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An Analysis of Robust Stability for Time Delay Systems Based on Lyapunov Type Operator Equation

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Abstract A robust stability problem for time delay systems is discussed by using a property of Lyapunov type operator equation. We propose a method to check the robust stability against the parameter perturbations occurring in both lumped parameter part and distributed delay element.

1. Introduction

In the design of robustly stable feedback systems, the key step is to analyze the stability of closed loop systems in case that nominal plant is perturbed. For lumped parameter systems, various methods for analyzing the robust stability have been proposed from both time domain approach and frequency domain approach.

In time delay systems, stability criteria for given system are proposed by Kamen[1], Mori et al.[2][3]. But check methods of robust stability for stabilized feedback system have not been so deeply studied as those for lumped parameter systems. A primary difficulty of analyzing the robust stability for time delay system comes from the fact that the system is infinite dimensional and has many kind of parameters.

In this paper, we propose a method to check the robust stability for time delay systems in case that the parameter perturbations occur in both lumped parameter part and distributed delay element. It is known that the properties of Lyapunov type operator equation of time delay systems are almost similar to those of finite dimensional case; for example, the time delay system is asymptotically stable if and only if there exists a unique positive definite solution to the Lyapunov type operator equation [4][5].

By using this property of Lyapunov type operator equation, we first derive an operator type condition of robust stability, in which the parameter perturbations are described as a bounded operator. Secondly, we investigate the relations between the magnitude of parameter perturbations and the bounded operator, and evaluate the upper bound of the operator which represents the parameter perturbations. By applying these preliminary results to the operator type condition, we lastly derive a norm condition for robust stability against parameter perturbations.

2. Systems and Preliminaries

Consider the nominal system

$$\dot{x}(t)=A_0 x(t)+A_1 x(t-h)+\int_{-h}^{0} A_{01}(\beta)x(t+\beta)d\beta \qquad (1)$$

with the initial condition

$$x(0)=\phi^0,\quad x(\beta)=\phi^1(\beta),\qquad -h\leqq\beta\leqq 0$$

where $x(t)\in R^n$, A_0, $A_1\in R^{n\times n}$, $A_{01}\in L_2(-h,0;R^{n\times n})$, $h>0$. The initial state

$\begin{bmatrix} \phi^0 \\ \phi^1 \end{bmatrix}$ is taken to be an element in $M_2 = R^n \times L_2(-h, 0; R^{n\times n})$, which is a Hilbert space with the inner product

$$\left\langle \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \begin{bmatrix} y^0 \\ y^1 \end{bmatrix} \right\rangle = x^{0\prime} y^0 + \int_{-h}^{0} x^{1\prime}(\beta) y^1(\beta) d\beta$$

We assume the nominal system (1) to be asymptotically stable. In the following we discuss the robust stability problem of system (1), focusing on the parameter perturbations which occur in both lumped parameter part and distributed delay element.

The perturbed system of (1) is described in the following way;

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} A_{01}(\beta) x(t+\beta) d\beta + \delta D_0 x(t) + \int_{-h}^{0} \delta D_1(\beta) x(t+\beta) d\beta \tag{2}$$

where $\delta D_0 \in R^{n\times n}$, $\delta D_1 \in L_2(-h, 0; R^{n\times n})$ represents the perturbation of lumped parameter part and distributed delay element respectively.

We can write the perturbed system (2) in the form of an evolution equation

$$\dot{z}(t) = (\mathcal{A} + \Pi \delta \mathcal{D}) z(t) \tag{3}$$

in M_2, where

$$z(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix}, \quad x_t(\beta) = x(t+\beta), \quad -h \leqq \beta \leqq 0$$

The operator $\mathcal{A}$ is an infinitesimal generator which describes the nominal system (1) and defined with the matrix of operators;

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} \phi^0 \\ \phi^1 \end{bmatrix} : \phi^1; \text{ absolutely continuous}, \ \phi^1 \in H^1(-h, 0; R^n), \ \phi^0 = \phi^1(0) \right\}$$

$$\mathcal{A} := \begin{bmatrix} A^{00} & A^{01} \\ 0 & A^{11} \end{bmatrix}, \quad \begin{matrix} A^{00} \in \mathcal{L}(R^n) \\ A^{01} \in \mathcal{L}(L_2, R^n) \\ A^{11} \in \mathcal{L}(L_2) \end{matrix} \tag{4}$$

$$A^{00}\phi^0 := A_0 \phi^0$$

$$A^{01}\phi^1 := A_1 \phi^1(-h) + \int_{-h}^{0} A_{01}(\beta) \phi^1(\beta) d\beta$$

$$(A^{11}\phi^1)(\alpha) := \frac{d\phi^1(\alpha)}{d\alpha}, \quad -h \leqq \alpha \leqq 0$$

The operator Π and $\delta \mathcal{D}$ are both bounded operators which describe the parameter perturbations and defined with the matrix of operators;

$$\Pi := \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad \Pi \in \mathcal{L}(R^n, M_2) \tag{5}$$

$$\delta \mathcal{D} := [\delta D^0 \ \ \delta D^1], \quad \begin{matrix} \delta D^0 \in \mathcal{L}(R^n) \\ \delta D^1 \in \mathcal{L}(L_2, R^n) \end{matrix} \tag{6}$$

$$\delta D^0 \phi^0 := \delta D_0 \phi^0$$

$$\delta D^1 \phi^1 := \int_{-h}^{0} \delta D_1(\beta) \phi^1(\beta) d\beta$$

It is well known that the perturbed infinitesimal generator also generates a strongly continuous semigroup in case the perturbation is bounded [6]. So the operator $\mathcal{A} + \Pi \delta \mathcal{D}$ of equation (3) generates a strongly continuous semigroup.

For time delay systems, the following properties of Lyapunov type operator equation is shown by Datko[4], Delfour et al.[5].

<u>Theorem 1.[4][5]</u> The time delay system of the form (1) is asymptotically stable if and only if there exists a unique positive definite solution $\mathcal{P} > 0$ in $\mathcal{L}(M_2)$ such that

$$\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P} = -\mathcal{Q} \tag{7}$$

where $\mathcal{Q} > 0$ in $\mathcal{L}(M_2)$.

Furthermore the solution $\mathcal{P} > 0$ of (7) is characterized by its matrix of operators;

$$\mathcal{P} := \begin{bmatrix} P^{00} & P^{01} \\ P^{10} & P^{11} \end{bmatrix}, \tag{8}$$

$$P^{00} \in \mathcal{L}(R^n), \quad P^{01} \in \mathcal{L}(L_2, R^n)$$
$$P^{10} \in \mathcal{L}(R^n, L_2), \quad P^{11} \in \mathcal{L}(L_2)$$

$$P^{00}\phi^0 := P_0 \phi^0$$

$$(P^{10}\phi^0)(\alpha) := P_1'(\alpha)\phi^0$$

$$P^{01}\phi^1 := \int_{-h}^{0} P_1(\beta)\phi^1(\beta)d\beta$$

$$(P^{11})\phi^1 := \int_{-h}^{0} P_2(\alpha,\beta)\phi^1(\beta)d\beta$$

$$-h \leq \alpha \leq 0 \quad \blacksquare$$

We call the triplet $\{P_0, P_1, P_2\}$ an integral kernel of operator $\mathcal{P}$.

3. Analysis of Robust Stability

In the previous section, it is shown that the perturbed system (2) can be written in the form of an evolution equation, in which the parameter perturbations are described as bounded operators Π, $\delta\mathcal{D}$. Our objective is now to derive a norm bound of parameter perturbation δD_0, δD_1 which guarantees the robust stability.

3.1 Fundamental Ideas

Let us consider Lyapunov type operator equation to nominal system (1)

$$\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P} = -2\mathcal{I} \qquad (9)$$

where the infinitesimal generator $\mathcal{A}$ is defined in (4). Since we assume that the nominal system (1) is asymptotically stable, there exists a positive definite solution $\mathcal{P} > 0$ to equation (9).

Using this solution $\mathcal{P} > 0$, we can obtain the following equality by rewriting equation (9).

$$\mathcal{P}(\mathcal{A} + \Pi\delta\mathcal{D}) + (\mathcal{A} + \Pi\delta\mathcal{D})^*\mathcal{P}$$
$$= -2\mathcal{I} + \mathcal{P}\Pi\delta\mathcal{D} + \delta\mathcal{D}^*\Pi^*\mathcal{P} \qquad (10)$$

Here we notice that the operator $\mathcal{A} + \Pi\delta\mathcal{D}$ represents the perturbed system (2). If we assume that the right hand side of equation (10) to be negative definite, the equality (10) can be seen as Lyapunov type operator equation defined with the perturbed system (2), and furthermore there always exists a positive definite solution $\mathcal{P} > 0$, which is the same solution to (9).

Hence a sufficient condition that the perturbed system $\mathcal{A} + \Pi\delta\mathcal{D}$ is asymptotically stable is derived in the following inequality;

$$\mathcal{P}\Pi\delta\mathcal{D} + \delta\mathcal{D}^*\Pi^*\mathcal{P} < 2\mathcal{I} \qquad (11)$$

where operator $\mathcal{P}$ is the positive definite solution to (9).

3.2 Norm-condition of Robust Stability

We now obtain a sufficient condition of robust stability in a form of operator inequality (11). So our next problem is to evaluate the upper bound of the operator $\delta\mathcal{D}, \mathcal{P}$ with the parameter $\{\delta D_0, \delta D_1\}$, $\{P_0, P_1, P_2\}$ defined in (6), (8) respectively.

<u>Theorem 2.</u> Let Π, $\delta\mathcal{D}$ and $\mathcal{P}$ be the operator defined in (5), (6) and (8) respectively. The bounded operator $\delta\mathcal{D}^*\delta\mathcal{D}$ and $(\Pi^*\mathcal{P})^*(\Pi^*\mathcal{P})$ are evaluated in the following way;

$$\delta\mathcal{D}^*\delta\mathcal{D} \leq d\,\mathcal{I} \qquad (12)$$

$$(\Pi^*\mathcal{P})^*(\Pi^*\mathcal{P}) \leq p\,\mathcal{I} \qquad (13)$$

where

$$d := \left\| \delta D_0 \delta D_0' + \int_{-h}^{0} \delta D_1(\beta)\delta D_1'(\beta)d\beta \right\|_2 \qquad (14)$$

$$p := \left\| P_0 P_0' + \int_{-h}^{0} P_1(\beta)P_1'(\beta)d\beta \right\|_2 \qquad (15) \quad \blacksquare$$

(See Appendix for proof)

By using the inequalities stated in Theorem 2, we obtain the main result on the robust stability against parameter perturbations.

<u>Theorem 3.</u> Let $\mathcal{P}$ be the positive definite solution to (9) and the triplet $\{P_0, P_1, P_2\}$ be an integral kernel of $\mathcal{P}$. The perturbed system (2) is asymptotically stable if the parameter perturbations δD_0 and δD_1 satisfy the following inequality.

$$\left\| \delta D_0 \delta D_0' + \int_{-h}^{0} \delta D_1(\beta)\delta D_1'(\beta)d\beta \right\|_2 < \frac{1}{p} \qquad (16)$$

where p is defined in (15). ■

(Proof) We first consider the sufficient condition of robust stability (11) described with the operators;

$$\mathcal{P}\Pi\,\delta\mathscr{D}+\delta\mathscr{D}^{*}\Pi^{*}\mathcal{P}<2\mathscr{I} \tag{11}$$

By adding $\beta(\Pi^{*}\mathcal{P})^{*}(\Pi^{*}\mathcal{P})$ and $(1/\beta)\delta\mathscr{D}^{*}\delta\mathscr{D}$ $(\beta>0)$ to the both sides of the inequality (11), we have the inequality;

$$\beta(\Pi^{*}\mathcal{P})^{*}(\Pi^{*}\mathcal{P})+\frac{1}{\beta}\delta\mathscr{D}^{*}\delta\mathscr{D}$$
$$<2\mathscr{I}+\{\sqrt{\beta}(\Pi^{*}\mathcal{P})-\frac{1}{\sqrt{\beta}}\delta\mathscr{D}\}^{*}$$
$$\times\{\sqrt{\beta}(\Pi^{*}\mathcal{P})-\frac{1}{\sqrt{\beta}}\delta\mathscr{D}\} \tag{17}$$

where $\beta>0$ is a free scalar parameter. Now from the inequality (17), we obtain a sufficient condition to (11) in the following way;

$$\beta(\Pi^{*}\mathcal{P})^{*}(\Pi^{*}\mathcal{P})+\frac{1}{\beta}\delta\mathscr{D}^{*}\delta\mathscr{D}<2\mathscr{I} \tag{18}$$

Inequality (18) means that, if there exists $\beta>0$ for given operator $\Pi^{*}\mathcal{P}$ and $\delta\mathscr{D}$, the condition of robust stability (11) is satisfied. Furthermore, by applying the result of Theorem 2 to inequality (18), we have a norm condition of parameter perturbations;

$$\beta\,p\,\mathscr{I}+\frac{1}{\beta}d\,\mathscr{I}<2\mathscr{I} \quad \text{i.e.} \quad \beta\,p+\frac{1}{\beta}d<2 \tag{19}$$

where scalar parameter p and d are defined in (14), (15) respectively.

In case $d=0$, inequality (19) always holds for sufficiently small $\beta>0$. In case $p>0$ and $d>0$, the left hand side of (19) is minimized by setting $\beta=\sqrt{d/p}$ for given parameter $p>0$, $d>0$.

Hence, by inserting $\beta=\sqrt{d/p}$ into the left hand side of inequality (19), we lastly obtain the condition of robust stability in the following way;

$$d<\frac{1}{p} \tag{20}$$

■

4. Example

Consider the system

$$\dot{x}(t)=A_0x(t)+A_1x(t-5) \tag{21}$$

$$A_0=\begin{bmatrix}-7.0 & 1.0\\ 0 & -8.0\end{bmatrix},\quad A_1=\begin{bmatrix}-0.3 & -0.1\\ -0.2 & -0.4\end{bmatrix}$$

which is asymptotically stable (Figure. 1). We shall analyze the robust stability of (21) by using Theorem 3.

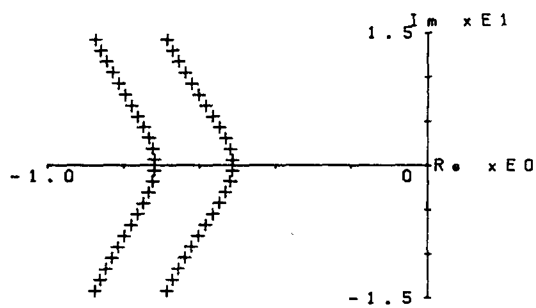

Fig. 1 Pole Configuration of Nominal System

From the calculation of the integral kernel to Lyapunov type operator equation (9) defined with (21), the scalar parameter p in (15) is calculated as follows;

$$p=0.19898 \tag{22}$$

So the upper bound of parameter perturbation is evaluated

$$\left\|\delta D_0\delta D_0'+\int_{-h}^{0}\delta D_1(\beta)\delta D_1'(\beta)d\beta\right\|_2<5.026 \tag{23}$$

Furthermore, in case perturbation occurs only in lumped parameter part, it is evaluated

$$\|\delta D_0\|_2<2.242 \tag{24}$$

from inequality (23).

If we give the parameter perturbation $\delta D_0=\mathrm{diag}[2.2\ 2.2]$ such that it satisfies (24), the pole configuration of perturbed system is as follows.

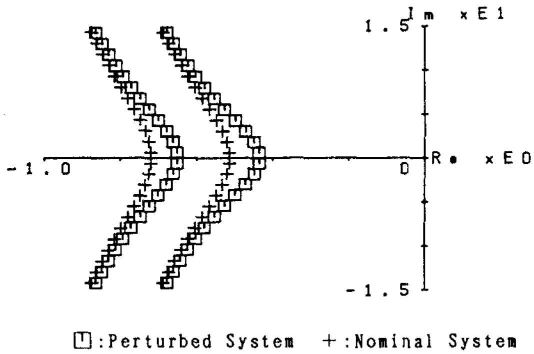

Fig. 2 Pole Configuration of Perturbed System

Figure 2. shows that the perturbed system is still asymptotically stable.

5. Conclusion

We have proposed a method to check the robust stability for time delay systems in case that parameter perturbations occur in both lumped parameter part and distributed delay element. Though this method needs a positive definite solution to Lyapunov type operator equation, Averaging method [7] can easily applied for the approximate calculation, and the computational work needed to obtain the solution can be said quite moderate.

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Appendix (Proof of Theorem 2)

The operator $\Pi^*\mathcal{P} = [P^{00} \; P^{01}]$ is similar to $\delta\mathcal{D}$ defined in (6). So, for simplicity, we give a proof only about inequality (12).

We shall break up the proof into two separate Lemmas.

<u>Lemma 2.1</u> The operator $\delta\mathcal{D}^*\delta\mathcal{D}$ is a compact self-adjoint operator. ■

(Proof) From the form of $\delta\mathcal{D}^*\delta\mathcal{D}$, it is immediately shown that $\delta\mathcal{D}^*\delta\mathcal{D}$ is self-adjoint.

By proving the following propositions, we show the compactness of $\delta\mathcal{D}^*\delta\mathcal{D}$.

(a) $\delta\mathcal{D}^*\delta\mathcal{D}$ is a bounded operator.
(b) $\delta\mathcal{D}^*\delta\mathcal{D}$ is finite dimensional.

<u>Proof of (a)</u> Since the operator $\delta\mathcal{D}$ defined in (6) is bounded, the operator $\delta\mathcal{D}^*\delta\mathcal{D}$ is a bounded operator.

<u>Proof of (b)</u> We show that the range of the operator $\delta\mathcal{D}^*\delta\mathcal{D}$ is finite dimensional.

For $\forall\phi \in M_2$, $\delta\mathcal{D}^*\delta\mathcal{D}\phi$ is calculated in the following way;

$$\delta\mathcal{D}^*\delta\mathcal{D}\phi$$
$$= \delta\mathcal{D}^*\left(\delta D_0\phi^0 + \int_{-h}^{0}\delta D_1(\beta)\phi^1(\beta)d\beta\right)$$

$$= \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix} \langle \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix} \begin{bmatrix} \phi^0 \\ \phi^1 \end{bmatrix} \rangle \quad (A1)$$

From the equation (A1), it is shown that the range of $\delta \mathscr{D}^* \delta \mathscr{D}$ is spanned with at most n columns of $\begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix}$. Hence the range of $\delta \mathscr{D}^* \delta \mathscr{D}$ is finite dimensional.

From the proof of proposition (a),(b), the compactness of $\delta \mathscr{D}^* \delta \mathscr{D}$ is proved. ■

Lemma 2.2 The operator $\delta \mathscr{D}^* \delta \mathscr{D}$ and the matrix

$$M(\delta \mathscr{D}) := \langle \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix}, \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix} \rangle$$

$$= \delta D_0 \delta D_0' + \int_{-h}^{0} \delta D_1(\beta) \delta D_1'(\beta) d\beta \quad (A2)$$

share the same nonzero eigenvalues. ■

(proof) Let λ, $x \in M_2$ be a nonzero eigenvalue and corresponding eigenvector of $\delta \mathscr{D}^* \delta \mathscr{D}$ respectively. It follows

$$\lambda \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix} \langle \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix}, \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} \rangle \quad (A3)$$

from the relation $\lambda x = \delta \mathscr{D}^* \delta \mathscr{D} x$. Pre-multiply (A3) by $\langle \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix}, \cdot \rangle$ and define

$v := \langle \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix}, \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} \rangle$, we obtain

$$\lambda v = M(\delta \mathscr{D}) v \quad (A4)$$

If v, i.e. $\langle \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix}, \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} \rangle$, were to equal zero, then so would λx from (A3). This is not possible (both λ and x are nonzero), so v is an eigenvector of $M(\delta \mathscr{D})$ and λ is an eigenvalue.

Conversely, let μ, $w \in R^n$ be a nonzero eigenvalue and corresponding eigenvector of $M(\delta \mathscr{D})$ respectively. It follows

$$\mu w = \langle \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix}, \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix} w \rangle \quad (A5)$$

from the relation $\mu w = M(\delta \mathscr{D}) w$. Pre-multiply (A5) by $\begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix}$ and define $z := \begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix} w$, we obtain

$$\mu z = \delta \mathscr{D}^* \delta \mathscr{D} z \quad (A6)$$

If z, i.e. $\begin{bmatrix} \delta D_0' \\ \delta D_1' \end{bmatrix} w$, were to equal zero, then so would μw from (A5). This is not possible (both μ and w are nonzero), so z is an eigenvector of $\delta \mathscr{D}^* \delta \mathscr{D}$ and μ is an eigenvalue.

Hence it is proved that the operator $\delta \mathscr{D}^* \delta \mathscr{D}$ and the matrix $M(\delta \mathscr{D})$ share the same nonzero eigenvalues. ■

Proof of Theorem 2

Since the operator $\delta \mathscr{D}^* \delta \mathscr{D}$ is compact and self-adjoint (Lemma 2.1), the upper bound $d (> 0)$ of inequality (12)

$$\delta \mathscr{D}^* \delta \mathscr{D} \leqq d \mathscr{I}$$

is evaluated with the maximum eigenvalue of $\delta \mathscr{D}^* \delta \mathscr{D}$. In Lemma 2.2, it is shown that the operator $\delta \mathscr{D}^* \delta \mathscr{D}$ and the matrix $M(\delta \mathscr{D})$ defined in (A2) share same nonzero eigenvalues.

Hence the upper bound $d (> 0)$ of inequality (12) is evaluated by the matrix norm of

$$d = \left\| \delta D_0 \delta D_0' + \int_{-h}^{0} \delta D_1(\beta) \delta D_1'(\beta) d\beta \right\|_2$$

which is equal to the maximum eigenvalue of semi-positive definite matrix $M(\delta \mathscr{D})$. ■