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APERIODICITY CONDITIONS FOR POLYNOMIALS WITH UNCERTAIN COEFFICIENT PARAMETERS

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Abstract: Aperiodicity of interval polynomials is studied. Aperiodicity is normally defined as a property such that all the roots are simple and negative real, while interval polynomials are referred to as polynomials with coefficients lying within specified closed intervals on the real axis. Several conditions for aperiodicity, including an exact one, are derived. Comments on them are given in contrast to the work by Soh and Berger, who also considered the problem with a modified definition of aperiodicity.

I. INTRODUCTION

One of possible approaches to tackle stabilty robustness problems is to express uncertain system characteristics by interval polynomials. Pertaining to the stability of interval polynomials, Kharitonov's theorem is seminal(Kharitonov, 1978). This paper studies a much stronger property than stability, namely, aperiodicity of interval polynomials. A necessary and sufficient condition for aperiodicity and two sufficient ones are provided. Several comments are given in contrast to the relevant work by Soh & Berger(1988a, b), who also considered aperiodicity problems for interval polynomials.

II. MAIN RESULTS

A polynomial is said to be aperiodic, if all the roots are simple and negative real(Jury, 1974). We will consider aperiodicity of interval polynomials given by:

$$f(s):=s^n+a_1s^{n-1}+\dots+a_n \quad \dots \quad (1)$$

$$c_i \leq a_i \leq d_i, \quad i=1,\dots,n \quad \dots \quad (2)$$

A key role played in the proof of main results is:

(Lemma 1)(Fuller 1955, Jury 1974)

A polynomial given by (1) is aperiodic, if and only if the polynomial defined by

$$g(s):=f(s^2)+s(df(s^2)/ds^2) \quad \dots \quad (3)$$

is a Hurwitz one.

We see that g(s) is a real polynomial of order 2n and is explicitly given by

$$g(s)=s^{2n}+b_1s^{2n-1}+b_2s^{2n-2}+\dots+b_{2n}, \quad \dots \quad (4)$$

where

$$b_{2k}=a_k,\ b_{2k-1}=(n-k+1)a_{k-1},\ k=1,\dots,n. \quad \dots \quad (5)$$

The coefficients b_{2k} and b_{2k-1} satisfy,

$$\underline{b}_{2k}=c_k \leq b_{2k} \leq d_k=\overline{b}_{2k} \text{ and}$$

$$\underline{b}_{2k-1}=(n-k+1)c_{k-1} \leq b_{2k-1} \leq (n-k+1)d_{k-1}=\overline{b}_{2k-1} \quad \dots \quad (6)$$

Thus, g(s) again is an interval polynomial and Kharitonov's theorem(Kharitonov, 1978), which gives a necessary and sufficient condition for the Hurwitz property of interval polynomials, seems to be applicable. However, as we see from (5), there certainly exists linear dependency between adjacent two coefficients. This renders the result of application rather conservative. To avoid this point, we notice that g(s) is a polytope of polynomials(see Barmish, 1988):

$$g(s)=\sum_{l=1}^{m} w_l g^l(s), \sum_{l=1}^{m} w_l=1, \quad w_l \geq 0 \quad \dots \quad (7)$$

where $g^l(s)$, $l=1,\dots,m(:=2^n)$ are extreme polynomials of the form (4) with:

$$b_{2k}=\underline{b}_{2k},\ b_{2k-1}=\underline{b}_{2k-1} \text{ or}$$

$$b_{2k}=\overline{b}_{2k},\ b_{2k-1}=\overline{b}_{2k-1},\ k=1,\dots,n \quad \dots \quad (8)$$

As to the root distribution of polytope of polynomials, at hand is the edge theorem(Bartlett et al., 1988), which claims that we have only to check edge polynomials. Invoking the theorem, together with Lemma 1, we obtain the first main result:

(Theorem 1)
We pick two different polynomials from $g^{1}(s)$ and form all the possible pairs. Then, interval polynomials (1) with (2) are aperiodic, if and only if convex combinations of two polynomials for each pair are Hurwitz.

The methods to check the Hurwitz property of convex combinations of two polynomials are elaborated in Fu and Barmish(1987) and Mori(1988). Utilizing , say, the former approach, we can obtain an exact condition for aperiodicity of convex combinations of two polynomials:

(Theorem 2)
Any convex combination of two polynomials,

$$f_i(s):=s^n+a_{i1}s^{n-1}+\ldots+a_{in}, \quad i=1,2, \quad \ldots \quad (9)$$

is aperiodic, if and only if $f_1(s)$ is aperiodic and the matrix $(H_1)^{-1}H_2$ has no eigenvalues in the open left half of the real axis. Here, H_i, $i=1,2$ are the Hurwitz matrices corresponding to $g_i(s)$, $i=1,2$, which are polynomials of the form (3) for $f_i(s)$ and given by:

$$H_i:=\begin{bmatrix} n & (n-1)a_{i1} & (n-2)a_{i2} & \cdots & . \\ 1 & a_{i1} & a_{i2} & \cdots & . \\ 0 & n & (n-1)a_{i1} & \cdots & . \\ 0 & 1 & a_{i1} & \cdots & . \\ . & 0 & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & a_{in} \end{bmatrix} \quad \ldots\ldots \quad (10)$$

Though Theorem 1 gives an exact aperiodicity condition for interval polynomials, a major drawback is that the number of polynomials to be checked grows explosively as n increases. A way to evade this at the expense of sharpness is returning to the preceding argument; application of Kharitonov's theorem to (4). This can be done by regarding each coefficient as independent. Application of the weak version and the strong one of Kharitonov's theorem yields the following two sufficient conditions for aperiodicity of interval polynomials.

(Theorem 3)
Denote the polynomials (4) whose coefficients take either of the equalities in (6) by G_1. Then, interval polynomials (1) with (2) are aperiodic, if all members of G_1 are Hurwitz.

(Theorem 4)
Define the following four polynomials associated with g(s).

$$g_1(s):=s^{2n}+\underline{b}_1s^{2n-1}+\overline{b}_2s^{2n-2}+\overline{b}_3s^{2n-3}+\underline{b}_4s^{2n-4}+\underline{b}_5s^{2n-5}+\ldots \quad (11)$$

$$g_2(s):=s^{2n}+\overline{b}_1s^{2n-1}+\overline{b}_2s^{2n-2}+\underline{b}_3s^{2n-3}+\underline{b}_4s^{2n-4}+\overline{b}_5s^{2n-5}+\ldots \quad (12)$$

$$g_3(s):=s^{2n}+\underline{b}_1s^{2n-1}+\underline{b}_2s^{2n-2}+\overline{b}_3s^{2n-3}+\overline{b}_4s^{2n-4}+\underline{b}_5s^{2n-5}+\ldots \quad (13)$$

$$g_4(s):=s^{2n}+\overline{b}_1s^{2n-1}+\underline{b}_2s^{2n-2}+\underline{b}_3s^{2n-3}+\overline{b}_4s^{2n-4}+\overline{b}_5s^{2n-5}+\ldots. \quad (14)$$

In order that interval polynomials (1) with (2) are aperiodic, the Hurwitz property of the polynomials $g_i(s)$, $i=1,2,3,4$ is sufficient.

In this way we have only to check four polynomials. Note that the cardinality of G_1 is m^2.

III. COMMENTS

Soh & Berger studied conditions for the roots of interval polynomials to be located in sectors on the complex plane(1988 a). Their results reduce to the aperiodicity condition when the sectors reduce to the real axis. A considerable difference lies, however, between their definition of aperiodicity and the one commonly accepted and employed here. This is due to the point that they allowed repeated roots. To make this difference clear, we write their definition as "aperiodicity". Then, with this modified definition, their result reads:

(Theorem 5)(Soh and Berger, 1988 a)
Let the polynomial family (1) with coefficients satisfying either of the equalities of (2) be denoted by F_1. Then, interval polynomials (1) with (2) are "aperiodic" if and only if every member of F_1 is "aperiodic".

Later, they also presented an improved version over the above using Kharitonov's theorem for polynomials with complex coefficients:

(Theorem 6)(Soh and Berger, 1988 b)
Interval polynomials (1) with (2) are "aperiodic", if and only if 8 members of F_1 are so.

Comparing the present results with these two theorems, we find that Theorem 3 and Theorem 4 almost parallel with Theorem 5 and Theorem 6, respectively. Note that Theorems 3 and 4 give only sufficient conditions, while Theorems 5 and 6 necessary and sufficient ones. However, the property required to be checked is the stability in the former theorems, while in the latter "aperiodicity", for which proposed methods to check are rather unwieldy(Jury and Pavlidis, 1962). The results of this section reveal the difference of the criteria derived from the different definitions.

IV. CONCLUDING REMARKS

Several conditions for interval polynomials to be aperiodic are provided. A necessary and sufficient condition for aperiodicity of convex combinations of two polynomials is also derived. Among them is an exact aperiodicity condition for interval polynomials, but the procedures for checking becomes rather messy as the order of polynomials increases. It seems that contriving much simpler conditions, without losing exactness, would be possible. This direction is currently being pursued.

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