Uniform ultimate boundedness를 보장하는 선형 시불변 되먹임 보상기 설계

최한호°, 유동상, 정명진
한국과학기술원, 전기 및 전자과

The Design of a Robust Linear Time-invariant Feedback Compensator Guaranteeing Uniform Ultimate Boundedness for Uncertain Multivariable Systems

Han-Ho Choi°, Dong Sang Yoo, and Myung Jin Chung
Dept. of E.E., KAIST

Abstract : In this paper, we propose a robust linear time-invariant feedback compensator design methodology for multivariable systems which have both matched and mismatched uncertainties. In order to attack the problem of designing robust compensators guaranteeing uniform ultimate boundedness of every closed-loop system response within an arbitrarily small ball centered at the zero state based solely on the knowledge of the upper norm-bounds of uncertainties, we use an approach based upon the comparison theorem which is an effective approach in studying augmented feedback control systems with both mismatched and matched uncertainties. Through the approach, we draw some sufficient conditions for robust stability, and we give a simple example.

I. Introduction

Recently, the problem of designing a robust controller which guarantees the desired performance and stability of multivariable systems whose mathematical models are subject to uncertainties has been occupied the attention of system theorists. Many researchers have attacked the problem from the deterministic point of view. The salient feature of their approaches is the fact that it is a deterministic treatment of uncertainties in that a certain deterministic performance is required in the presence of uncertain information.

Some design methodologies have been developed in the time domain by using a Lyapunov approach. Roughly speaking, a Lyapunov function of a stable nominal system is employed as a Lyapunov function candidate for the actual uncertain system and a control function is then chosen such that the Lyapunov function decrease along every possible trajectory of the uncertain dynamic system, at least outside a neighborhood of the zero state. Therefore, through that approach uniform ultimate boundedness is obtained for all possible uncertainties.[1-6]

But the methodologies using the Lyapunov approach are based on the assumption that nonlinear uncertainties satisfy the so-called "matching conditions" and/or the assumption that the actual system state should be available directly and/or the assumption that output has no uncertainties. In addition, in those methodologies if the nominal systems of dynamic systems are not stable then a preliminary stabilization of nominal systems should be performed. Even though Chen and Leitmann, in [4], studied the robustness of a controlled system when the uncertainties do not satisfy the matching conditions, through their methodology we can get only the maximum allowable bound of mismatched uncertainties such that satisfactory system behavior can be guaranteed for a given controller coping with matched uncertainties. Chen , in [5-6], studied robust output feedback controller design for uncertain dynamical systems, but his study is also based upon the assumption that uncertainties satisfy matching conditions and the proposed design methodes are rather complicated and through his methods it is too difficult to arbitrarily design the radius of the small ball centered at the zero state within which one wants to guarantee uniform ultimate boundedness of every system response. In most practical situations, dynamic systems may have uncertainties which do not satisfy matching conditions and the actual system state is not available directly.

In this paper, taking account of these problems, we propose a robust linear time-invariant feedback compensator design methodology for multivariable systems which have both mismatched and matched nonlinear time varying model uncertainties. We use an approach based on the comparison theorem, which is a more effective approach than the Lyapunov approach in studying augmented feedback control systems with both mismatched and matched uncertainties. According to the proposed methodology one does not have to stabilize the nominal system preliminarily. In order to make the state enter an arbitrarily small ball centered at the zero state in finite time and remain within it thereafter (i.e. to guarantee uniform ultimate boundedness of all possible system responses within an arbitrarily small ball centered at the zero state) one has only to shift the nominal closed-loop poles to the the left of a vertical line in the complex plane which is determined by norm-bounds on the uncertainties and the radius of the ball centered at the zero state and/or norms involving the parameter of both compensator and system model. Therefore, the control system design can be well performed through eigenstructure assignment.

Before proceeding further, we will give some notations. If x is a real vector, then $\|x\|_p$ is the norm defined by $\|x\|_p = \{\sum |x_i|^p\}^{1/p}$ where x_i denotes the element of the vector x and $p=1,2,\infty$. If A is a matrix, then $\|A\|_{ip}$ is the induced matrix norm corresponding to the vector norm and $\mu_{ip}(A)$ is the corresponding matrix measure. Details on the norms and on matrix measures may be found in [7].

II. Problem Formulation

Let the actual plant to be controlled be represented by the equations

$$\dot{x}(t) = Ax(t) + Bu(t) + \eta_1(t,x(t)) \qquad x(0)=x_0 \tag{1}$$
$$y(t) = Cx(t) + \eta_2(t,x(t))$$

where $x \in R^n$, $u \in R^m$, $y \in R^r$ are the state, input, and output respectively, and A, B, and C are constant real matrices with appropriate dimensions. Nonlinear time-varying uncertainties $\eta_1(t,x(t))$ and $\eta_2(t,x(t))$ are Caratheodory functions with the following known upper norm-bounds:

$$\|\eta_1(t,x(t))\|_p \le \beta_1 + \beta_2\|x\|_p \tag{2}$$
$$\|\eta_2(t,x(t))\|_p \le \beta_3 + \beta_4\|x\|_p$$

where β_i, $i=1,\dots,4$ are nonnegative constants.

Without loss of generality, we assume that the triple (A,B,C) is controllable and observable. Suppose that only the output vector y is directly available. Consider an output feedback compensator given by

$$\dot{v} = K_{21}y + K_{22}v \qquad v(0)=v_0 \tag{3}$$

$$u = K_{11}y + K_{12}v \tag{4}$$

where K_{11}, K_{12}, K_{21}, and K_{22} have appropriate dimensions, and (3) is a dynamic compensator of order s; $0 \le s \le n$. The extreme case $s=0$ represents static gain output feedback.

Thus, our design problem is formulated as choosing the parameters K_{11}, K_{12}, K_{21}, and K_{22} of (3) and (4) such that all the closed-loop system responses of (1), (3) and (4) satisfies uniform ultimate boundedness within an arbitrarily small ball centered at the zero state.

III. Robust Output Compensator Control

Let $\bar{x}^T = [x^T \; v^T]$, then the closed-loop system is given by

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{\eta}_1(t,\bar{x}) \qquad \bar{x}(0) = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{5}$$
$$y = \bar{C}\bar{x} + \bar{\eta}_2(t,\bar{x})$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0_s \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I_s \end{bmatrix}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} C & 0 \\ 0 & I_s \end{bmatrix}$$

$$= \begin{bmatrix} A+BK_{11}C & BK_{12} \\ K_{21}C & K_{22} \end{bmatrix} \qquad \bar{C} = [C \;\; 0] \tag{6}$$

$$\bar{\eta}_1(t,\bar{x}) = \begin{bmatrix} \eta_1(t,x)+BK_{11}\eta_2(t,x) \\ K_{21}\eta_2(t,x) \end{bmatrix} \tag{7}$$

$$\bar{\eta}_2(t,\bar{x}) = \eta_2(t,x)$$

Associated with (5) we get an approximate closed-loop feedback system as follows:

$$\dot{\bar{x}} = \bar{A}\bar{x} \qquad \bar{x}(0) = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}. \tag{8}$$
$$y = \bar{C}\bar{x}$$

Let us define the transition matrix $\Phi(t)$ of (8) and suppose (8) is asymptotically stable, then it is clear that for some finite constants $\alpha>0$, $m>0$

$$\|\Phi(t)\|_p \le m\exp(-\alpha t) \qquad t \ge 0 \tag{9}$$

Now, we are ready to present a theorem which will be used for establishing a robust output feedback compensator design methodology.

Theorem 1: If we choose the control parameters of (3) and (4) such that the system (8) is asymptotically stable and one of the two following inequalities is satisfied:

$$\alpha - m\rho_2 - m\rho_1/\delta > 0, \tag{10}$$

$$\mu_{ip}(\bar{A}) + \rho_2 + \rho_1/\delta < 0, \tag{11}$$

where δ is the radius of an arbitrarily small ball centered at $\bar{x}=0$ within which we want to guarantee uniform ultimate boundedness of all the closed-loop system (5) responses, and

$$\rho_1 = \beta_1 + \beta_3(\|BK_{11}\|_{ip} + \|K_{21}\|_{ip})$$

$$\rho_2 = \beta_2 + \beta_4(\|BK_{11}\|_{ip} + \|K_{21}\|_{ip})$$

then the following properties hold:

1) Uniform Boundedness: Given any $r \in [0,\infty)$, there exists a $d(r)<\infty$ such that $\|\bar{x}_0\|_p \le r$ implies $\|\bar{x}(t)\|_p \le d(r)$, $\forall t \ge 0$

2) Uniform Ultimate Boundedness: Given any $\bar{\delta}>\delta$ and any $r \in [0,\infty)$, there is a $T(\bar{\delta},r) \in [0,\infty)$ such that $\|\bar{x}_0\|_p \le r$ implies $\|\bar{x}(t)\|_p \le \bar{\delta}$, $\forall t \ge T(\bar{\delta},r)$

3) Uniform Stability: Given any $\bar{\delta}>\delta$, there is a $D(\bar{\delta})>0$ such that $\|\bar{x}_0\|_p \le D(\bar{\delta})$ implies $\|\bar{x}(t)\|_p \le \bar{\delta}$, $\forall t \ge 0$

Proof : We can prove using the comparison theorem (See [12]), but we do not present here because of the space limitations.

From Theorem 1, it is clear that if all the eigenvalues of $\bar{A}$ are in the left half plane then (9) is satisfied. Negative α is equal to or larger than the real part of the eigenvalue nearest to the imaginary axis. So, robustness margins are given by the real part of the eigenvalue nearest to the imaginary axis.

Theorem 1 says that for arbitrarily β_1, β_2, β_3, β_4, and δ, if a controller satisfies one of (10) and (11), then the controller guarantees uniform ultimate boundedness of every closed-loop system response within the ball $\{\bar{x} \in R^{s+n} \mid \|\bar{x}\|_p \leq \delta\}$. Because both ρ_1 and ρ_2 of (10) or (11) are dependent on the norms of BK_{11} as well as K_{21}, it may not be easy to satisfy one of (10) and (11). But, if the upper norm-bound of output uncertainties is small, i.e. $\beta_3 \approx 0$ and $\beta_4 \approx 0$, then $\rho_1 \approx \beta_1$ as well as $\rho_2 \approx \beta_2$, and it is sure that by choosing appropriately not only δ but also the gain matrices of (3) and (4), because for arbitrary β_1 and β_2 of (2) we can easily satisfy one of the two inequalities (10) and (11) without seriously caring about the norms of the parameters K_{11} and K_{21}, we can achieve an arbitrary transient response as well as an arbitrary good steady-state response of the closed-loop system (5). The above inequalities both (10) and (11) are only sufficient conditions. So, we cannot say that robust controllers necessarily satisfy the inequalities. From the above inequality conditions (10) and (11), it is recommended in robust controller design that one should choose the parameters of a controller with minimum induced matrix norms. According to the chosen norm and the corresponding matrix measure, the above sufficient conditions can be more conservative or less conservative, i.e. the sharpness of the conditions will varies with the chosen norm and matrix measure.

Remark 1: If $K_{12} = K_{21} = K_{22} = 0$, i.e. the static output feedback controller case, then the condition (10) and (11) become respectively $\alpha > m\left[\beta_2 + \beta_4\|BK_{11}\|_{ip} + (\beta_1 + \beta_3\|BK_{11}\|_{ip})/\delta\right]$ and $-\mu_{ip}(\bar{A}) > \left[\beta_2 + \beta_4\|BK_{11}\|_{ip} + (\beta_1 + \beta_3\|BK_{11}\|_{ip})/\delta\right]$.

Remark 2: If $K_{12} = K_{21} = K_{22} = 0$, $C = I_n$ and $\beta_3 = \beta_4 = 0$, i.e. the linear full state feedback case, then the condition (10) and (11) are respectively reduced to $\alpha > m\left[\beta_2 + \beta_1/\delta\right]$ and $-\mu_{ip}(\bar{A}) > \left[\beta_2 + \beta_1/\delta\right]$. From these inequalities, we can see that in the linear full state feedback case there is no restriction of K_{11}, β_1, and β_2. So, it is sure that in the linear full state feedback case both arbitrary transient response and arbitrary good steady state response can be obtained by choosing appropriate δ and the static gain matrix K_{11} in spite of arbitrarily large uncertainties. □

Suppose both $\|\eta_1(t,x)\|_p$ and $\|\eta_2(t,x)\|_p$ of (2) are bounded by linear functions of $\|x\|_p$, i.e. $\beta_1 = \beta_3 = 0$, then we can show that the following inequalities are satisfied:

$$\|\bar{x}(t)\|_p \leq m\,\|\bar{x}_0\|_p \exp[-(\alpha - m\rho_2)t] \tag{12}$$

$$\|\bar{x}(t)\|_p \leq \|\bar{x}_0\|_p \exp[(\mu_{ip}(\bar{A}) + \rho_2)t] \tag{13}$$

Thus, the following consequent corollary on the Theorem 1 can be established.

Corollary 2: Consider (5) with $\beta_1 = \beta_3 = 0$. If we choose of (3) and (4) such that (8) is asymptotically stable and one of the two following inequalities is satisfied:

$$\alpha - m\rho_2 > 0, \tag{14}$$

$$\mu_{ip}(\bar{A}) + \rho_2 < 0, \tag{15}$$

where

$$\rho_2 = \beta_2 + \beta_4(\|BK_{11}\|_{ip} + \|K_{21}\|_{ip})$$

Then the closed-loop system (5) with $\beta_1 = \beta_3 = 0$ is asymptotically stable.

Proof : Immediate from (12), (13), (14) and (15). □

From the above corollary, we can see that if β_1 as well as β_3 are zero and output measurements have no uncertainty then we can obtain an arbitrary transient response as well as asymptotic stability of the closed-loop system by satisfying easily either (14) or (15) without any restriction of β_2 and the norms of controller parameters. The inequalities (14) and (15) are similar to the results of Chen and Wong [8], Sobel *et al.* [9], and Zak [10]. Because their results are reduced to the special cases of our results, we can say that our results are general ones.

From the preceding analysis, different controller design procedures can be established. And from (6), it can be seen that compensator design problem is equivalent to a static output feedback problem, which has been treated by several authors. Especially, Kwon and Youn, in [11], drew the necessary and sufficient conditions for eigenstructure assignment by output feedback and gave a simple procedure for eigenstructure assignment by output feedback. In order to guarantee uniform ultimate boundedness of all possible system responses within an arbitrarily small ball centered at the zero state one has only to shift the nominal closed-loop poles to the the left of a vertical line in the complex plane which is determined by norm-bounds on the uncertainties and the radius of the ball centered at the zero state and/or norms involving the parameter of both compensator and system model. Therefore, the control system design can be well performed through eigenstructure assignment by output feedback.

IV. Example

To illustrate the preceding results, we give an example.

Example : Consider the following dynamic system:

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$+\begin{bmatrix} 0.5x_3\cos(x_3)+0.1\cos(u_2) \\ 0.5\sin(x_3)+0.1\cos(u_1) \\ 0.5x_1\sin(u_2) \end{bmatrix} \quad x_0 = \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} -0.1\sin(x_2) \\ 0.1x_2\sin(u_1) \end{bmatrix}$$

Find a robust static gain output feedback controller which assures not only that every closed-loop system response $x(t)$ will reach $\exp[-1]$ times x_0 within 1 sec, i.e. the time constant is less than 1 sec, but also uniform ultimate boundedness of every closed-loop system response within the ball of $\{x \in R^3 \mid \|x\|_\infty \le 0.05\}$.

Solution: From the above dynamic equations, we get $\|\eta_1(t,x(t))\|_\infty \le 0.1+0.5\|x\|_\infty$ and $\|\eta_2(t,x(t))\|_\infty \le 0.1\|x\|_\infty$, i.e. $\beta_1 = 0.1$, $\beta_2 = 0.5$, $\beta_3 = 0$, $\beta_4 = 0.1$. Using eigenstructure assignment by output feedback, we get a controller as

$$u = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix} y$$

then $\mu_{i\infty}(\bar{A}) = -3$, $\mu_{i\infty}(\bar{A}) + \rho_2 + \rho_1/\delta = -3 + (0.5 + 0.1\times 4 + 0.1/0.05) = -0.1$, and the following inequality holds:

$$\|x(t)\|_\infty \le 4.95\exp[-2.1t] + 0.05, \ \forall t \ge 0$$

Every system response will reach $\exp[-1]$ times x_0 within 1/2.1 sec. The design specifications are satisfied.

V. Conclusion

In this paper, we propose a robust linear time-invariant compensator design methodology for multivariable systems which have both matched and mismatched nonlinear time-varying model uncertainties with known upper norm-bounds and our analysis is restricted to uncertain dynamical systems where the nominal systems are linear time-invariant. In order to design a robust output feedback compensator guaranteeing uniform ultimate boundedness of every system response within an arbitrary small ball centered at the zero state, we use an approach based upon the comparison theorem which is an effective approach in studying augmented feedback control systems with both mismatched and matched uncertainties. Through the approach, we draw the sufficient conditions for robust stability, and we give a simple example.

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