete problem satisfies the properties guaranteeing the existence of a solution (see Boukas and Haurie [1988] or Boukas [1987]). This solution may be obtained by a successive approximation method (for a recent presentation of the method see Bertsekas [1987]).

The successive approximation algorithm operates as follows:

For a given finite difference interval h:

step 1 : Choose $\varepsilon \in \mathbb{R}_+$, set: $n := 1$, and $(V_h^\beta)^n(z) := 0$, $\forall\beta\in E$, $\forall z \in G$

step 2 : set $(V_h^\beta)^{n-1}(z) := (V_h^\beta)^n(z)$, $\forall\beta\in E$, $\forall z\in G$

step 3 : determine the policy π^n such that: $T_{\pi^n}(V_h)(\beta, z) = T^*(V_h)(\beta, z)$, $\forall\beta\in E$, $\forall z \in G$

step 4 : Test:

$$\bar{c} := \min_{\substack{\beta\in E\\ z\in G}}\{(V_h^\beta)^n(z) - (V_h^\beta)^{n-1}(z)\}$$
$$\underline{c} := \max_{\substack{\beta\in E\\ z\in G}}\{(V_h^\beta)^n(z) - (V_h^\beta)^{n-1}(z)\}$$
$$c_{min} := \frac{\rho}{1-\rho}\bar{c}$$
$$c_{max} := \frac{\rho}{1-\rho}\underline{c}$$

if $|c_{max} - c_{min}| \le \varepsilon$ then stop $\pi^* = \pi^n$; else let $n = n+1$, and go the step 2.

4.2 The data

For our work station system we have used the following data:

1. control constraints: $u \in [0, .19]$, $v \in [0, 6\,10^{-3}]$
2. demand rate: $d = .18$
3. jump rates:
$\lambda_{12}(a, v) = .02(1 - e^{-8\times10^{-4}a})$ represents the transition rate from the state "operational" to the state "failure", of the machine;
$\lambda_{21}(a, v) = 7.33\,10^{-2}$ represents the transition rate from the state "failure" to the state "operational", of the machine;
$\lambda_{31}(a, v) = 1.46\times10^{-2}$ represents the transition rate from the state "preventive maintenance" to the state "operational", of the machine;
4. discount rate: $\rho = .01$
5. instantaneous cost: $\psi^\beta(a, x, \pi) = c^+x^+ + c^-x^- + c^\beta$, $\forall\beta\in E$
where the coefficients are given in Table 1.

c^+	c^-	c^1	c^2	c^3
1	2	0	10	1

Table 1: Cost rates.

4.3 Implementation of the numerical method

The implementation of the approximation technique necessitates the use of a finite grid G. one has thus to impose some boundary conditions to describe the behavior of the system at the border of G. These boundary conditions are described below and correspond to an additional approximation of the original problem. However it is felt that these boundary conditions are realistic and that the influence of this approximation will be negligible. The grid G associated with our example is a subset of $\mathbb{R}^2$. For a given finite difference interval h, with $(n_1 \times n_2)$ points, this grid is defined by:

$$G = \{z^i \in \mathbb{R}^2, i = (i_1, i_2)\,|$$
$$z_1^i = z_1^0 + (i_1 - 1)h_1 : z_1^0 = 0,\ z_1^{n_1} = 200,\ i_1 \in \{1\cdots n_1\}$$
$$z_2^i = z_2^0 + (i_2 - 1)h_2 : z_2^0 = -80,\ z_2^{n_2} = 80,\ i_2 \in \{1\cdots n_2\}\}$$

To approximate the solution of the infinite horizon problem, we impose some boundary conditions. For each $\beta \in E$, when a component of z reaches its boundary, the control is imposed. This control forces the system to jump with probability one to a specified state.

Note that the boundary conditions imposed below are not the outcome of the optimization problem, and if the grid is large enough, they do not affect the solution.

We have used for our work station example, the following boundary conditions.

A. The work station is operational:

1. $x(t)$ is at its lower bound:
   - if the age of the work station is not at its upper bound, force this work station to produce with its maximum production rate.
   - if the age of the work station is at its upper bound, send this work station to preventive maintenance.
2. $x(t)$ is at its upper bound:
   - if the age of the work station is not at its upper bound, force this work station not to produce.
   - if the age of the work station is at its upper bound, send this work station to preventive maintenance.
3. $x(t)$ is neither at its upper bound nor its lower bound:
   - if the age of the work station is at its upper bound, send this work station to preventive maintenance.

if the age of the work station is neither at its upper bound nor at its lower bound use the previous equations to determinate the control law.

B. work station is not operational:

In this case the constraint associated control set is empty.

4.4 Results

The successive approximation algorithm has been implemented to approximate the solution of the work station system with the data and grid G presented at sub-sections 4.2 and 4.3. The grid considered has $(41 \times 41 \times 3)$ points. The results obtained in the case when the work station is operationnal are presented in Figures 4.1 to 4.3.

From the results we can conclude the following:

A. Control policy

When the stock level is negative which represents a backlog, operational work station is forced to produce at its maximum production rate even if it is aged, and no preventive maintenance action is permitted.

When the stock level is positive which represents a surplus, the preventive maintenance action of the operational work station can be considered. This action will depend on the age of this work station.

From these results, we state that the production rate control is similar to Akella and Kumar's policy [1986]. The preventive maintenance rate is a function of the age of the work station. The preventive maintenance activity in considered only if the stock level is positive.

B. Value function V

Figure 4.3 show the evolution of the value function V vs the age a of the work station and the stock level x in each discrete system state. This figure indicates that the form of value function can be approximated by a quadratic form. This confirms the assumption used by the researchers of the MIT. Therefore the method used first by Kimemia and Gershwin [1983], could also be adopted in our case.

5. Conclusion

We have formulated the optimal control of the production and preventive maintenance rates in the case of a simple system. The optimality conditions are established in both cases: finite and infinite horizon. These optimality conditions are translated by a hyperbolic partial differential system. To compute the solution of this system in the case of tthe infinite horizon, we have proposed an heuristic approach. A validation of our results is illustrated by the numerical results. Concerning, the production rate, our results are similar to those obtained by Akella and Kumar [1986]. The preventive maintenance acitivity is a function of the age of the work station and the level of the stock.

6. References

R. A. Akella and P. R. Kumar, 1986, "Optimal Control of Production Rate in Failure Prone Manufacturing system", *IEEE, Trans. on Automatic Control*, Vol. AC-31, no. 2, pp. 116-126.

D. P. Bertsekas, 1987, "Dynamic Programming: Deterministic and stochastic Models", Prentice Hall, Inc.

T. Bielecki and P. R. Kumar,1987, "Optimality of Zero-Inventory Policies for Unreliable Manufacturing Systems", Operations research (special issue in Manufacturing System).

E. K. Boukas, 1987, "Commande optimale stochastique appliquée aux systèmes de production", Ph. D Thesis, Ecole polytechnique de Montréal, Université de Montréal.

E. K. Boukas and A. Haurie, 1987, "Optimality Conditions For Continuous Time Systems With Controlled Jump Markov Disturbances: Application To Production And Maintenance Scheduling ", Proceeding INRIA 8^{th} International Conference Springer Verlag on *Analysis and Optimization of Systems*, Antibes June 1988.

J. Kimemia, S. B. Gershwin, 1983, "An algorithm for computer control of a flexible manufacturing system". *IIE. Transactions*, Dec., vol. 15, no. 4, pp. 353-362.

O. Z. Maimon and S. B. Gershwin, 1987, "Dynamic Scheduling And Routing For Flexible Manufacturing Systems", IEEE. Internationnal Conference on Robotics and Automation, 1987.

G. J. Olsder, R. Suri, 1980, "Time optimal of parts-routing in a manufacturing system with failure prone machines". Proc. 19th IEEE. Conference on decision and control Alburquerque, New Mexico, pp. 722-727.

R. Rishel, 1975, "Control of systems with jump markov disturbances", *IEEE Transactions on Automatic Control*, April, pp. 241-244.

D. Vermes, 1985, "Optimal Control of Piecewise Deterministic Markov Proces", Stochastics, 14, pp. 165-208.

S. B. Gershwin R. Akella and Y. F. Choong, 1985 "Short term production sheduling of an automated manufacturing facility", IBM Journal of Research and Development, vol. 29, No. 4, July.

P. Algoet, 1988 "Flow balance equations for the steady-state ditribution of a flexible manufacturing system", To appear, IEEE, Trans. on AC, July 1989.

A. Sharifnia, 1988,"Production Control of a Manufacturing system with Multiple Machine states", IEEE Trans. on AC, vol. 33, no. 7, july 1988.

M. Caramanis and A. Sharifnia, 1988, "Design of Near-Optimum Flow Controllers For Flexible Manufacturing Systems." Technical Report, Boston University Department od Manufacturing Engineering, Feb. 1988.

R. Malhamé and E.K. Boukas, " Transient and steady-states of the statistical flow balance equations in Manufacturing systems", submitted for publications, 1989.

Wonham W. M., Random differentiel equations in control theory, in A. T. Bharucha-Reid, NY, Academic Press, 1969.

Davis M.H.A. Piecewise Deterministic Markov Process: A General Class of Non Diffusion Models", J. Royal Statist. Soc. (B), 46(3), 1984.

Krasovskii N.N. and E.A. Lidski, "Analytical design of Controllers in Stochastic System with Velcity Limited Controlling action",Appl. Math. Mech. vol. 25, pp. 627-643, 1961.

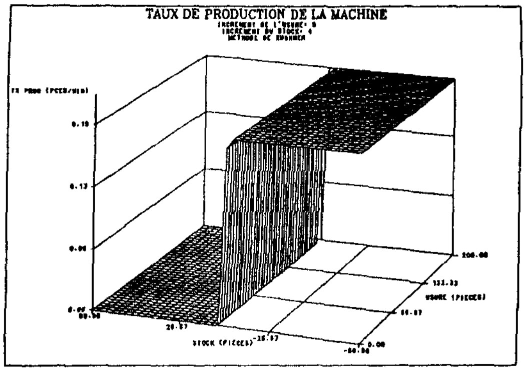

Fig. 4.1 - The work station is in operational state.

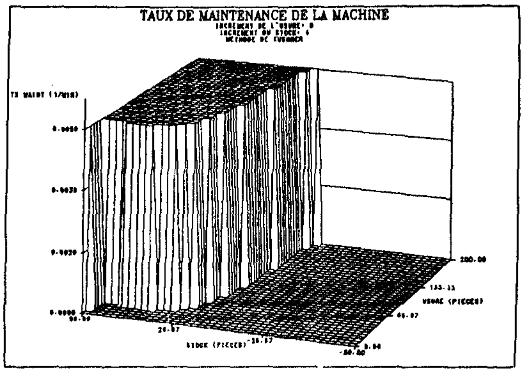

Fig. 4.2 - The work station is in operational state.

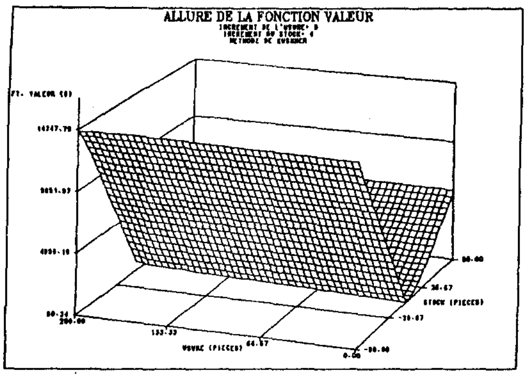

Fig. 4.3 - The work station is in operational state.