'89 KACC 1989. 10. 27～28

A Stability Condition of Minimal Variance Control with Mismatch of Time Delay

H.Hashimoto Y.Takenami K.Akizuki

Faculty of Science and Engineering, Waseda University
3-4-1 OHKUBO, SHINJUKU-KU, TOKYO 169, JAPAN

Abstract: This paper presents a stability condition for Åström's minimal variance control(MVC) with mismatch of time delay for a SISO ARMAX model containing time delay. The proof of the condition presented here is based on the characteristic equation in the feedback system and its magnitude. This condition, from easy numerical calculation, is able to find the stability of the feedback system without knowing the real time delay.

1 Introduction

The minimal variance control(MVC) strategy for single input single output discrete time linear stochastic system containing time delay is studied by Åström[1]. This strategy is an extremely simple algorithm which can provide effective regulation or tracking when the parameters of the system and disturbance structures are known exactly. However, the time delay used in a control system does not coincide in general with that in the real system. Therefore, the feedback system designed by the MVC law, having some mismatch of the time delay, may not guarantee stability of the feedback system.

This paper presents a stability condition for MVC with mismatch of time delay. The proof of the condition presented here uses the characteristic equation in the feedback system and its magnitude. This condition, from easy calculation, is able to find the stability of the feedback system without knowing the real time delay. Comparing the presented condition with that studied by another author's, it is found theoretically that the presented condition has much effectiveness in terms of sufficiency. Furthermore, a simple design procedure based on the presented condition is demonstrated to stabilize the unstable feedback system caused by mismatch of time delay.

2 Problem Statement

Consider the minimal variance control (MVC) of a SISO ARMAX system representation of the form

$$A(z^{-1})y(k) = z^{-d}B(z^{-1})u(k) + C(z^{-1})w(k) \quad (1)$$

where $y(k)$ and $u(k)$ denote the output and input respectively, and $\{w(k)\}$ is a sequence of independent, equally distributed normal $(0,\sigma_w^2)$ random variables. $A(z^{-1}), B(z^{-1})$ and $C(z^{-1})$ are scalar polynomials in the unit delay operator z^{-1} . Then

$$A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_{n_a} z^{-n_a}$$

$$B(z^{-1}) = b_0 + b_1 z^{-1} + \cdots + b_{n_b} z^{-n_b}$$

$$C(z^{-1}) = 1 + c_1 z^{-1} + \cdots + c_{n_c} z^{-n_c}$$

The following assumption will be made about the system (1).

[Assumption 1]

1. The degrees and coefficients of $A(z^{-1}), B(z^{-1})$ and $C(z^{-1})$ are known.

2. $B(z^{-1}), C(z^{-1})$ are stable polynomials.

3. The time delay d is greater than or equal to 1.

4. The reference signal $y_R(k)$ is bounded, and the d step ahead time value of it can be used.

In the case of knowing the time delay d , under Assumption 1, the admissible control input $u(k)$ that minimizes

$$J = E\{(y(k) - y_R(k))^2\} \tag{2}$$

is generated by following [1]

$$u(k) = \frac{C(z^{-1})y_R(k+d) - S(z^{-1})y(k)}{B(z^{-1})R(z^{-1})} \tag{3}$$

where polynomials $R(z^{-1})$ and $S(z^{-1})$, which are of degrees $d-1$ and $n_a - 1$ respectively, are determined uniquely by the following synthesis equation [2]

$$C(z^{-1}) = A(z^{-1})R(z^{-1}) + z^{-d}S(z^{-1}) \tag{4}$$

where

$$deg\{R(z^{-1})\} = d-1,\ deg\{S(z^{-1})\} = n_a - 1$$

and

$$\begin{aligned} R(z^{-1}) &= 1 + r_1 z^{-1} + \cdots + r_{d-1} z^{-(d-1)} \\ S(z^{-1}) &= s_0 + s_1 z^{-1} + \cdots + s_{n_a-1} z^{-(n_a-1)} \end{aligned}$$

Next, consider the MVC law with mismatch of time delay. Let $\hat{d}$ be the time delay used in the design of the MVC, then $\hat{R}(z^{-1})$ and $\hat{S}(z^{-1})$, similar in its calculation to eq.(4), are

$$C(z^{-1}) = A(z^{-1})\hat{R}(z^{-1}) + z^{-\hat{d}}\hat{S}(z^{-1}) \tag{5}$$

where

$$deg\{\hat{R}(z^{-1})\} = \hat{d}-1,\ deg\{\hat{S}(z^{-1})\} = n_a - 1$$

From eqs.(4) and (5), when $\hat{d}$ is not equal to d, $\hat{R}(z^{-1}) \neq R(z^{-1})$ and $\hat{S}(z^{-1}) \neq S(z^{-1})$. In this case, the control input is generated by

$$u(k) = \frac{C(z^{-1})y_R(k+\hat{d}) - \hat{S}(z^{-1})y(k)}{B(z^{-1})\hat{R}(z^{-1})} \tag{6}$$

Note that it is not guranteed that the polynomial $R(z^{-1})$ (or($\hat{R}(z^{-1})$)) defined by eq.(4)(or (5)) is stable.

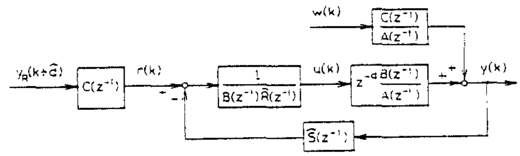

Figure 1: The feedback system with mismatch of time delay.

3 A Stability Condition with Mismatch of Time Delay

The feedback system designed by the MVC law with mismatch of time delay is shown in Fig.1. The closed loop transfer function from $r(k)$ to $y(k)$ is defined by

$$G_{CL}(z^{-1}) = \frac{z^{-d}B(z^{-1})}{B(z^{-1})[A(z^{-1})\hat{R}(z^{-1}) + z^{-d}\hat{S}(z^{-1})]} \tag{7}$$

In spite of the time delay $\hat{d}$ not appearing explicitly in the right side of (7), $G_{CL}(z^{-1})$ will become stable or unstable according to the value of $\hat{d}$, because $\hat{R}(z^{-1})$ and $\hat{S}(z^{-1})$ are determined depending on $\hat{d}$. The poles of $G_{CL}(z^{-1})$ cannot be examined directly, since the real time delay d in the right side of (7) is unknown. Therefore, we will consider the derivation of the stability condition without knowing the real time delay. The characteristic equation of the feedback system is defined by

$$\psi(z^{-1}) = B(z^{-1})\{A(z^{-1})\hat{R}(z^{-1}) + z^{-d}\hat{S}(z^{-1})\} \tag{8}$$

To examine the stability of $\psi(z^{-1})$, it is sufficient to examine the roots of $[A(z^{-1})\hat{R}(z^{-1}) + z^{-d}\hat{S}(z^{-1})]$ shown in eq.(8), since $B(z^{-1})$ is a stable polynomial from Assumption 1. From this characteristic equation, we will get the theorem concerned with the stability of the feedback system with mismatch of time delay.

Theorem 1 *Under Assumption 1,suppose that $A(z^{-1})\hat{R}(z^{-1})$ has its all poles inside the unit circle and satisfies the following inequality*

$$|\hat{S}(z^{-1})| < |A(z^{-1})\hat{R}(z^{-1})|,\ at\ |z^{-1}| = 1 \tag{9}$$

then the feedback system is stable.

In order to prove this theorem, we need the following Lemma 1 and Lemma 2.

Lemma 1 *Let the nth polynomial $f(z)$, with real coefficients, have all its roots outside the unit circle, then*

$$|f(z)| = |f^*(z)|, \quad at\ |z| = 1 \tag{10}$$

where

$$f(z) = f_0 + f_1 z + \cdots + f_n z^n \tag{11}$$

and, $f^*(z)$ is the reciprocal polynomial defined by

$$f^*(z) = z^n f(z^{-1}) \tag{12}$$

(Proof) see [3]

Lemma 2 *Consider the following equations*

$$\begin{aligned} f(z^{-1}) &= A(z^{-1})\hat{R}(z^{-1}) + z^{-d}\hat{S}(z^{-1}) && (13)\\ g(z^{-1}) &= A(z^{-1})\hat{R}(z^{-1}) && (14)\end{aligned}$$

$$\begin{aligned} deg\{f(z^{-1})\} &= \begin{cases} n_a + \hat{d} - 1 & (\hat{d} - d \geq 0) \\ n_a + d - 1 & (d - \hat{d} > 0) \end{cases} \\ deg\{g(z^{-1})\} &= n_a + \hat{d} - 1 \end{aligned}$$

The reciprocal polynomials of the above equations are given by

(a) $\hat{d} \geq d$

$$f^*(z) = (A(z)\hat{R}(z))^* + z^{\hat{d}-d}\hat{S}^*(z) \tag{15}$$
$$deg\{f^*(z)\} = n_a + \hat{d} - 1$$

(b) $d > \hat{d}$

$$f^*(z) = z^{d-\hat{d}}(A(z)\hat{R}(z))^* + \hat{S}^*(z) \tag{16}$$
$$deg\{f^*(z)\} = n_a + d - 1$$

and

$$g^*(z) = (A(z)\hat{R}(z))^* \tag{17}$$
$$deg\{g^*(z)\} = n_a + \hat{d} - 1$$

where $(A(z)\hat{R}(z))^*$ and $\hat{S}^*(z)$ are the reciprocal polynomials of $(A(z)\hat{R}(z))$ and $\hat{S}(z)$, respectively.

(Proof) From (8), the coefficient of the lowest order of $\hat{S}(z^{-1})$ is added to the dth of $A(z^{-1})\hat{R}(z^{-1})$. Note this equal power addition and that the orders of $A(z^{-1})\hat{R}(z^{-1})$ and $\hat{S}(z^{-1})$ are $n_a + \hat{d} - 1$ and $n_a - 1$ respectively, and thus we find that the order of $f(z^{-1})$ depends on the sizes of d and $\hat{d}$.

Consider the case of $\hat{d} \geq d$

$$\begin{aligned} deg\{f(z^{-1})\} &= deg\{A(z^{-1})\hat{R}(z^{-1})\} \\ &= n_a + \hat{d} - 1 && (18)\end{aligned}$$

From the definition of the reciprocal polynomial

$$\begin{aligned} (A(z)\hat{R}(z))^* &= z^{n_a+\hat{d}-1}A(z^{-1})\hat{R}(z^{-1}) && (19)\\ \hat{S}^*(z) &= z^{n_a-1}\hat{S}(z^{-1}) && (20)\end{aligned}$$

and taking into account (12), we get the reciprocal polynomial of $f(z^{-1})$ as follows

$$\begin{aligned} f^*(z) &= z^{n_a+\hat{d}-1}f(z^{-1}) \\ &= z^{n_a+\hat{d}-1}\{A(z^{-1})\hat{R}(z^{-1}) + z^{-d}\hat{S}(z^{-1})\} \\ &= (A(z)\hat{R}(z))^* + z^{\hat{d}-d}\hat{S}^*(z) && (21)\end{aligned}$$

Next, consider the case of $d > \hat{d}$

$$\begin{aligned} deg\{f(z^{-1})\} &= deg\{\hat{S}(z^{-1})\} + d \\ &= n_a - 1 + d && (22)\end{aligned}$$

Taking into account (12),(19) and (20), from (22), we get

$$\begin{aligned} f^*(z) &= z^{n_a+d-1}f(z^{-1}) \\ &= z^{n_a+d-1}\{A(z^{-1})\hat{R}(z^{-1}) + z^{-d}\hat{S}(z^{-1})\} \\ &= z^{d-\hat{d}}(A(z)\hat{R}(z))^* + \hat{S}^*(z) && (23)\end{aligned}$$

This compeletes the proof of Lemma2.

Using Lemma 1 and 2, we prove the theorem.

(Proof of Theorem 1) Since $B(z^{-1})$ is assumed to be a stable polynomial, we only examine the stability of $f(z^{-1})$ instead of the characteristic equation $\psi(z^{-1})$ in (8). The stability of $f(z^{-1})$ is determined by the locations of the roots of $f^*(z)$,so we examine the number of the roots inside the unit circle by using $g^*(z) = (A(z)\hat{R}(z))^*$. From the assumption of Theorem 1, all roots of $g^*(z)$

exist inside the unit circle. Then, if the following inequality is satisfied on the unit circle $|z| = 1$

$$|f^*(z) - g^*(z)| < |g^*(z)| \tag{24}$$

then from Rouchè's Theorem[3], $f^*(z)$ has the same number of roots inside the unit circle as $g^*(z)$. As shown in Lemma 2, the order of $f^*(z)$ in the case of $\hat{d} \geq d$ differs from that in the case of $d > \hat{d}$, so we consider them separately. First, consider the case of $\hat{d} \geq d$, from (14) and (18), then

$$deg\{f^*(z)\} = deg\{g^*(z)\} = n_a + \hat{d} - 1$$

$f^*(z)$ has the same number of roots as $g^*(z)$. Therefore, if all roots of $g^*(z)$ exist inside the unit circle and the inequality in (24) is satisfied, then all roots of $f^*(z)$ exist inside the unit circle. Note that from Lemma 1 the following equations are satisfied on the unit circle

$$|f(z^{-1}) - g(z^{-1})| = |\hat{S}(z^{-1})| = |\hat{S}^*(z)| \tag{25}$$

$$|g(z^{-1})| = |A(z^{-1})\hat{R}(z^{-1})| = |(A(z)\hat{R}(z))^*| \tag{26}$$

Thus, for $|z^{-1}| = 1$, eq.(9) is equivalent to eq.(24). Therefore, if the inequality in (9) is satisfied, then the characteristic equation (8) is stable. Secondly, we consider the case of $d > \hat{d}$. In this case the order of $f^*(z)$ is $n_a + d - 1$ from (22). Consider the polynomial $z^{d-\hat{d}}g^*(z)$ having the same order as $f^*(z)$. This polynomial has $(d - \hat{d})$ roots at the origin and $n_a + d - 1$ stable roots except the origin, so it is a stable polynomial. Hence, if the following inequality is satisfied

$$|f^*(z) - z^{d-\hat{d}}g^*(z)| < |z^{d-\hat{d}}g^*(z)| \tag{27}$$

then all roots of $f^*(z)$ exist inside the unit circle. Because of

$$|f^*(z) - z^{d-\hat{d}}g^*(z)| = |\hat{S}^*(z)| \tag{28}$$

$$|z^{d-\hat{d}}g^*(z)| = |(A(z)\hat{R}(z))^*| \tag{29}$$

so we get the same result as in the case of $\hat{d} \geq d$.

As the result, when eq.(9) is satisfied, all roots of $\psi(z^{-1})$ exist inside the unit circle, i.e., the feedback system shown in (7) is stable. This is the required result.

From eq.(5) and the proof of Theorem 1, we find that this condition for stability does not require the real time delay. When $A(z^{-1}), B(z^{-1})$ and $C(z^{-1})$ are known, it is expected that there are some values of $\hat{d}$ that satisfies the stability condition in (9). But, the values depend on $A(z^{-1}), B(z^{-1})$ and $C(z^{-1})$ only, not on the real time delay.

The usefulness of this theorem is that from (9) the stability of the characteristic equation $\psi(z^{-1})$ can be verified directly, because $\hat{R}(z^{-1})$ and $\hat{S}(z^{-1})$ can be calculated from eq.(5) without knowing the real time delay.

One of the methods that examines the inequality in (9) is to calculate the both sides of eq.(9) numerically, that is, setting $z^{-1} = exp(-j\omega)$, they can be calculated easily.

4 Discussion

In order to confirm the effectiveness of the derived condition in terms of sufficiency, we will compare it with that studied by Gawthrop [4]. The Gawthrop's condition for stability of the feedback system requires

$$g_{FF} \times g_{FB} < 1 \tag{30}$$

where g_{FF} and g_{FB} are the gains of feedforward and feedback blocks respectively ,shown in Fig.1 and defined by

$$\begin{aligned} g_{FF} &= \max_{\omega}\left|\frac{z^{-d}}{A(z^{-1})\hat{R}(z^{-1})}\right| \\ &= \max_{\omega}\left|\frac{1}{A(z^{-1})\hat{R}(z^{-1})}\right| \qquad (31) \\ g_{FB} &= \max_{\omega}\left|\hat{S}(z^{-1})\right| \qquad (32) \end{aligned}$$

Note that the above condition is based on the Small Gain Theorem[5] and does not require the real time delay, also.

Since the value of $A(z^{-1})\hat{R}(z^{-1})$ is non-zero except in special cases, it follows from eqs.(9),(31) and (32) that

$$1 > \frac{\left|\hat{S}(z^{-1})\right|}{\left|A(z^{-1})\hat{R}(z^{-1})\right|} \geq \frac{\max_\omega \left|\hat{S}(z^{-1})\right|}{\min_\omega \left|A(z^{-1})\hat{R}(z^{-1})\right|}$$

$$= \max_{\omega}\left|\hat{S}(z^{-1})\right| \times \max_{\omega}\left|\frac{1}{A(z^{-1})\hat{R}(z^{-1})}\right| \quad (33)$$

The above result points out that the derived condition has more effectiveness, in terms of sufficiency, than Gawthrop's. The stability condition for general modeling error was investigated by Åström [6]. This condition, however, requires the actual knowledge of the modeling error, so the knowledge of the real time delay, also. In respect of knowledge of the real time delay, the derived condition does not require it. For these reasons, the derived condition seems to be useful in practice.

5 Examples

[Example 1]

Consider the system

$$\begin{aligned} A(z^{-1}) &= 1 - 1.5z^{-1} + 0.7z^{-2} \\ B(z^{-1}) &= 1 + 0.3z^{-1} \\ C(z^{-1}) &= 1 - 0.3z^{-1} + 0.35z^{-2} \\ d &= 2 \end{aligned}$$

Table 1. *The results of the roots of $\psi(z^{-1})$,(9) and the roots of $\hat{R}(z^{-1})$.*

$\hat{d}$	ψ	(9)	Roots of $\hat{R}(z^{-1})$
1	U	×	...
2	S	×	-1.2
3	U	×	$-0.6 \pm j1.044$
4	U	×	$-1.04, -0.0799 \pm j1.13$
5	U	×	$-0.7751 \pm j0.5678, 0.1751 \pm j1.019$
6	S	×	-0.8252
7	S	○	$-0.4567 \pm j0.7398, 0.2692 \pm j0.8968$ $-0.5241 \pm j0.07797, 0.2651 \pm j0.8499$ $-0.3411 \pm j0.6803$
8	S	×	$0.3535, 0.3244 \pm j0.8798$ $-0.3123 \pm j0.8505, -0.7889 \pm j0.3465$
9	S	×	$0.5924, -0.9053, -0.1612 \pm j0.9425$ $0.4102 \pm j0.8586, -0.6925 \pm j0.603$
10	S	×	$0.7101, 0.4742 \pm j0.8128$ $-0.5406 \pm j0.7621, -0.8694 \pm j0.2886$ $-0.01925 \pm j0.9648$
11	S	×	$0.7752, -0.9021$ $0.09503 \pm j0.9473, -0.3849 \pm j0.8458$ $-0.7611 \pm j0.4964, 0.5144 \pm j0.7611$
12	S	○	$0.8109, -0.8451 \pm j0.2282$ $0.1776 \pm j0.9070, -0.2463 \pm j0.8734$ $-0.6249 \pm j0.6276, 0.5332 \pm j0.7131$

The above system satisfies Assumption 1. For this system, change $\hat{d}$ used in the MVC design from 1 to 12. In this case, Table 1 shows the sets of results are the roots of $\psi(z^{-1})$, the stability condition in (9) and the roots of $\hat{R}(z^{-1})$. The symbols shown in Table 1 denote that the S represents $\psi(z^{-1})$ be stable and the U be unstable, and ○ represents eq.(9) is satisfied and × is not.

In the case of $\hat{d}$ = 7 and 12, the derived condition (9) is satisfied and $\psi(z^{-1})$ is a stable polynomial. In the other cases, eq.(9) is not satisfied but $\psi(z^{-1})$ is a stable polynomial. The reasons for this are that the derived condition provides sufficient condtion only, and $\hat{R}(z^{-1})$ has the unstable roots in the case of $\hat{d}$ =2-5 in spite of $\psi(z^{-1})$ being stable.

[Example 2]

Consider the system

$$\begin{aligned} A(z^{-1}) &= 1 + 0.2z^{-1} + 0.44z^{-2} - 0.408z^{-3} \\ B(z^{-1}) &= 0.5 + 0.35z^{-1} \\ C(z^{-1}) &= 1 - 0.4z^{-1} + 0.4z^{-2} \\ d &= 3 \end{aligned}$$

The above system satisfies Assumption 1. The set of results are shown in Table 2. Looking over the results in the case of $\hat{d}$ =8 and 11, it is found that the derived condition is superior to the condition in (30).

Table 2. *The results of the roots of $\psi(z^{-1})$,(9) and (30).*

$\hat{d}$	$\psi(z^{-1})$	Eq.(9)	Eq.(30)
1	U	×	×
2	U	×	×
3	S	×	×
4	U	×	×
5	S	×	×
6	U	×	×
7	S	×	×
8	S	○	×
9	S	×	×
10	S	○	○
11	S	○	×
12	S	○	○
13	S	○	○

Now, using the presented condition, we give a simple design procedure to stabilize the unstable feedback system caused by mismatch of time delay. From Theorem 1, if all roots of the polynomial $A(z^{-1})\hat{R}(z^{-1})$ exist inside the unit circle and eq.(9) is satisfied, then the feedback system is stable. Therefore, not using $\hat{S}(z^{-1})$ in (5) directly, if we can select the appropriate value α such that a value $\alpha|\hat{S}(z^{-1})|$ instead of $|\hat{S}(z^{-1})|$ satisfies the derived condition in (9), the stabilized feedback system can be constructed.

For an example, we consider the case of $\hat{d} = 2$ in Example.2. In this case, $A(z^{-1})\hat{R}(z^{-1})$ has all its poles inside the unit circle, so this satisfies the assumption of Theorem 1. Fig.2 shows the trajectries of $\alpha|\hat{S}(z^{-1})|$ and $|A(z^{-1})\hat{R}(z^{-1})|$ v.s. $\omega = 0 - 2\pi$, with $z^{-1} = exp(-j\omega)$. In the case of $\alpha = 1$, the two trajectories of $\alpha|\hat{S}(z^{-1})|$ and $|A(z^{-1})\hat{R}(z^{-1})|$ intersect at six points, and the characteristic equation $\psi(z^{-1})$ is unstable. In the case of $\alpha = 0.3$, the trajectory of $\alpha|\hat{S}(z^{-1})|$ is under that of $|A(z^{-1})\hat{R}(z^{-1})|$ for $\omega = 0 - 2\pi$, and thus,from the presented condition, $\psi(z^{-1})$ is stable.

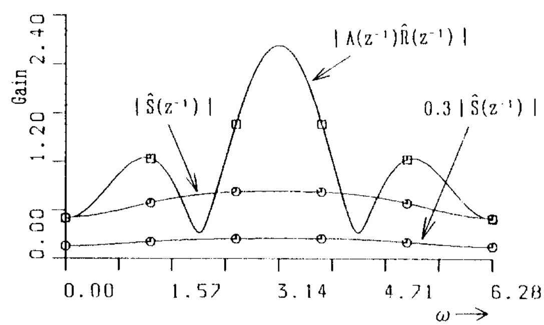

Figure 2: The trajectries of $|A(z^{-1})\hat{R}(z^{-1})|$ and $\alpha|\hat{S}(z^{-1})|$ v.s. $\omega = 0 - 2\pi$.

6 Conclusion

In this paper, we have presented a stability condition based on the characteristic equation for MVC with mismatch of time delay. Here, the stability conditions require the actual knowledge of the system in general, so they can not be used in practice. On the other hand, limiting in respect of knowledge for the real time delay, the presented condition does not require it. Though the presented condition is sufficient, it is found from the theoretical investigation that the presented condition is superior to that in the sense of the Small gain theorem. Furthermore, this condition can be verified easily by numerical calculation. Finally, even though the actual value of the real time delay is not known, a simple design procedure, based on the presented condition, which can stabilize the unstable feedback system caused by mismatch of time delay is shown.

References

[1] K.J.Åström, Introduction to Stochastic Theory, Academic Press,1970

[2] J.Ackerman, Sampled-dara Control Systems, Springer-Verlag, 1985

[3] A Book being related to complex analysis

[4] P.J.Gawthrop, Robustness of Self-Tuning Controllers, IEE Proc, vol.129,Pt.d,No.1,21/29,1982

[5] C.Desoer and M.Vidyasage, Feedback Systems Input-Output Properties, Academic Press New York

[6] K.J.Åström, Robustness of a Design Method Based on Assignment of Poles and Zeros, IEEE Trans.Automatic Control, AC-25,3,588/591,1980