'89 KACC 1989. 10. 27～28

Design of the Multi-Input Deadbeat State Regulator with the Minimal Input Energy

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This paper discusses the regulator problem of minimizing the input energy for the multi-input linear time-invariant discrete-time system with the zero terminal state. The optimal inputs are expressed by the state feedback form and they are made up of three phases. The optimal feedback gains are independent of the initial state.

1 . Introduction

The fixed terminal state regulator problem, which fixes the terminal state to zero at a specified finite time and minimizes a performance index, is an important problem in design of the control system. For the digital control system formulated by the discrete-time system, the input of satisfying the fixed terminal state condition can be obtained by the state feedback, and its gain does not depend on the initial state. It is called the state deadbeat control (finite time settling control) [1], and this property is a significant advantage that the analogue control system expressed by the continuous-time system does not have. We have obtained the general deadbeat inputs which are restricted to the constant gain state feedback [2]. Furthermore, we have obtained *the deadbeat principle* when the inputs are not restricted to the constant gain state feedback [3][4]. The state feedback law which minimizes the input energy has been obtained for a single-input system [3], but it has not obtained for a multi-input system yet.

This paper discusses the regulator problem of minimizing the input energy for the multi-input linear time-invariant discrete-time system with the zero terminal state. The performance index is the sum of the scalar products of the input vectors. By using *the deadbeat principle*, the system is transformed into the new time-variant system with the time-variant structure of the new input. The performance index is transformed into the sum of the quadratic forms of the new state vectors and the new input vectors with the cross terms, and the new terminal state is free. The optimal inputs are expressed by the state feedback form and they are made up of three phases. The optimal feedback gains are independent of the initial state.

2 . Problem Formulation

Consider a reachable discrete-time linear time-invariant system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad (1)$$

where $\mathbf{x}(t)$ is an $n \times 1$ state vector, $\mathbf{u}(t)$ is an $m \times 1$ input vector, $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ and $n \times m$ real constant matrices respectively, and t's are zero or positive integers. It is assumed that the matrix $\mathbf{B}$ is of full rank m.

At the terminal time τ, we fix the terminal state $\mathbf{x}(\tau)$ to zero. That is, we add the fixed terminal state condition (state deadbeat condition)

$$\mathbf{x}(\tau) = \mathbf{0}. \qquad (2)$$

And we consider the problem of minimizing the input energy

$$J(\tau) = \frac{1}{2}\sum_{t=0}^{\tau-1} \mathbf{u}(t)'\mathbf{u}(t). \qquad (3)$$

If we apply the Lagrange multiplier method to this problem directly, the optimal solution is given as the solution of the two-point boundary value problem (TPBV problem). Generally speaking, it is difficult to solve this TPBV problem, and even if the solution is found, it is obtained by the open-loop form. Therefore, we describe *the deadbeat principle* in order to solve this TPBV problem.

3 . The Deadbeat Principle

First, we describe the lemma with respect to the Luenberger reachable canonical form [5].

[*Lemma*]

By the assumption that the system (1) is reachable and $\mathbf{B}$ is of full rank m, the positive integers $n_i (i = 1, \cdots, m)$ called the reachability indices are determined uniquely when $\mathbf{A}$ and $\mathbf{B}$ are given. The reachability indices n_i's are numbered as follows for the sake of convenience:

$$n_1 + \cdots + n_m = n, \qquad (4)$$

$$n \geq n_1 \geq \cdots \geq n_m \geq 1. \qquad (5)$$

Furthermore, there exist coordinate transformations of state vectors and input vectors

$$\mathbf{x}(t) = \mathbf{S_1}\bar{\mathbf{x}}(t), \quad \mathbf{u}(t) = \mathbf{S_2}\bar{\mathbf{u}}(t), \qquad (6)$$

such that the system (1) is transformed into the following Luenberger reachable canonical form :

$$\bar{\mathbf{x}}(t+1) = (\mathbf{J_0} + \mathbf{EA^*})\bar{\mathbf{x}}(t) + \mathbf{E}\bar{\mathbf{u}}(t), \qquad (7)$$

where $\mathbf{J_0}$ is a nilpotent Jordan matrix constructed from $n_i \times n_i$ nilpotent Jordan blocks $(i = 1, \cdots, m)$ as follows :

$$\mathbf{J_0} = \text{block diag}\,\{\mathbf{J}(n_1), \cdots, \mathbf{J}(n_m)\}, \qquad (8)$$

where

$$\mathbf{J}(n_i) = \begin{bmatrix} 0 & 1 & & & \\ & \cdot & \cdot & \mathbf{0} & \\ & & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & 1 \\ & & & & 0 \end{bmatrix}. \qquad (9)$$

And $\mathbf{E}$ is an $n \times m$ matrix such that the $(n_1 + \cdots + n_i, i)$ elements $(i = 1, \cdots, m)$ are 1 and the others are 0. Furthermore, $\mathbf{A^*}$ is an $m \times n$ matrix determined uniquely by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{S_1}$ and $\mathbf{S_2}$. □

From the fixed terminal state condition (2) [1][4], τ must be

$$\tau \geq n_1. \qquad (10)$$

Since the minimum settling time is n_1, let the settling time τ be

$$\tau = n_1 + q, \qquad (11)$$

where q is zero or a positive integer. Then, we obtain the following principle :

[*The deadbeat principle* (using the Luenberger reachable canonical form)] [4]

For an arbitrary initial state $\bar{\mathbf{x}}(0)$, the general deadbeat inputs $\bar{\mathbf{u}}(t)$ of satisfying the state deadbeat condition

$$\bar{\mathbf{x}}(n_1 + q) = \mathbf{0} \qquad (12)$$

must be equivalent to the following **1)** $\sim$ **3)** :

1) $0 \leq t \leq q-1$

$\bar{\mathbf{u}}(t)$ are arbitrary inputs.

2) $q \leq t \leq n_1 - n_m + q - 1$

$$\bar{u}_i(t) = \begin{cases} \text{arbitrary inputs} \\ \qquad (n_i \leq n_1 + q - 1 - t) \\ -\mathbf{e}_m(i)\mathbf{A^*}\bar{\mathbf{x}}(t) \\ \qquad (n_i \geq n_1 + q - t) \end{cases} \qquad (13)$$

where $\bar{u}_i(t)$ is the i -th element of $\bar{\mathbf{u}}(t)$, and $\mathbf{e}_m(i)$ is a $1 \times m$ unit vector such that the only i -th element is 1 and the others are 0.

3) $n_1 - n_m + q \leq t \leq n_1 + q - 1$

$$\bar{\mathbf{u}}(t) = -\mathbf{A^*}\bar{\mathbf{x}}(t). \qquad (14)$$

When $q = 0$ (minimum settling time), the case **1)** does not exist. □

That is, when $\tau = n_1 + q$, the general deadbeat inputs $\bar{\mathbf{u}}(t)(t = 0, \cdots, n_1 + q - 1)$ using the Luenberger reachable canonical form consist of the following three phases :

1) First phase $(t = 0, \cdots, q-1)$
All $\bar{\mathbf{u}}(t)$'s are arbitrary inputs .

2) Second phase $(t = q, \cdots, n_1 - n_m + q - 1)$
Some elements of $\bar{\mathbf{u}}(t)$'s are the special constant gain state feedback, and the others are arbitrary.

3) Third phase $(t = n_1 - n_m + q, \cdots, n_1 + q - 1)$
$\bar{\mathbf{u}}(t)$'s are the special constant gain state feedback.

Therefore, transforming back to the original coordinate system, we obtain the following principle :

[*The deadbeat principle*] [4]

For an arbitrary initial state $\mathbf{x}(0)$, the general deadbeat inputs $\mathbf{u}(t)$ of satisfying the state deadbeat condition such that

$$\mathbf{x}(n_1 + q) = \mathbf{0} \tag{15}$$

must be equivalent to the following **1)** $\sim$ **3)** :

1) $0 \leq t \leq q - 1$

$\mathbf{u}(t)$'s are arbitrary inputs.

2) $q \leq t \leq n_1 - n_m + q - 1$

$$\mathbf{u}(t) = \mathbf{S_2}\bar{\mathbf{u}}(t). \tag{16}$$

where $\bar{\mathbf{u}}(t)$ is given by (13).

3) $n_1 - n_m + q \leq t \leq n_1 + q - 1$

$$\mathbf{u}(t) = -\mathbf{S_2}\mathbf{A}^*\mathbf{S_1}^{-1}\mathbf{x}(t). \tag{17}$$

□

The second phase of the general deadbeat inputs in the original coordinate system are given by the linear combinations of the special constant gain state feedback and arbitrary elements by matrix $\mathbf{S_2}$.

It is apparent that the second phase does not exist when the reachability indices $n_i (i = 1, \cdots, m)$ are equal to each other as follows :

$$n_1 = n_2 = \cdots = n_m = \frac{n}{m} \quad . \tag{18}$$

Thus, we have the following corollary :

[*Corollary*]

When the reachability indices $n_i (i = 1, \cdots, m)$ are equal to each other, the general deadbeat inputs must be equivalent to

$$\mathbf{u}(t) = \begin{cases} \text{arbitrary inputs} \\ \qquad (0 \leq t \leq q - 1) \\ -\mathbf{S_2}\mathbf{A}^*\mathbf{S_1}^{-1}\mathbf{x}(t) \\ \qquad (q \leq t \leq n_1 + q - 1) \end{cases} . \tag{19}$$

□

By using *the deadbeat principle* , the fixed terminal state condition (2) can be removed.

4 . Minimization of the Input Energy

By using the Luenberger reachable canonical form, the performance index (3) is transformed into

$$J(n_1 + q) = \frac{1}{2} \sum_{t=0}^{n_1+q-1} \bar{\mathbf{u}}(t)'\bar{\mathbf{R}}\bar{\mathbf{u}}(t), \tag{20}$$

where

$$\bar{\mathbf{R}} = (\mathbf{S_2}^{-1})'\mathbf{S_2}^{-1}. \tag{21}$$

It is difficult to find the optimal solution of the feedback form with the initial state $\bar{\mathbf{x}}(0)$ and the terminal state $\bar{\mathbf{x}}(n_1 + q) = \mathbf{0}$. However, we can transform the fixed terminal state problem into a new problem with a free terminal state by *the deadbeat principle* .

From the additivity of the performance index, $J(n_1 + q)$ can be divided into the following three parts according to the three phases of *the deadbeat principle* :

$$J(n_1 + q) = J_1(n_1 + q) + J_2(n_1 + q) + J_3(n_1 + q), \tag{22}$$

$$J_1(n_1 + q) = \frac{1}{2} \sum_{t=0}^{q-1} \bar{\mathbf{u}}(t)'\bar{\mathbf{R}}\bar{\mathbf{u}}(t), \tag{23}$$

$$J_2(n_1 + q) = \frac{1}{2} \sum_{t=q}^{n_1-n_m+q-1} \bar{\mathbf{u}}(t)'\bar{\mathbf{R}}\bar{\mathbf{u}}(t), \tag{24}$$

$$J_3(n_1 + q) = \frac{1}{2} \sum_{t=n_1-n_m+q}^{n_1+q-1} \bar{\mathbf{u}}(t)'\bar{\mathbf{R}}\bar{\mathbf{u}}(t). \tag{25}$$

From the third phase of *the deadbeat principle* , (25) depends on only $\bar{\mathbf{x}}(n_1 - n_m + q)$. From (14), we obtain

$$\bar{\mathbf{x}}(t + 1) = \mathbf{J_0}\bar{\mathbf{x}}(t), \tag{26}$$

and

$$\bar{\mathbf{x}}(t) = \mathbf{J_0}^{t-n_1+n_m-q}\bar{\mathbf{x}}(n_1 - n_m + q), \tag{27}$$

$$\bar{\mathbf{u}}(t) = -\mathbf{A}^*\mathbf{J_0}^{t-n_1+n_m-q}\bar{\mathbf{x}}(n_1 - n_m + q). \tag{28}$$

From (28), $J_3(n_1+q)$ becomes as follows regardless of the value of $\bar{\mathbf{x}}(n_1-n_m+q)$:

$$J_3(n_1+q) = \frac{1}{2}\bar{\mathbf{x}}(n_1-n_m+q)'\bar{\mathbf{Q}}_3\bar{\mathbf{x}}(n_1-n_m+q), \tag{29}$$

where

$$\bar{\mathbf{Q}}_3 = \sum_{t=n_1-n_m+q}^{n_1+q-1} (\mathbf{J_0}^{t-n_1+n_m-q})'(\mathbf{A}^*)'\bar{\mathbf{R}} \times \mathbf{A}^*\mathbf{J_0}^{t-n_1+n_m-q}. \tag{30}$$

It is apparent that $\bar{\mathbf{Q}}_3$ is an $n \times n$ positive semi-definite symmetric matrix.

Let the number of the arbitrary inputs of (13) in the second phase of *the deadbeat principle* be m_t. Then, (13) can be rewritten as

$$\bar{\mathbf{u}}(t) = \mathbf{F}(t)\bar{\mathbf{x}}(t) + \mathbf{W_t}(t)\bar{\mathbf{v}}_\mathbf{t}(t), \tag{31}$$

where

$$\mathbf{F}(t) = -\begin{bmatrix} \mathbf{e_m}(1) \\ \vdots \\ \mathbf{e_m}(m-m_t) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{A}^*, \tag{32}$$

$$\mathbf{W_t}(t) = -\begin{bmatrix} \mathbf{e_m}(m-m_t+1) \\ \vdots \\ \mathbf{e_m}(m) \end{bmatrix}, \tag{33}$$

and $\mathbf{F}(t)$ is an $m \times n$ matrix, $\mathbf{W_t}(t)$ is an $m \times m_t$ matrix and $\bar{\mathbf{v}}_\mathbf{t}(t)$ is an $m_t \times 1$ arbitrary input vector. The orders of $\bar{\mathbf{v}}_\mathbf{t}(t)$ and $\mathbf{W_t}(t)$ are time-variant, and

$$\text{rank } \mathbf{W_t}(t) = m_t \ \ \text{(full rank)}. \tag{34}$$

Substituting (31) into (7), we have

$$\bar{\mathbf{x}}(t+1) = \bar{\mathbf{A}}(t)\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}_\mathbf{t}(t)\bar{\mathbf{v}}_\mathbf{t}(t), \tag{35}$$

where $n \times n$ matrix $\bar{\mathbf{A}}(t)$ and $n \times m_t$ matrix $\bar{\mathbf{B}}_\mathbf{t}(t)$ are as follows respectively :

$$\bar{\mathbf{A}}(t) = \mathbf{J_0} + \mathbf{E}[\mathbf{A}^* + \mathbf{F}(t)], \tag{36}$$

$$\bar{\mathbf{B}}_\mathbf{t}(t) = \mathbf{E}\mathbf{W_t}(t). \tag{37}$$

That is, the system (1) is transformed into a new time-variant system that the order of the input is time-variant. Then, from (24), $J_2(n_1+q)$ becomes as

$$\begin{aligned} J_2(n_1+q) &= \frac{1}{2}\sum_{t=q}^{n_1-n_m+q-1} [\bar{\mathbf{x}}(t)'\bar{\mathbf{Q}}(t)\bar{\mathbf{x}}(t) \\ &\quad +2\bar{\mathbf{x}}(t)'\bar{\mathbf{S}}_\mathbf{t}(t)\bar{\mathbf{v}}_\mathbf{t}(t) \\ &\quad +\bar{\mathbf{v}}_\mathbf{t}(t)'\bar{\mathbf{R}}_\mathbf{t}(t)\bar{\mathbf{v}}_\mathbf{t}(t)], \end{aligned} \tag{38}$$

where

$$\bar{\mathbf{Q}}(t) = \mathbf{F}(t)'\bar{\mathbf{R}}\mathbf{F}(t), \tag{39}$$
$$\bar{\mathbf{S}}_\mathbf{t}(t) = \mathbf{F}(t)'\bar{\mathbf{R}}\mathbf{W_t}(t), \tag{40}$$
$$\bar{\mathbf{R}}_\mathbf{t}(t) = \mathbf{W_t}(t)'\bar{\mathbf{R}}\mathbf{W_t}(t). \tag{41}$$

Apparently, $\bar{\mathbf{Q}}(t)$ is an $n \times n$ positive semi-definite symmetric matrix, $\bar{\mathbf{S}}_\mathbf{t}(t)$ is an $n \times m_t$ matrix, and $\bar{\mathbf{R}}_\mathbf{t}(t)$ is an $m_t \times m_t$ positive definite matrix.

All the inputs of the first phase of *the deadbeat principle* are arbitrary. However, we use the same expression as in the second phase for convenience. Then, we have

$$\mathbf{F}(t) = \mathbf{0}, \quad \mathbf{W_t}(t) = \mathbf{I_m}, \quad \bar{\mathbf{v}}_\mathbf{t}(t) = \bar{\mathbf{u}}(t), \tag{42}$$

$$\bar{\mathbf{A}}(t) = \mathbf{J_0} + \mathbf{E}\mathbf{A}^*, \quad \bar{\mathbf{B}}_\mathbf{t}(t) = \mathbf{E}, \tag{43}$$

$$\begin{aligned} J_1(n_1+q) &= \frac{1}{2}\sum_{t=0}^{q-1}[\bar{\mathbf{x}}(t)'\bar{\mathbf{Q}}(t)\bar{\mathbf{x}}(t) \\ &\quad +2\bar{\mathbf{x}}(t)'\bar{\mathbf{S}}_\mathbf{t}(t)\bar{\mathbf{v}}_\mathbf{t}(t) \\ &\quad +\bar{\mathbf{v}}_\mathbf{t}(t)'\bar{\mathbf{R}}_\mathbf{t}(t)\bar{\mathbf{v}}_\mathbf{t}(t)], \end{aligned} \tag{44}$$

where

$$\bar{\mathbf{Q}}(t) = \bar{\mathbf{S}}_\mathbf{t}(t) = \mathbf{0}, \quad \bar{\mathbf{R}}_\mathbf{t}(t) = \bar{\mathbf{R}}. \tag{45}$$

For simplicity, let

$$n_1 - n_m + q - 1 = p. \tag{46}$$

Then, the new system $(t = 0, \cdots, p)$ is

$$\bar{\mathbf{x}}(t+1) = \bar{\mathbf{A}}(t)\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}_\mathbf{t}(t)\bar{\mathbf{v}}_\mathbf{t}(t),$$

and the new performance index is

$$\begin{aligned} J(n_1+q) &= \frac{1}{2}\sum_{t=0}^{p}[\bar{\mathbf{x}}(t)'\bar{\mathbf{Q}}(t)\bar{\mathbf{x}}(t) \\ &\quad +2\bar{\mathbf{x}}(t)'\bar{\mathbf{S}}_\mathbf{t}(t)\bar{\mathbf{v}}_\mathbf{t}(t) \\ &\quad +\bar{\mathbf{v}}_\mathbf{t}(t)'\bar{\mathbf{R}}_\mathbf{t}(t)\bar{\mathbf{v}}_\mathbf{t}(t)] \\ &\quad + \frac{1}{2}\bar{\mathbf{x}}(p+1)'\bar{\mathbf{Q}}_3\bar{\mathbf{x}}(p+1). \end{aligned} \tag{47}$$

Here, (42) $\sim$ (45) are in the first phase, and (36) $\sim$ (41) are in the second phase. That is, the fixed terminal state problem of minimizing the input energy is transformed into the new free terminal state problem that the new terminal state $\bar{\mathbf{x}}(q)$ is free. In this new LQ problem, the system is time-variant with the variable structure of the input, and the performance index is the quadratic form with the time-variant weight matrices and with cross terms.

This new problem can be solved by using the Lagrange multipliers method. First, using the $n \times 1$ Lagrange multiplier vectors $\bar{\lambda}(t)(t = 1, \cdots, p+1)$, the Lagragian function $\bar{L}$ becomes

$$\bar{L} = J(n_1 + q) + \sum_{t=0}^{p} \bar{\lambda}(t+1)'[\bar{\mathbf{A}}(t)\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}_t(t)\bar{\mathbf{v}}_t(t) - \bar{\mathbf{x}}(t+1)]. \quad (48)$$

Let the gradients $\partial\bar{L}/\partial\bar{\mathbf{x}}(t)$, $\partial\bar{L}/\partial\bar{\mathbf{v}}_t(t)$ and $\partial\bar{L}/\partial\bar{\mathbf{x}}(p+1)$ be $\mathbf{0}$, then we obtain

$$\bar{\lambda}(t) = \bar{\mathbf{Q}}(t)\bar{\mathbf{x}}(t) + \bar{\mathbf{S}}_t(t)\bar{\mathbf{v}}_t(t) + \bar{\mathbf{A}}(t)'\bar{\lambda}(t+1) \quad (1 \le t \le p), \quad (49)$$

$$\bar{\mathbf{S}}_t(t)'\bar{\mathbf{x}}(t) + \bar{\mathbf{R}}_t(t)\bar{\mathbf{v}}_t(t) + \bar{\mathbf{B}}_t(t)'\bar{\lambda}(t+1) = \mathbf{0} \quad (0 \le t \le p), \quad (50)$$

$$\bar{\mathbf{Q}}_3\bar{\mathbf{x}}(p+1) = \bar{\lambda}(p+1) \quad (51)$$

As $\bar{\mathbf{R}}_t(t)$ is non-singular from (41) and (45), from (50) we obtain

$$\bar{\mathbf{v}}_t(t) = -\bar{\mathbf{R}}_t(t)^{-1}\bar{\mathbf{S}}_t(t)'\bar{\mathbf{x}}(t) - \bar{\mathbf{R}}_t(t)^{-1}\bar{\mathbf{B}}_t(t)'\bar{\lambda}(t+1). \quad (52)$$

For simplicity, let

$$\bar{\mathbf{Q}}_0(t) = \bar{\mathbf{Q}}(t) - \bar{\mathbf{S}}_t(t)'\bar{\mathbf{R}}_t(t)^{-1}\bar{\mathbf{S}}_t(t)', \quad (53)$$

$$\bar{\mathbf{A}}_0(t) = \bar{\mathbf{A}}(t) - \bar{\mathbf{B}}_t(t)'\bar{\mathbf{R}}_t(t)^{-1}\bar{\mathbf{S}}_t(t)'. \quad (54)$$

From (52), (49) becomes

$$\bar{\lambda}(t) = \bar{\mathbf{Q}}_0(t)\bar{\mathbf{x}}(t) + \bar{\mathbf{A}}_0(t)'\bar{\lambda}(t+1), \quad (55)$$

and (35) becomes

$$\bar{\mathbf{x}}(t+1) = \bar{\mathbf{A}}_0(t)\bar{\mathbf{x}}(t) - \bar{\mathbf{B}}_t(t)\bar{\mathbf{R}}_t(t)^{-1}\bar{\mathbf{B}}_t(t)'\bar{\lambda}(t+1). \quad (56)$$

From (52), (55) and (56), the new LQ control problem that the new terminal state $\bar{\mathbf{x}}(p+1)$ is free is transformed into a new TPBV problem

$$\begin{bmatrix} \mathbf{I}_n & \bar{\mathbf{B}}_t(t)\bar{\mathbf{R}}_t(t)^{-1}\bar{\mathbf{B}}_t(t)' \\ \mathbf{0} & \bar{\mathbf{A}}_0(t)' \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}(t+1) \\ \bar{\lambda}(t+1) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_0(t) & \mathbf{0} \\ -\bar{\mathbf{Q}}_0(t) & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}(t) \\ \bar{\lambda}(t) \end{bmatrix}, \quad (57)$$

with boundary conditions $\bar{\mathbf{x}}(0)$ and $\bar{\lambda}(p+1)$. Here, we apply the following Riccati transformation by the $n \times n$ symmetric matrix $\bar{\mathbf{P}}(t)$:

$$\bar{\lambda}(t) = \bar{\mathbf{P}}(t)\bar{\mathbf{x}}(t). \quad (58)$$

Then, (56) becomes

$$[\mathbf{I_n} + \bar{\mathbf{B}}_t(t)\bar{\mathbf{R}}_t(t)^{-1}\bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)]\bar{\mathbf{x}}(t+1) = \bar{\mathbf{A}}_0(t)\bar{\mathbf{x}}(t). \quad (59)$$

If $\bar{\mathbf{P}}(t+1)$ is assumed to be a positive semi-definite matrix , we have

$$[\mathbf{I_n} + \bar{\mathbf{B}}_t(t)\bar{\mathbf{R}}_t(t)^{-1}\bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)]^{-1} = \mathbf{I_n} - \bar{\mathbf{B}}_t(t)[\bar{\mathbf{R}}_t(t) + \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{B}}_t(t)]^{-1} \times \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1). \quad (60)$$

Then, (59) becomes

$$\bar{\mathbf{x}}(t+1) = \{\mathbf{I}_n - \bar{\mathbf{B}}_t(t)[\bar{\mathbf{R}}_t(t) + \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{B}}_t(t)]^{-1} \times \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\}\bar{\mathbf{A}}_0(t)\bar{\mathbf{x}}(t). \quad (61)$$

From (55), (58) and (61), we obtain

$$\{\bar{\mathbf{P}}(t) - \bar{\mathbf{Q}}_0(t) - \bar{\mathbf{A}}_0(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{A}}_0(t) + \bar{\mathbf{A}}_0(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{B}}_t(t) \times [\bar{\mathbf{R}}_t(t) + \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{B}}_t(t)]^{-1} \times \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{A}}_0(t)\}\bar{\mathbf{x}}(t) = \mathbf{0}. \quad (62)$$

So as (62) to hold for any $\bar{\mathbf{x}}(t)$, $\bar{\mathbf{P}}(t)$ must satisfy the nonstationary Riccati equation

$$\bar{\mathbf{P}}(t) = \bar{\mathbf{Q}}_0(t) + \bar{\mathbf{A}}_0(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{A}}_0(t) - \bar{\mathbf{A}}_0(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{B}}_t(t) \times [\bar{\mathbf{R}}_t(t) + \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{B}}_t(t)]^{-1} \times \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{A}}_0(t). \quad (63)$$

From (51) and (58), the terminal condition is

$$\bar{\mathbf{P}}(p+1) = \bar{\mathbf{Q}}_3. \quad (64)$$

As $\bar{\mathbf{Q}}_3$ is a positive semi-definite matrix from (30), $\bar{\mathbf{P}}(t+1)$ is also a positive semi-definite matrix as the assumption.

From (52), (58) and (61), the optimal input of the new LQ problem $\bar{\mathbf{v}}_t^*(t)(t = 0, \cdots, q-1)$ is

$$\bar{\mathbf{v}}_t^*(t) = -[\bar{\mathbf{R}}_t(t) + \bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{B}}_t(t)]^{-1} \times [\bar{\mathbf{B}}_t(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{A}}(t) + \bar{\mathbf{S}}_t(t)']\bar{\mathbf{x}}(t). \quad (65)$$

Then, the optimal input in the first phase and the second phase is expressed as

$$\bar{\mathbf{u}}^*(t) = \mathbf{K}(t)\bar{\mathbf{x}}(t) + \mathbf{W}_t(t)\bar{\mathbf{v}}_t{}^*(t). \quad (66)$$

Specially in the first phase, from (42), (43) and (45), we have

$$\bar{\mathbf{u}}^*(t) = -[\bar{\mathbf{R}} + \mathbf{E}'\bar{\mathbf{P}}(t+1)\mathbf{E}]^{-1}\mathbf{E}' \times \bar{\mathbf{P}}(t+1)\bar{\mathbf{A}}(t)\bar{\mathbf{x}}(t), \quad (67)$$

$$\bar{\mathbf{P}}(t) = \bar{\mathbf{A}}(t)'\bar{\mathbf{P}}(t+1)\bar{\mathbf{A}}(t) - \bar{\mathbf{A}}(t)' \times\bar{\mathbf{P}}(t+1)\mathbf{E}[\bar{\mathbf{R}} + \mathbf{E}'\bar{\mathbf{P}}(t+1)\mathbf{E}]^{-1} \times \mathbf{E}'\bar{\mathbf{P}}(t+1)\bar{\mathbf{A}}(t). \quad (68)$$

The optimal input of the third phase is given by (14).

Furthermore, the minimal value $J^*(n_1+q)$ of $J(n_1+q)$ is

$$\begin{aligned} J^*(n_1+q) &= \frac{1}{2}\sum_{t=0}^{p-1}[\bar{\mathbf{x}}(t)'\bar{\mathbf{P}}(t)\bar{\mathbf{x}}(t) \\ &\quad -\bar{\mathbf{x}}(t+1)'\bar{\mathbf{Q}}(t+1)\bar{\mathbf{x}}(t+1) \\ &\quad +\frac{1}{2}\bar{\mathbf{x}}(p+1)'\bar{\mathbf{Q}}_3\bar{\mathbf{x}}(p+1) \\ &= \frac{1}{2}\bar{\mathbf{x}}(0)'\bar{\mathbf{P}}(0)\bar{\mathbf{x}}(0). \end{aligned} \quad (69)$$

That is, the optimal inputs $\bar{\mathbf{u}}^*(t)$ consist of the following three phases :

1) First phase $(t = 0, \cdots, q-1)$
$\bar{\mathbf{u}}^*(t)$'s are variable gain state feedback obtained from (67) and (68).

2) Second phase $(t = q, \cdots, n_1 - n_m + q - 1)$
$\bar{\mathbf{u}}^*(t)$'s are obtained from (63) ~ (66), and some elements are the constant gain state feedback, and the others are variable gain state feedback.

3) Third phase $(t = n_1 - n_m + q, \cdots, n_1 + q - 1)$
$\bar{\mathbf{u}}^*(t)$'s are the constant gain state feedback obtained from (14).

Furthermore, the following properties are obtained:

1) The constant feedback gain and the variable feedback gains do not depend on the initial state $\bar{\mathbf{x}}(0)$.

2) Let $\bar{\mathbf{K}}^*(n_1+q, t)$ be the optimal feedback gain at t when the terminal time is n_1+q. The following equality holds:

$$\bar{\mathbf{K}}^*(n_1+q, t) = \bar{\mathbf{K}}^*(n_1+q+1, t+1). \quad (70)$$

3) Since $J^*(n_1+q)$ is the special case of $J(n_1+q+1)$, the following inequality holds:

$$J^*(n_1+q) \geq J^*(n_1+q+1). \quad (71)$$

The transformation to the original coordinate system is trivial. In that case, the same properties are held.

5 . Conclusion

This paper discusses the regulator problem of minimizing the input energy for the multi-input linear time-invariant discrete-time system with the zero terminal state. By using *the deadbeat principle* , the system is transformed into the new time-variant system, and the fixed terminal state problem is transformed into the new free terminal state problem. In this new LQ problem, the system is time-variant with the variable structure of the input, and the performance index is the quadratic form with the time-variant weight matrices and with cross terms.

The optimal inputs are mede up of three phases. They are given by the variable gain state feedback in the first phase, by linear combinations of the constant gain state feedback and the variable gain state feedback in the second phase, and by the constant gain state feedback in the third phase. The variable gains are obtained by using the special nonstationary Riccati equation, and all gains are independent of the initial state of the system.

Acknowledgment – A part of this research was supported by the INAMORI FOUNDATION.

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