Lyapunov 방정식의 해의 고유치 및 트레이스의 범위

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Eigenvalues and Trace Bounds on the Solutions of Lyapunov Equations

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요 약

본 논문에서는 연속 및 이산 Lyapunov 방정식의 해의 고유치 및 트레이스의 범위를 시스템 행렬의 고유치 및 고유벡터 행렬을 이용하여 표시한다. 이산 시스템의 경우 시스템 행렬의 최대 특이치가 1 보다 큰 경우나 연속 시스템의 경우 시스템 행렬의 대칭행렬이 불안정한 경우에도 상한 값이 항상 계산 가능한 범위가 제시된다. 본 논문에서 제시된 범위들은 몇가지 조건을 갖고 다른 문헌에서 제시된 것들 보다 정확하며, 더우기 특정한 시스템 행렬에 대해서는 범위의 상한과 하한이 일치한다.

1. INTRODUCTION

The continuous and the discrete Lyapunov equations play a fundamental role in various areas of engineering theory, particularly in control theory. In the last decade, considerable research about the continuous Lyapunov equation such as

$$A^TP + PA = -Q \tag{1.1}$$

as well as the discrete one such as

$$A^TPA - P = -Q \tag{1.2}$$

has been carried out to find estimates for some scalar quantities that express the 'extent' or 'size' of the solution of these equations. Here A, P, and Q are real n by n matrices. P and Q are symmetric and Q is supposed positive semidefinite. As such measures, the eigenvalues (especially the smallest and largest one), the determinant, the trace and some norm of the solution P were proposed. Since the computation of these quantities causes some difficulty when the dimension of the matrices involved increases, one wants to find bounds for these quantities. Mori and Derese [1] gave a comprehensive summary of the relevant bounds as well as their possible applications. Since Mori and Derese, many authors suggested the relevant bounds on the solution P in both cases [2-7]. In the continuous case, Mori and Derese [1] reported the upper and lower bounds of the maximum and minimum eigenvalues and only the lower bounds of the remaining eigenvalues. Wang *et al* [5] suggested the upper and lower bounds of the trace. But the upper bounds of the maximum eigenvalue in [1] and the trace in [5] can be calculated only when $(A+A^T)Q^{-1}$ or $(A+A^T)/2$ is stable. In the discrete case, Mori and Derese [1] reported the upper and lower bounds of all eigenvalues and the trace. Garloff [4] also suggested the upper and lower bounds of the trace. But all the upper bounds in [1] and [4] can be calculated only when the maximum singular value of A, $\sigma_1(A)$ is less than 1. Mori *et al* [3,6] suggested eigenvalue bounds for both the continuous and the discrete cases which can be always calculated. Troch [7] improved these bounds by using the similar method. But the calculation of these bounds is very complex including the calculations of differential equation, integral equation, linear equation, etc. This paper suggests the eigenvalues and trace bounds on the solutions of both the continuous and the discrete Lyapunov equation which can be always calculated in terms of eigenvalues and eigen vectors of A. It is shown that under some conditions these bounds can express the bounds of the solution P well.

The organization of the paper is as follows. In Section 2, the bounds for the continuous matrix Lyapunov equation are obtained. In Section 3, the bounds for the discrete one are also obtained. Finally, Section 4 makes conclusions.

2. CONTINUOUS CASE

It is well known that the solution of the continuous matrix Lyapunov equation (1.1) is expressed [8] by

$$P = \int_0^\infty \exp(A^Tt)Q\exp(At)dt. \tag{2.1}$$

A can be represented by

$$A = \Gamma\Lambda\Gamma^{-1}, \tag{2.2}$$

where $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \cdot \cdot \cdot, \Lambda_m)$, and Λ_i is the $m_i \times m_i$ Jordan block matrix defined as

$$\Lambda_i = \begin{bmatrix} \alpha_i & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \alpha_i & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \alpha_i \end{bmatrix} \tag{2.3}$$

and α_i is an eigenvalue of A with the multiplicity m_i and $m_1 + \cdot \cdot \cdot + m_m = n$. $\Gamma = [\Gamma_1, \Gamma_2, \cdot \cdot \cdot, \Gamma_n]$ is the corresponding eigen vector matrix of A. Then P of (2.1) can be rewritten as

$$P = \int_0^\infty \Gamma^T \exp(\Lambda^T t)\Gamma^T Q \Gamma \exp(\Lambda t)\Gamma^{-1} dt. \tag{2.4}$$

We define G_c and F_c as follows.

$$G_c := \int_0^\infty \exp(\Lambda^T t)\exp(\Lambda t)\,dt$$
$$= \mathrm{diag}\left[\int_0^\infty \exp(\Lambda_i^T t)\exp(\Lambda_i t)\,dt,\ i=1,\cdots,m\right]$$
$$=: \mathrm{diag}[g_{c1}, \cdot \cdot \cdot, g_{cm}]$$

$$F_c := \int_0^\infty \exp(\Lambda t)\exp(\Lambda^T t)\,dt$$
$$= \mathrm{diag}\left[\int_0^\infty \exp(\Lambda_i t)\exp(\Lambda_i^T t)\,dt,\ i=1,\cdots,m\right]$$
$$=: \mathrm{diag}[f_{c1}, \cdot \cdot \cdot, f_{cm}]$$

Lemma 1 : $g_{ci} = (g_{ci}(l,k))$ and $f_{ci} = (f_{ci}(l,k))$ have the following values, respectively, where $l=1, \cdot \cdot \cdot, m_i$ and $k=1, \cdot \cdot \cdot, m_i$.

$$g_{ci}(l,k) = \begin{cases} \displaystyle\sum_{\gamma=1}^{l} \frac{(l+k-2\gamma)!}{(l-\gamma)!\,(k-\gamma)!\,(-2\alpha_i)^{l+k+1-2\gamma}} & \text{if } k \geq l \\ g_{ci}(k,l) & \text{if } k < l \end{cases} \tag{2.5}$$

$$f_{ci}(l,k) = \begin{cases} \displaystyle\sum_{\gamma=1}^{m_i+1-k} \frac{(2\gamma+k-l-2)!}{(\gamma-1)!(\gamma+k-l-1)!(-2\alpha_i)^{2\gamma+k-l-1}} & \text{if } k \geq l \\ f_{ci}(k,l) & \text{if } k < l \end{cases} \tag{2.6}$$

Where $(\cdot)!$ denotes the factorial of $(\cdot)$. When A has distinct eigenvalues, $G_c = F_c = -0.5\Lambda^{-1}$.

Proof : We define $\Lambda_i(s)$ as $(sI_{mi} - \Lambda_i)^{-1}$. Then,

$$\Lambda_i(s) = \begin{bmatrix} (s-\alpha_i)^{-1} & (s-\alpha_i)^{-2} & \cdots & (s-\alpha_i)^{-mi} \\ 0 & (s-\alpha_i)^{-1} & \cdots & (s-\alpha_i)^{1-mi} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & & (s-\alpha_i)^{-1} \end{bmatrix}$$

Since $\exp(\Lambda_i t)$ is the inverse Laplace transform of $\Lambda_i(s)$,

$$\exp(\Lambda_i t) = \begin{bmatrix} 1 & t & t^2/2! & \cdot\cdot\cdot & t^{mi-1}/(m_i-1)! \\ 0 & 1 & t & \cdot\cdot\cdot & t^{mi-2}/(m_i-2)! \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ 0 & 0 & 0 & & 1 \end{bmatrix} \exp(\alpha_i t).$$
$$=: X_i(t)\exp(\alpha_i t).$$

S_{gci} and S_{fci} are defined as $X_i^T(t) \cdot X_i(t)$ and $X_i(t) \cdot X_i^T(t)$, respectively. Then,

$$S_{gci}(l,k) = \begin{cases} \displaystyle\sum_{\gamma=1}^{l} \frac{t^{l+k-2\gamma}}{(l-\gamma)!\,(k-\gamma)!} & (k \geq l) \\ S_{gci}(k,l) & (k<l) \end{cases}$$

$$S_{fci}(l,k) = \begin{cases} \displaystyle\sum_{\gamma=1}^{m_i+1-k} \frac{t^{2\gamma+k-l-2}}{(\gamma-1)!\,(\gamma+k-l-1)!} & (k \geq l) \\ S_{fci}(k,l) & (k<l) \end{cases}$$

Since $g_{ci} = \int_0^\infty S_{gci}\exp(2\alpha_i t)dt$ and $f_{ci} = \int_0^\infty S_{fci}\exp(2\alpha_i t)dt$, we can obtain g_{ci} of (2.5) and f_{ci} of (2.6) by using the partial integration method. If A has discint eigenvalues, Λ is diagonal.

Thus, $G_c = F_c = \int_0^\infty \exp(2\Lambda t)dt = -0.5\Lambda^{-1}$.

This completes the proof.

Remarks

1) $g_{ci}(l,k) = f_{ci}(m_i+1-k, m_i+1-l)$. Thus it is sufficient to calculate one of them.
2) If $\gamma_1 = \gamma_2 + 1$, $g_{ci|mi=\gamma 2}$ is the principal minor with order γ_2 of $g_{ci|mi=\gamma 1}$. Thus, only the last row and last column of g_{ci} are needed to be calculated when m_i increases by one.

Theorem 1 : For the solution of the continuous Lyapunov equation (1.1), the following inequalities hold.

$$\lambda_1(\Gamma^{-T}G_c\Gamma^{-1})\lambda_n(\Gamma^T Q\Gamma) \leq \lambda_i(P) \leq \lambda_i(\Gamma^{-T}G_c\Gamma^{-1})\lambda_1(\Gamma^T Q\Gamma) \quad (2.7)$$

$$\max\{\lambda_n(F_c)\,tr(\Gamma^T Q\Gamma)/\lambda_1(\Gamma^T\Gamma),\ tr(\Gamma^{-T}G_c\Gamma^{-1})\lambda_n(\Gamma^T Q\Gamma)\} \leq tr(P) \leq \min\{\lambda_1(F_c)\,tr(\Gamma^T Q\Gamma)/\lambda_n(\Gamma^T\Gamma),\ tr(\Gamma^{-T}G_c\Gamma^{-1})\lambda_1(\Gamma^T Q\Gamma)\} \quad (2.8)$$

Where λ_1, λ_n, and λ_i denote the maximum, minimum, and i-th eigenvalue, respectively, and $tr(\cdot)$ the trace of $(\cdot)$.

Proof : From (2.4) and the definition of G_c, we can obtain the following inequality.

$$\lambda_n(\Gamma^T Q\Gamma)\Gamma^{-T}G_c\Gamma^{-1} \leq P \leq \lambda_1(\Gamma^T Q\Gamma)\Gamma^{-T}G_c\Gamma^{-1}$$

By calculating the eigenvalue and trace of both sides of the above inequality, we can easily obtain the inequality (2.7) and the following inequality.

$$\lambda_n(\Gamma^T Q\Gamma)\,tr(\Gamma^{-T}G_c\Gamma^{-1}) \leq tr(P) \leq \lambda_1(\Gamma^T Q\Gamma)\,tr(\Gamma^{-T}G_c\Gamma^{-1}) \quad (2.9)$$

By taking the trace of both sides of (2.4), we can obtain the following equation.

$$tr(P) = tr[\int_0^\infty \Gamma^{-T}\exp(\Lambda^T t)\Gamma^T Q\Gamma\exp(\Lambda t)\Gamma^{-1}dt]$$

$$= tr[(\Gamma^T\Gamma)^{-1}\int_0^\infty \exp(\Lambda^T t)\Gamma^T Q\Gamma\exp(\Lambda t)dt]$$

From this equation, we can also obtain the following inequality.

$$tr[\int_0^\infty \exp(\Lambda^T t)\Gamma^T Q\Gamma\exp(\Lambda t)dt]/\lambda_1(\Gamma^T\Gamma) \leq tr(P) \leq tr[\int_0^\infty \exp(\Lambda^T t)\Gamma^T Q\Gamma\exp(\Lambda t)dt]/\lambda_n(\Gamma^T\Gamma)$$

This inequality and the definition of F_c yield the following inequality.

$$\lambda_n(F_c)\,tr(\Gamma^T Q\Gamma)/\lambda_1(\Gamma^T\Gamma) \leq tr(P) \leq \lambda_1(F_c)\,tr(\Gamma^T Q\Gamma)/\lambda_n(\Gamma^T\Gamma) \quad (2.10)$$

From the inequality (2.9) and (2.10), the inequality (2.6) can be obtained. This completes the Proof.

Remarks

1) When A is symmetric, Λ is diagonal and Γ is a unitary matrix. Therefore, if $Q = I_n$, we can obtain the following equations.

$$P = -0.5A^{-1},\ \lambda_i(P) = \lambda_i(-0.5A^{-1}),$$

and $tr(P) = tr(-0.5A^{-1})$

2) The bounds of (2.7) and (2.8) can be calculated even though $(A+A^T)/2$ is not stable.
3) The smaller the condition number of the eigen vector matrix of A is, the tighter the bounds of (2.7) and (2.8).

The following example compares bounds of this paper with those of Troch [7] and Wang *et al* [5].

Example 1 [7] : $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$,

where Λ = diag (−1, −1, −1) and $\Gamma = I_3$. Assume that $Q = \mathrm{diag}(I_2, \alpha)$ and α = 1, 2, and 3. From (2.7), the lower bound of each eigenvalue is {.345, .5, .904} for all α and the upper bound {.345, .5, .904}, {.691, 1, 1.809}, and {1.382, 2, 3.618} for α = 1, 2, and 4, respectively. The Troch's lower bound for each eigenvalue is {.059, .086, .155}, {.059, .155, .172}, and {.059, .155, .343} and the upper bound {2.014, 2.914, 5.272}, {2.014, 5.272, 5.828}, and {2.013, 5.272, 11.677} for α = 1, 2, and 4, respectively. Eigenvalues of P for each α are {.345, .5, .904}, and {.345, .904, 1}, and {.345, .904, 2}, respectively. From (2.8), the lower and the upper bounds for the trace of P for each α are {1.75, 1.75}, {1.75, 3.5}, and {2.073, 5.427}, Troch's trace bounds {.300, 10.20}, {.386, 13.11}, and {.558, 18.95}, Wang's ones {1, 3}, {1.667, 5}, and {2, 6}, and trace of P is 1.75, 2.25, and 3.25, respectively.

From these results, it is noted that if the eigen vector matrix of A is well-conditioned and $\sigma_n(Q)$ is near to $\sigma_1(Q)$, the method of this paper considerably reduces the conservatism on the estimate of the bounds on the eigenvalues and trace of P. Especially, when the eigen vector matrix is orthonormal and Q identity, the method of this paper offers exact bounds.

3. DISCRETE CASE

The solution of the discrete matrix Lyapunov equation (1.2) is described [8] by

$$P = \sum_{i=0}^{\infty}(A^T)^i QA^i. \quad (3.1)$$

By replacing A in (3.1) with $\Gamma\Lambda\Gamma^{-1}$, P of (3.1) can be rewritten as

$$P = \sum_{l=0}^{\infty} \Gamma^{-T}(\Lambda^T)^l \Gamma^T Q \Gamma \Lambda^l \Gamma^{-1}. \qquad (3.2)$$

G_d and F_d are defined as follows.

$$G_d := \sum_{k=0}^{\infty} (\Lambda^T)^k \Lambda^k = \mathrm{diag}[\sum_{k=0}^{\infty} (\Lambda_i^T)^k \Lambda_i^k, \quad i=1,\cdots,m]$$

$$F_d := \sum_{k=0}^{\infty} \Lambda^k (\Lambda^T)^k = \mathrm{diag}[\sum_{k=0}^{\infty} \Lambda_i^k (\Lambda_i^T)^k, \quad i=1,\cdots,m]$$

We define S_{gdi} and S_{fdi} as $(\Lambda_i^T)^k \Lambda_i^k$ and $\Lambda_i^k (\Lambda_i^T)^k$, respectively. Then,

$$G_d = \mathrm{diag}[\sum_{k=0}^{\infty} S_{gdi}, \quad i=1,\cdots,m]$$

$$=: \mathrm{diag}[g_{d1}, \cdot\cdot\cdot, g_{dm}] \text{ and} \qquad (3.3)$$

$$F_d = \mathrm{diag}[\sum_{k=0}^{\infty} S_{fdi}, \quad i=1,\cdots,m]$$

$$=: \mathrm{diag}[f_{d1}, \cdot\cdot\cdot, f_{dm}]. \qquad (3.4)$$

When A has distinct eigenvalues, $G_d = F_d = (I - \Lambda^2)^{-1}$. It is known that [9]

$$\Lambda_i^k = \alpha_i^k \cdot \begin{bmatrix} 1 & k\alpha_i^{-1} & \frac{k!\alpha_i^{-2}}{2!(k-2)!} & \cdots & \frac{k!\alpha_i^{1-m_i}}{(m_i-1)!(k-m_i+1)!} \\ 0 & 1 & k\alpha_i^{-1} & \cdots & \frac{k!\alpha_i^{2-m_i}}{(m_i-2)!(k-m_i+2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k\alpha_i^{-1} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Using this equation, the following lemma is obtained.

Lemma 2 : $S_{gdi} = (S_{gdi}(l,j))$ and $S_{fdi} = (S_{fdi}(l,j))$ have the following values, respectively, where $l=1, \cdot\cdot\cdot, m_i$ and $j=1, \cdot\cdot\cdot, m_i$.

$$S_{gdi}(l,j) = \begin{cases} \sum_{\gamma=1}^{l} \frac{k!k!\alpha_i^{2\gamma-l-j}\cdot\alpha_i^{2k}}{(l-\gamma)!(j-\gamma)!(k-l+\gamma)!(k-j+\gamma)!} & \text{if } j \geq l \\ S_{gdi}(j,l) & \text{if } j < l \end{cases} \qquad (3.5)$$

$$S_{fdi}(l,j) = \begin{cases} \sum_{\gamma=1}^{m_i+l-1} \frac{k!k!\alpha_i^{j-l-2\gamma+2}\cdot\alpha_i^{2k}}{(\gamma-1)!(l-j+\gamma-1)!(k-\gamma+1)!(k-l+j-\gamma-1)!} & \text{if } l \geq j \\ S_{fdi}(j,l) & \text{if } j < l \end{cases} \qquad (3.6)$$

It should be noted that remarks after Lemma 1 also hold for g_{di} and f_{di}. By using (3.3), (3.4), (3.5), and (3.6), we can obtain g_{di} and f_{di} when $m_i = 2$ and 3 as follows.

$$g_{di|mi=2} = \begin{bmatrix} \delta_i^2 & \alpha_i\delta_i \\ \alpha_i\delta_i & 2-\alpha_i^2+\alpha_i^4 \end{bmatrix} / \delta_i^3,$$

$$f_{di|mi=2} = \begin{bmatrix} 2-\alpha_i^2+\alpha_i^4 & \alpha_i\delta_i \\ \alpha_i\delta_i & \delta_i^2 \end{bmatrix} / \delta_i^3,$$

$$g_{di|mi=3} \cdot \delta_i^5 = \begin{bmatrix} \multicolumn{2}{c}{g_{di|mi=2} \cdot \delta_i^5} & \begin{matrix} 2\alpha_i^2\delta_i^2 \\ \alpha_i\delta_i(3-\alpha_i^2+\alpha_i^4) \end{matrix} \\ 2\alpha_i^2\delta_i^2 & \alpha_i\delta_i(3-\alpha_i^2+\alpha_i^4) & 3-\alpha_i^2+6\alpha_i^4-3\alpha_i^6+\alpha_i^8 \end{bmatrix},$$

$$f_{di|mi=3} \cdot \delta_i^5 = \begin{bmatrix} 3-\alpha_i^2+6\alpha_i^4-3\alpha_i^6+\alpha_i^8 & \alpha_i\delta_i(3-\alpha_i^2+\alpha_i^4) & 2\alpha_i^2\delta_i^2 \\ \begin{matrix} \alpha_i\delta_i(3-\alpha_i^2+\alpha_i^4) \\ 2\alpha_i^2\delta_i^2 \end{matrix} & \multicolumn{2}{c}{f_{di|mi=2} \cdot \delta_i^5} \end{bmatrix},$$

where $\delta_i = 1/(1-\alpha_i^2)$.

Using the similar method to Theorem 1, we can obtain the following theorem for the discrete Lyapunov equation.

Theorem 2 : For the solution of the discrete Lyapunov equation (1.2), the following inequalities hold.

$$\lambda_1(\Gamma^{-T}G_d\Gamma^{-1})\lambda_n(\Gamma^T Q\Gamma) \leq \lambda_i(P) \leq \lambda_i(\Gamma^{-T}G_d\Gamma^{-1})\lambda_1(\Gamma^T Q\Gamma) \qquad (3.7)$$

$$\max\{\lambda_1(F_d)\mathrm{tr}(\Gamma^T Q\Gamma)/\lambda_1(\Gamma^T\Gamma), \ \mathrm{tr}(\Gamma^{-T}G_d\Gamma^{-1})\lambda_n(\Gamma^T Q\Gamma)\}$$
$$\leq \mathrm{tr}(P) \leq$$
$$\min\{\lambda_n(F_d)\mathrm{tr}(\Gamma^T Q\Gamma)/\lambda_n(\Gamma^T\Gamma), \ \mathrm{tr}(\Gamma^{-T}G_d\Gamma^{-1})\lambda_n(\Gamma^T Q\Gamma)\} \qquad (3.8)$$

It should be also noted that remarks after Theorem 1 hold here. When A is symmetric and $Q = I_n$, we can obtain the following equations.

$$P = (I_n - A^2)^{-1}, \quad \lambda_i(P) = \lambda_i(I_n - A^2)^{-1},$$

and $\mathrm{tr}(P) = \mathrm{tr}[(I_n - A^2)^{-1}]$.

Via an example, the bounds of (3.7) and (3.8) are compared with those of Troch [7] and of Garloff [4].

Example 2 : $A = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$,

where $\Lambda = diag(0.5, 0.5, 0.25)$ and $\Gamma = I_3$.

Assume that $Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$ and α = 1, 2, and 4.

Then we can obtain the following bounds on the eigenvalues and trace of P.

< Eigenvalue Bounds on P >

α		1	2	4
(3.7)	LB	4.0743E-1 4.1524E-1 1.7351	4.0743E-1 4.1524E-1 1.7351	4.0743E-1 4.1524E-1 1.7351
	UB	2.7926 2.8461 11.892	2.7926 2.8461 11.892	4.2667 4.3485 18.170
Troch	LB	5.4953E-1 1.0664 6.8880	5.4953E-1 2.1328 6.8880	5.4953E-1 4.2656 6.8880
	UB	3.0851 5.9868 38.669	3.0851 11.974 38.669	3.0851 23.947 38.669
$\lambda_i(P)$		6.0714E-1 1.0667 8.1336	6.0714E-1 2.1333 8.1336	6.0714E-1 4.2667 8.1336

(LB : Lower bound, UB : Upper bound)

< Trace bound on P >

α		1	2	4
(3.8)	LB	4.2667	5.3333	7.4667
	UB	17.5310	17.5310	26.7850
Troch	LB	8.5039	9.5703	11.7036
	UB	47.7410	53.7280	65.7010
Wang and et al	LB	4.0074	5.0092	7.0129
	UB	can not be calculated		
Tr.(P)		1.75	2.25	3.25

From the these results, it is noted that the same arguments after Example 1 hold here too.

4. CONCLUSIONS

Some bounds on the eigenvalues and trace of the solutions of both the continuous and the discrete Lyapunov equations were obtained in terms of the eigenvalues and eigen vectors of the system matrix A. These bounds are always calculated regardless of the stability of $(A+A^T)/2$ in the continuous case and the maximum singular value of A in the discrete case. It was noted that the bounds of this paper reduced the conservatism in the estimated bounds on the solution of the both Lyapunov equations when the system matrix A is well-conditioned and Q is the identity matrix. Especially, when A is symmetric and Q is the identity matrix, the bounds of this paper offer the exact estimates on the eigenvalues and trace of the solutions of both Lyapunov equations.

Troch [7] also suggests the bounds of all eigenvalues which can be always calculated, which requires a heavy computational burden. However, only the eigenvalues and eigen vectors of A are necessary to calculate the bounds of this paper.

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