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POLE-PLACEMENT WITHIN SPECIFIED REGIONS USING *LQ*-DESIGN

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A new procedure is presented for optimally placing closed-loop poles of multivariable continuous-time systems in specified regions via linear-quadratic *(LQ)* state-feedback design. This method has the advantages of pole-placement and *LQ*-design. In addition, it provides minimum feedback gains in the control law.

1. Introduction

The method of *LQ*-design of control systems provides excellent robustness properties and can tolerate a good deal of system nonlinearity, see for example [4]. However, the transient responses of a *LQ*-designed system may be heavily influenced by the choice of the weighting matrices Q and R. These matrices are in general determined by trial and error. Pole placement techniques may be used to place the closed-loop poles of a system at desirable locations corresponding to desirable responses. It is, however, too restrictive to place poles at exact locations. To retain both of the desirable features, namely, good transient responses and robustness properties, several *LQ*-design methods have been proposed to place the closed-loop poles within a specified region [1]-[3],[5]-[7].

Anderson and Moore [1] utilized a shifted system matrix to place poles in the open left-hand side of a vertical line on the negative real axis. Shieh et al. [5] placed poles within a vertical strip. Wittenmark et al. [7] and Furuta et al. [2] assigned poles within a circle using the *LQ*-design method for a transformed system and using the discrete Riccati equation, respectively. Kawasaki and Shimemura [3] proposed an iterative method for locating poles in an open hyperbola. Shien et al. [6] modified this iterative method and placed poles within an open sector.

A common feature of the methods mentioned above is that they place all the poles in one common region and have no control on the locations of the individual poles. Consequently, the design of systems with more specific dynamic performances may not be achieved easily; for example, it may be desirable to have the damping ratio of a second-order *SISO* system or a higher-order *SISO* system with a pair of dominant closed-loop poles lie within a specific range, such as between 0.4 and 0.8, in order to achieve satisfactory responses [8].

A new and versatile method for optimally placing the closed-loop poles of multivariable linear systems in specified regions using the *LQ*-design method is presented in this paper. The proposed method has not only the flexibility in dealing with individual poles, but also has no restriction on the locations of the regions. In addition, the method can be used to find a set of feedback gains which minimizes a linear quadratic performance measure via gain optimization. Therefore, robust feedback control systems with the following properties can be designed:
(1).Pole placement in specific regions,
(2).Robustness properties of LQ-design,
(3).Feedback gain optimization.
The effectiveness of the method will be demonstrated by two examples.

2. Pole-Placement within Specified Regions

Problem Formulation

Consider the linear time-invariant controllable system described by

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0, \tag{2.1}$$

$$y(t) = Cx(t), \tag{2.2}$$

where
$x(t) \in R^n$, $u(t) \in R^r$, $y(t) \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times r}$, and $C \in R^{m\times n}$.
The problem investigated in this paper is to determine the optimal LQ control $u(t)=-Kx(t)$, which minimizes the following quadratic performance measure, such that all closed-loop

poles are placed within specified regions for desirable responses:

$$J_1 = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt, \qquad (2.3)$$

where R and Q are symmetric positive definite matrices. The elements of R may be pre-determined to weigh the relative importance of the elements of the input. The weighting matrix Q can be determined during the design process according to the desirable pole locations. The relationship between Q and the optimal control $u(t) = -Kx(t)$ which minimizes J_1 is given by the well known matrix Riccati equation

$$PBR^{-1}B^TP - A^TP - PA - Q = 0, \qquad (2.4)$$

$$K = R^{-1}B^TP. \qquad (2.5)$$

According to the Lyapunov theory, the linear time-invariant system described by

$$\dot{x}(t) = A'x(t), \qquad (2.6)$$

is stable if and only if there exists a symmetric positive definite solution of the following Lyapunov equation for any given symmetric positive definite matrix Q

$$A'^TP + PA' = -Q. \qquad (2.7)$$

If all the eigenvalues of A' are distinct, then there exits only one solution P [8]. It is known that there exits such a solution P of the matrix Riccati equation (2.4) if and only if the matrices R and Q are symmetric positive definite, and the closed-loop system $A-BK$ is stable. Therefore, we can obtain a solution P of (2.4) by satisfying the following conditions:

$$R > 0,$$

$$Q = PBR^{-1}B^TP - A^TP - PA > 0,$$

$$Re[\lambda_i(A-BK)] < 0, \qquad i=1,\ldots,n$$

where $\lambda_i[.]$ denotes the ith eigenvalue of $[.]$, and $Re\lambda_i[.]$ and $Im\lambda_i[.]$ denote the real and imaginary parts of $\lambda_i[.]$, respectively.

To achieve pole-placement within desirable regions, the following inequality constraints, which describe the boundary condition of the regions, must be satisfied:

$$f_i[\lambda_i(A-BK)] \geq 0. \qquad i=1,\ldots,n$$

For example, consider a circular region with origin at (a, jb) and a radius of r. Then, we have

$$f_i = r - \sqrt{[Re\lambda_i(A-BK) - a]^2 + [Im\lambda_i(A-BK) - b]^2}$$

and $\lambda_i(A-BK)$ lies within the ith region for

$$f_i[\lambda_i(A-BK)] \geq 0.$$

The problem investigated in this paper can be posed as an optimization problem

$$\min \; J_2 = 0.5 \times \sum_{i=1}^{n\times r} k_i^2, \qquad (2.8)$$

with respect to the $n(n+1)/2$ elements of P, and subject to

$$\lambda_i(Q) > 0, \qquad i=1,..,n \qquad (2.9)$$

$$f_i[\lambda_i(A-BK)] \geq 0. \qquad i=1,..,n \qquad (2.10)$$

where k_i, $i=1,\ldots,r\times n$ are the elements of the control gain matrix K,, and Q and K are determined by

$$Q = PBR^{-1}B^TP - A^TP - PA, \qquad (2.11)$$

$$K = R^{-1}B^TP. \qquad (2.12)$$

During the design process, the elements of the matrix P are adjusted to satisfy (2.8),(2.9) and (2.10).

It is not possible to obtain analytical derivatives for the two constraints given by (2.9) and (2.10) because of the complicated relationships between $\lambda_i(Q)$, $\lambda_i(A-BK)$ and the matrix P. Consequently, we have to use either numerical derivatives or direct search methods which avoid the evaluation of derivatives, either analytically or numerically.

To provide a quantitative measure of (2.9) and (2.10), we define the variable y_i as

$$y_i = \begin{cases} 0 & \lambda_i(Q) > 0 \\ \alpha & \text{otherwise} \end{cases} \qquad i=1,\ldots,n \qquad (2.13)$$

and the symbol $<f_i>$ as

$$<f_i> = \begin{cases} -f_i & f_i < 0 \\ 0 & f_i \geq 0 \end{cases} \qquad i=1,\ldots,n \qquad (2.14)$$

where $\alpha > 0$ also serves as a scaling factor for removing improper differences in magnitude between values of y_i and $<f_i>$, which sometimes may cause difficulty in finding feasible solutions, particularly when penalty functions are used.

Several types of penalty functions can be attached to unconstrained minimization methods to ensure that constraints are satisfied [9]. For example,

$$Jp = J_2 + \gamma \sum_{i=1}^{n} <f_i> + \gamma \sum_{i=1}^{n} y_i , \qquad (2.15)$$

where Jp is an artificial objective function, J_2 is the objective function given in (2.8), and γ is a fixed large number.

The optimum solution K obtained may not be the global minimum point with respect to the objective function in (2.8). However, as long as the constraints (2.9) and (2.10) are satisfied, the closed-loop poles will be placed within the desirable regions, and the optimum solution K will be the global minimum point with respect to the quadratic performance measure given in (2.3). It is usually not possible to find a global minimum for a general nonlinear constrained optimization problem, but this is rarely an impediment to the satisfactory solutions of practical problems.

Computation Procedure

The computation procedure is described as follows:

Step 1: Select an initial symmetric positive definite matrix P_0, and specify R according to the desirable weight to the input.

Step 2: Compute Q according to (2.11). Call a subroutine to compute the eigenvalues of Q. The result is used to determine y_i in (2.13).

Step 3: Compute the control gain matrix K according to (2.12). Compute the closed-loop system matrix $A-BK$ and call a subroutine to compute the eigenvalues $\lambda_i(A-BK)$. Reorder the eigenvalues to ensure that each eigenvalue lying outside the regions will be moved into the nearest region. Compute f_i according to specified pole locations. The result is used to determine $<f_i>$ in (2.14).

Step 4: The results from Step 2 and Step 3 are used to determine the value of the artificial objective function, for example, the one given in (2.15). Call a minimization subroutine to minimize the value of Jp by adjusting the matrix P.

Step 2, 3, and 4 are iterated automatically via the minimization subroutine until an optimum solution is found. Different local minimum points with respect to the objective function in (2.8) may be obtained by providing different starting points, and the one with the smallest value of J_2 should be chosen as the final solution.

3. Examples

Example 1: SISO System

Let the system shown in Figure 3.1 be described by the linear nth order state space model

$$\dot{x}(t) = Ax(t) + bu(t), \; x(0) = x_0, \qquad (3.1)$$

$$y(t) = Cx(t), \qquad (3.2)$$

and the control is given by

$$u(t) = -kx(t) + r(t) \qquad (3.3)$$

where k is the feedback gain vector,and

$$A = \begin{bmatrix} 0.25 & 1.10 & -4.45 \\ 0.40 & -1.00 & -2.40 \\ 1.45 & -0.90 & -1.65 \end{bmatrix},$$

$$b = \begin{bmatrix} 1.00 \\ 2.00 \\ 3.00 \end{bmatrix}, \quad c^T = \begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix},$$

the eigenvalues of A are

$$\lambda(A) = \{-0.20 \pm j2.00, -2.00\}.$$

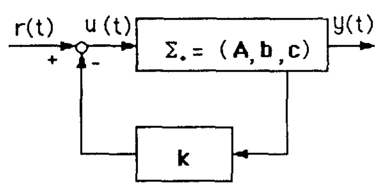

Figure 3.1 Control System of Example 1

The objective is to find a linear feedback control such that

(1).The closed-loop frequency response meets the following specifications:

Damping ratio: $0.4 \le \zeta \le 0.8$,
Time to first peak: $t_s \le 2.2$ second,
System bandwidth: $\omega_b \le 10$;

(2).The control is optimal with respect to (2.3); and

(3).The control is also optimal with respect to (2.8).

For a $SISO$ system given by (3.1) and (3.2), the transient responses depend mainly upon the locations of a pair of dominant complex conjugate poles if they exist, and, therefore, this system can be treated as a second order system, for which the performances can be expressed in terms of the pole locations as follows

$$t_s = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad (3.4)$$

$$\omega_b = \omega_n\sqrt{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}}, \quad (3.5)$$

$$|\lambda_1| = |\lambda_2| = \omega_n, \quad (3.6)$$

$$\theta = cos^{-1}\zeta, \quad (3.7)$$

where ω_n denotes natural frequency.

Substituting the required specifications into (3.4)–(3.7), we obtain

$$36.9^o \le \theta \le 66.4^o, \quad and \quad 2.38 \le |\lambda_i| \le 7.28, i=1,2 .$$

The desirable regions can be determined as a circle with origin at (-2.0,j2.4) and a radius of 0.7; a circle with origin at (-2.0,-j2.4) and a radius of 0.7; and on the real axis left of $s = -10$, assuming that poles in this region have negligible effect on the closed-loop transient responses. The specified regions are as shown in Figure 3.2.

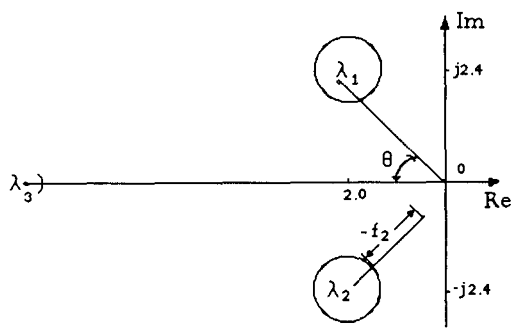

Figure 3.2 Specified Regions for Example 1

The optimization subroutine used is *SEEK* [9], which is based on Hook and Jeeves direct search method. The subroutine for computing the penalty function is *OPTIM 5* [9], which is based on the method given by Schuldt et al.

We follow the proposed design procedure of Section 2. Let $R = I_1$ and the initial matrix

$$P_0 = \begin{bmatrix} 2.10 & 0.04 & -0.30 \\ 0.04 & 0.70 & -0.40 \\ -0.30 & -0.40 & 0.70 \end{bmatrix}.$$

Under the initial condition, Q_0 is not a positive definite matrix, and the closed-loop poles are not located within the specified regions, i.e.,

$$\lambda(A-bk) = \{ -7.80, -0.14, 0.78\}.$$

The optimum solution is found to be

$$k = [-5.56 \quad 3.83 \quad 2.97],$$

$$Q = \begin{bmatrix} 34.31 & -31.65 & -7.10 \\ -31.65 & 31.40 & 6.51 \\ -7.10 & 6.51 & 3.34 \end{bmatrix},$$

$$J_2 = 0.5 \times \sum_{i=1}^{3} k_i^2 = 27.23.$$

The closed-loop eigenvalues are

$$\lambda(A-bk) = \{-1.63 \pm j2.57, -10.14\}.$$

Example 2: MIMO System

Consider the dynamic system in (2.1) and (2.2), and assume $\tilde{y}$ to be the output of a known second order system

$$\dot{z}(t) = Fz(t), \qquad z(0) = z_0 \quad (3.8)$$

$$\tilde{y}(t) = Hz(t), \quad (3.9)$$

where $z(t) \in R^p$, $\tilde{y}(t) \in R^m$, $F \in R^{p \times p}$, and $H \in R^{m \times p}$.

The objective is to find an optimal tracking control such that the output of $\Sigma_0 = (A,B,C)$ follows the response of $\Sigma = (F,H)$, and such that the closed-loop poles of $\Sigma_0 = (A,B,C)$ are placed within specified regions to achieve good transient responses. The quadratic performance measure is

$$J = \int_0^\infty (x^T(t)Q_1x(t) + u^T(t)Ru(t) + [y(t) - \tilde{y}(t)]^T Q_2 [y(t) - \tilde{y}(t)])dt. \quad (3.10)$$

For a completely observable system $\Sigma = (F,H)$, (3.10) can be rewritten as

$$J=\int_0^{\infty}\{x^T(t)Q_1x(t)+u^T(t)Ru(t)+[x(t)-\tilde{x}(t)]^TQ_3[x(t)-\tilde{x}(t)]\}dt. \quad (3.11)$$

where $Q_3=C^TQ_2C$, and $\tilde{x}(t)=C^T(CC^T)^{-1}\tilde{y}(t)$. The tracking system is as shown in Figure 3.3.

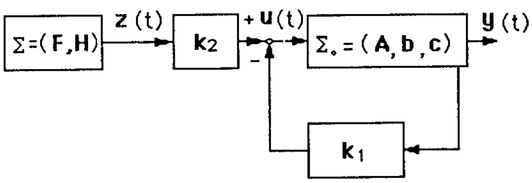

Figure 3.3 Control System of Example 2

To apply the result of LQ regulator design, we define an augmented system as

$$\hat{x}(t)=\begin{bmatrix}x(t)\\ z(t)\end{bmatrix},\quad \hat{A}=\begin{bmatrix}A & 0\\ 0 & F\end{bmatrix},\quad \hat{B}=\begin{bmatrix}B\\ 0\end{bmatrix},\ \text{and}$$

$$\hat{Q}=\begin{bmatrix}Q_1+Q_3 & -Q_3LH\\ -H^TL^TQ_3 & H^TL^TQ_3LH\end{bmatrix},$$

where $L=C^T(CC^T)^{-1}$. Then we obtain

$$\dot{\hat{x}}(t)=\hat{A}\hat{x}(t)+\hat{B}\hat{u}(t),\quad \hat{x}(0)=\hat{x}_0, \qquad \text{and}$$

$$J=\int_0^{\infty}[\hat{x}^T(t)\hat{Q}\hat{x}(t)+u^T(t)R\,u(t)]dt. \quad (3.12)$$

The optimal control is

$$u(t)=R^{-1}B^T\hat{P}\hat{x}(t),$$

where $\hat{P}$ is the solution of the matrix Reccati equation for the augmented system. Let

$$\hat{P}=\begin{bmatrix}P & P_{21}^T\\ P_{21} & P_{22}\end{bmatrix},$$

then P,P_{21}, and P_{22} are the solution of the following matrix Riccati equations, respectively:

$$PBR^{-1}B^TP-A^TP-PA-(Q_1+Q_3)=0, \quad (3.13)$$

$$P_{21}BR^{-1}B^TP-F^TP_{21}-P_{21}A+H^TL^TQ_3=0, \quad (3.14)$$

$$P_{21}BR^{-1}B^TP_{21}-F^TP_{22}-P_{22}F-H^TL^TQ_3LH=0. \quad (3.15)$$

The optimal control is $u(t)=-K_1x(t)+K_2z(t)$, where $K_1=R^{-1}B^TP$, $K_2=-R^{-1}B^TP_{21}^T$.

The computation procedure is as follows. First, follow the procedure of Section 2 to solve the Riccati equation (3.10), and obtain P, $Q=Q_1+Q_3$, K_1, and $\lambda(A-BK)$. Then, partition Q into Q_1 and Q_3 such that $\hat{Q}$,Q_1 and Q_3 are symmetric positive definite matrices. Finally, solve the linear equations (3.11) and (3.12) to obtain P_{21}, P_{22} and K_2.

Consider the systems with matrices A,B,C,F, and H as

$$A=\begin{bmatrix}0.25 & 1.10 & -4.45\\ 0.40 & -1.00 & -2.40\\ 1.45 & -0.90 & -1.65\end{bmatrix},$$

$$B=\begin{bmatrix}-1.00 & 1.00\\ -1.00 & -1.00\\ 1.00 & -1.00\end{bmatrix},$$

$$C=\begin{bmatrix}1.00 & 1.00 & 0.00\\ 0.00 & 1.00 & 1.00\end{bmatrix},$$

$$F=\begin{bmatrix}-1.00 & 0.00\\ 0.00 & -1.00\end{bmatrix},\quad H=\begin{bmatrix}1.00 & 0.00\\ 0.00 & 1.00\end{bmatrix}.$$

The closed-loop poles are to be placed within the following regions to achieve good transient response: a circle with origin at (-1.5, j1.8) and a radius of 0.6; a circle with origin at (-1.5, -j1.8) and a radius of 0.6; and on the real axis left of $s=-8$, respectively.

The results are as follows

$$R=I_2,$$

$$K_1=\begin{bmatrix}-1.437 & -0.870 & 2.303\\ 1.734 & -2.125 & -3.261\end{bmatrix},$$

$$K_2=\begin{bmatrix}-0.087 & -0.012\\ -0.006 & -0.035\end{bmatrix},$$

$$Q_1=\begin{bmatrix}4.132 & -5.584 & -6.432\\ -5.584 & 9.029 & 11.630\\ -6.432 & 11.630 & 27.242\end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0.222 & 0.000 & 0.000 \\ 0.000 & 0.424 & 0.000 \\ 0.000 & 0.000 & 0.330 \end{bmatrix},$$

The optimal control $u_1(t)=-K_1x(t)$ places the closed-loop poles within the desirable regions. The poles are

$$\lambda(A-BK) = \{-10.097, -2.016 \pm j1.660\}.$$

4. Conclusion

This paper presents a method for optimally placing closed-loop poles of multivariable continuous-time systems in desirable regions via *LQ*-design and gain minimization. The problem is posed as a constrained optimization problem. The feedback gains are minimized subject to the matrix Riccati equations and the boundary conditions of the specified regions of the closed-loop poles.

Robustness properties of *LQ*-design, and desirable transient and steady-state responses, as well as minimum feedback gains in the control law are achieved by implementing the proposed method. The method has the following advantages: (1).It deals with poles individually; (2).It has no restriction on the locations of the regions; and (3).It leads to minimum feedback gains in the control law.

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