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Generalized $\mathcal{H}^\infty$ Control Theory

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Abstract

In this paper we formulate and solve a generalized $\mathcal{H}^\infty$ control problem. The conventional formulation of $\mathcal{H}^\infty$ problem has some constraints in application, e.g. it can not deal with the servo problem. This is due to the superfluous requirement of internal stability of the augmented system. In this paper, we alleviate the stability of the augmented system to admit pole-zero cancellation on the imaginary axis outside the feedback loop of G_{22} and K. After such generalization, the servo problem is naturally incorporated into the $\mathcal{H}^\infty$ synthsis.

Notation

Let X^L, X^R, $X^\perp$ denote a left inverse, a right inverse and an annihilator of matrix X. $\mathbf{C}^0$, $\mathbf{C}^-$, $\mathbf{C}^+$, $\bar{\mathbf{C}}^-$ and $\bar{\mathbf{C}}^+$ denote the imaginary axis, the open left half plane, the open right half plane, the closed left half plane and the closed right half plane respectively. $\rho(U)$ is the spectral radius of U, $\lambda[U]$ denotes the spectrum of U or a member of the spectrum, $\bar{\sigma}(U)$ denotes the maximal singular value of U. $\mathcal{RH}^\infty_{m\times r}$ is the set of stable real rational proper matrices of dimension $m\times r$, $\mathcal{BH}^\infty_{m\times r}$ is a subset $\mathcal{RH}^\infty_{m\times r}$ with $\mathcal{L}^\infty$ norm less than 1. $m.uc.(A\,,\,B)$ denotes the maximal uncontrollable subspace of $(A\,,\,B)$ in $\bar{\mathbf{C}}^-$, $m.uo.(C\,,\,A)$ denotes the maximal unobservable subspace of $(C\,,\,A)$ in $\bar{\mathbf{C}}^-$.

$$HM([\Theta \quad V]\,,\,Q) := [\Theta_{11}Q + \Theta_{12} \quad V_1][\Theta_{21}Q + \Theta_{22} \quad V_2]^{-1} \tag{1}$$

$$DHM(\begin{bmatrix} \Psi \\ W \end{bmatrix}\,,\,Q) := \begin{bmatrix} \Psi_{11} + Q\Psi_{21} \\ W_1 \end{bmatrix}^{-1} \begin{bmatrix} \Psi_{12} + Q\Psi_{22} \\ W_2 \end{bmatrix} \tag{2}$$

where Θ_{ij}, Ψ_{ij} is the (i, j)-block of Θ and Ψ, $V = [V_1^T \quad V_2^T]^T$, $W(s) = [W_1 \quad W_2]$.

$$J_{m,r} := diag[I_m\,, \quad I_r]$$

$$G^\sim(s) := G^T(-s)$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = [A\,,\,B\,,\,C\,,\,D] := D + C(sI - A)^{-1}B$$

1 Introduction

The $\mathcal{H}^\infty$ control theory, as a unified frequency domain synthesis methodology and for its close relation with robust control problems, has been studied vigorously and widely. The elegant two-Riccati equation state space solution has been established in recent years [3, 1, 6, 4, 9]. In all these papers, the $\mathcal{H}^\infty$ control problem is formulated as the problem of finding a controller $K(s)$ such that the feedback system of Fig. 1 is *internally stable* and $\|\Phi\|_\infty < 1$ (or ≤ 1) where $\Phi(s)$ is the closed loop transfer function matrix

$$\Phi = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}\,, \tag{3}$$

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \tag{4}$$

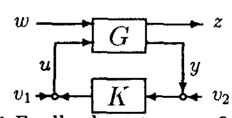

Fig.1 Feedback system configuration

However, such formulation of $\mathcal{H}^\infty$ control problem is restrictive in application, e.g. the servo problem can not be solved within this formulation. We observe that the constraint arises from the requirement of *internal stability* of the whole system of Fig. 1. We note that the control system of Fig. 1 is an abstracted system that embraces weighting functions. In Fig. 1, G_{22} corresponds to the real plant and the feedback loop of G_{22}, K corresponds to the real feedback system, while other parts are related to weighting functions. What is really necessary is the internal stability of the feedback loop of G_{22} and K that requires the stability of the four transfer functions from v_1, v_2 to y, u. However, the internal stability of the whole system of Fig. 1 requires the stability of the nine transfer functions from w, v_1, v_2 to z, y, u which is obviously superfluous.

Now let us clarify to what extent should we generalize the problem. Firstly, when weighting functions have $\mathbf{C}^+$ poles, there arise two kind of problems: one is to cancel the $\mathbf{C}^+$ poles of weighting functions with the zeros of the real feedback system and solve an $\mathcal{H}^\infty$ optimization problem; another is to leave those poles intact and to minimize the

$\mathcal{L}^\infty$ norm of the closed loop system. The first problem corresponds to embedding those $\mathbf{C}^+$ modes into the poles (or zeros) of the controller, which is of little sense for control engineering. The second problem can be converted into a problem with stable weighting functions via multiplication of suitable unitary matrices. From these arguments, it is clear that the choice of weighting functions with $\mathbf{C}^+$ poles offers no advantage. So we need only extend the formulation to admit closed loop poles on the imaginary axis. Secondly, we note poles of $\Phi(s)$ on the imaginary axis must be hidden modes, otherwise it is impossible for $\Phi(s)$ to have bounded $\mathcal{L}^\infty$ norm.

Let the plant G in Fig. 1 be given by a minimal realization

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \begin{array}{c} \begin{array}{ccc} n & r & p \end{array} \\ \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \end{array} \begin{array}{c} n \\ m \\ q \end{array} \quad (5)$$

Here we emphasize that this realization is *minimal*, it concerns with the existence of Riccati equations. Throughout this paper, it is implicitly assumed that the real plant has *no unstable hidden modes*.

Based on the preceding observations, we generalize $\mathcal{H}^\infty$ control problem as follows.

Definition 1 *The generalized $\mathcal{H}^\infty$ control problem is to find the necessary and sufficient condition for the existence of a controller satisfying*

C1: *All closed loop poles of the system of Fig. 1 lie in $\bar{\mathbf{C}}^-$,*

C2: *The feedback loop of G_{22} and K is internally stable,*

C3: *{unstable closed loop pole}={uncontrollable mode of $(A,\ B_2)$ and unobservable mode of $(C_2,\ A)$ on $\mathbf{C}^0$},*

C4: $\Phi(s) \in \mathcal{BH}^\infty_{m\times r}$.

Now let us have a further look into this definition. C1 and C4 means that all unstable closed loop poles are hidden modes. An uncontrollable mode of $(A,\ B_2)$ and unobservable mode of $(C_2,\ A)$ may be a pole of G_{11}, G_{12} and G_{21}, but not G_{22}. So C3 implies that only unstable pole-zero cancellation outside the feedback loop of G_{22} and K is allowed, which happens between the poles of G_{11}, G_{12}, G_{21}(weighting functions) and the zeros of the feedback loop of G_{22} and K. Therefore C4 implies the stability of $(I - G_{22}K)^{-1}$, which in turn implies the internal stability of the feedback loop of G_{22} and K. That is, C1, C3 and C4 actually implies C2. Here C2 is stated merely for emphasis.

We make the following assumptions on $G(s)$.

A1: (A, B_2, C_2) is controllable and observable in $\mathbf{C}^+$,

A2: $D_{12} = [0 \quad I_p]^T$, $D_{21} = [0 \quad I_q]$,

A3: G_{12} and G_{21} do not have any transmission zeros on $\mathbf{C}^0$ and for all ω satisfying $|j\omega I - A| = 0$, $rank\,[j\omega I - A\ ,\ B_2] = n$ and $rank\,[j\omega I - A^T\ ,\ C_2^T] = n$, $\begin{bmatrix} j\omega I - A & B_2 \\ C_1 & D_{12} \end{bmatrix}$, $\begin{bmatrix} j\omega I - A & B_1 \\ C_2 & D_{21} \end{bmatrix}$ are of fullrank.

Note subject to A3, uncontrollable modes of $(A,\ B_2)$ and unobservable modes of $(C_2,\ A)$ on $\mathbf{C}^0$ are the only $\mathbf{C}^0$ invariant zeros of G_{12} and G_{21}. The implication of A1 is that weighting functions are permitted to have $\mathbf{C}^0$ poles, and $\mathbf{C}^+$ poles that are both controllable from B_2 and observable from C_2. The first part of A3 generically implies that weighting functions do not have transmission zeros on $\mathbf{C}^0$ which is usually satisfied; meanwhile the second part implies that if G_{22} has a pole on $\mathbf{C}^0$, it also has to be a pole of G_{11}, G_{12} and G_{21}. When A3 does not hold, we face the so-called singular problems which we do not consider here.

Compared with the assumptions made in the standard $\mathcal{H}^\infty$ problem, the controllability and the observability on $\mathbf{C}^0$ is not required and assumptions on the invariant zeros of G_{12} and G_{21} are relaxed to permit unstable cancellations outside the feedback loop.

Encouragement for such generalization of $\mathcal{H}^\infty$ problems came from the work of Zhang, et al.[11]. They showed that for the mixed-sensitivity problem, an $\mathcal{H}^\infty$ servosystem can be constructed by inserting the reference signal generator into the weighting function of the sensitivity function.

It will be shown the solvability condition and the structure of the generalized $\mathcal{H}^\infty$ control system are analogous to those of the standard $\mathcal{H}^\infty$ problem except that we need different kind of solution of Riccati equation which we call the quasi-stabilizing solution.

We disclose that in a generalized $\mathcal{H}^\infty$ control system all invariant zeros of G_{12} and G_{21} in $\bar{\mathbf{C}}^-$ are hidden modes of the whole closed loop system. Thus the time response of an $\mathcal{H}^\infty$ control system can be improved by adjusting weighting functions and control criteria to avoid undesired cancellation.

2 Riccati equation

Let us consider the following differential game type Riccati equation.

$$\begin{aligned} &(A - BR^{-1}S^T)X + X(A - BR^{-1}S^T)^T \\ &-X(Q - SR^{-1}S^T)X + BR^{-1}B^T = 0\,, \qquad (DGR) \end{aligned}$$

$$\hat{A} = A - BR^{-1}S^T - X(Q - SR^{-1}S^T)\,. \qquad (6)$$

Definition 2 *When (DGR) has a symmetric solution X such that the spectrum of $\hat{A}$ lies in $\bar{\mathbf{C}}^-$, $\lambda[\hat{A}] \in \mathbf{C}^0$ is uncontrollable from X and $m.uc.(\hat{A}\ ,\ X) = m.uc.(A\ ,\ B)$, X is called a quasi-stabilizing solution of (DGR).*

We can prove that the quasi-stabilizing solution of (DGR) is unique. When $(A\ ,\ B)$ has uncontrollable modes on $\mathbf{C}^0$, the Hamiltonian matrix associated with (DGR) has eigenvalues on $\mathbf{C}^0$. In this case, it is technically difficult to calculate X directly. However, the following lemma offers us a way of calculating the solution of (DGR).

Lemma 1 *Make the following decompositions*

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

$$T^TQT = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \quad T^TS = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

in which A_{22} is the maximal uncontrollable mode of A in $\bar{\mathbf{C}}^-$. Then, (DGR) has a positive semidefinite quasi-stabilizing solution iff

$$\begin{aligned}&(A_{11} - B_1R^{-1}S_1^T)X_{11} + X_{11}(A_{11} - B_1R^{-1}S_1^T)^T \\ &-X_{11}(Q_{11} - S_1R^{-1}S_1^T)X_{11} + B_1R^{-1}B_1^T = 0 \quad (7)\end{aligned}$$

has a symmetric positive definite solution such that

$$\hat{A}_{11} = A_{11} - B_1R^{-1}S_1^T - X_{11}(Q_{11} - S_1R^{-1}S_1^T)$$

is stable. The solution for (DGR) is given by

$$X = Tdiag[X_{11}, \quad 0]T^T. \qquad (8)$$

3 Generalized $\mathcal{H}^\infty$ control

Partition D_{11} in the following way

$$\begin{aligned} D_{11} &= \begin{matrix} & r-q & q & \\ \Big[& D_{1111} & D_{1112} & \Big] \; m-p \\ \Big[& D_{1121} & D_{1122} & \Big] \; p \end{matrix} \\ &= \begin{bmatrix} D_{11r1} \\ D_{11r2} \end{bmatrix} \begin{matrix} m-p \\ p \end{matrix} = \overset{\quad r-q \qquad q}{[D_{11c1} \quad D_{11c2}]}. \end{aligned}$$

Owing to Assumption 2, when $max\{\bar{\sigma}(D_{11r1}), \bar{\sigma}(D_{11c1}^T)\} < 1$, we can define

$$\begin{aligned}
D_{12}^L &= [D_{11r2}D_{11r1}^T(I - D_{11r1}D_{11r1}^T)^{-1} \quad I] \\
D_{12}^\perp &= [(I - D_{11r1}D_{11r1}^T)^{-1/2} \quad 0] \\
D_{21}^R &= [D_{11c2}^TD_{11c1}(I - D_{11c1}^TD_{11c1})^{-1} \quad I]^T \\
D_{21}^\perp &= [(I - D_{11c1}^TD_{11c1})^{-1/2} \quad 0]^T \\
A_a &= A - B_2D_{12}^LC_1, \quad A_b = A - B_1D_{21}^RC_2 \qquad (9) \\
S_a &= -B_1D_{11}^T(D_{12}^\perp)^T, \quad S_b = -C_1^TD_{11}D_{21}^\perp \\
Q_a &= B_2D_{12}^L(B_2D_{12}^L)^T \\
&\quad -(B_1 - B_2D_{12}^LD_{11})(B_1 - B_2D_{12}^LD_{11})^T \\
Q_b &= (D_{21}^RC_2)^TD_{21}^RC_2 \\
&\quad -(C_1 - D_{11}D_{21}^RC_2)^T(C_1 - D_{11}D_{21}^RC_2).
\end{aligned}$$

Since

$$\begin{bmatrix} A_a - sI & 0 \\ D_{12}^\perp C_1 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -B_2D_{12}^L \\ 0 & D_{12}^\perp \\ 0 & D_{12}^L \end{bmatrix} \bullet \begin{bmatrix} A - sI & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} I & 0 \\ -D_{12}^LC_1 & I \end{bmatrix} \qquad (10)$$

$$\begin{bmatrix} A_b - sI & B_1D_{21}^\perp & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & -B_1D_{21}^R \\ 0 & I \end{bmatrix} \bullet \begin{bmatrix} A - sI & B_1 \\ C_2 & D_{21} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -D_{21}^RC_2 & D_{21}^\perp & D_{21}^R \end{bmatrix} \qquad (11)$$

the invariant zeros of G_{12} and G_{21} are exactly the unobservable modes of $(D_{12}^\perp C_1, A_a)$ and the uncontrollable modes of $(A_b, B_1D_{21}^\perp)$.

3.1 Necessary condition

Obviously

$$G_{12}^\perp[I - \Phi\Phi^\sim](G_{12}^\perp)^\sim = G_{12}^\perp[I - G_{11}G_{11}^\sim](G_{12}^\perp)^\sim.$$

So, when the generalized $\mathcal{H}^\infty$ problem has a solution, we have

$$\Gamma_a(s) = G_{12}^\perp[I - G_{11}G_{11}^\sim](G_{12}^\perp)^\sim > 0 \quad \forall s = j\omega.$$

Let $\omega \to \infty$, we obtain $\bar{\sigma}(D_{11r1}) < 1$. Simple manipulation yields

$$\begin{aligned}\Gamma_a(s) = \; & [D_{12}^\perp C_1(sI - A_a)^{-1} \quad I] \begin{bmatrix} Q_a & S_a \\ S_a^T & I \end{bmatrix} \\ & \bullet \begin{bmatrix} (-sI - A_a^T)^{-1}(D_{12}^\perp C_1)^T \\ I \end{bmatrix}.\end{aligned}$$

According to the spectral factorization theory of [7], the following Riccati equation

$$\begin{aligned}&X(A_a - S_aD_{12}^\perp C_1) + (A_a - S_aD_{12}^\perp C_1)^TX \\ &-X(Q_a - S_aS_a^T)X + (D_{12}^\perp C_1)^TD_{12}^\perp C_1 = 0 \quad (Ric1)\end{aligned}$$

has a symmetric positive semidefinite solution such that

$$\lambda[\hat{A}_a] \subset \bar{\mathbf{C}}^-, \; \hat{A}_a = A_a - S_aD_{12}^\perp C_1 - (Q_a - S_aS_a^T)X$$

$\lambda[\hat{A}_a] \in \mathbf{C^0}$ is unobservable from X and $m.uo.(X, \hat{A}_a) = m.uo.(D_{12}^\perp C_1, A_a)$.

Similarly, from

$$\begin{aligned}\Gamma_b(s) &= (G_{21}^\perp)^\sim[I - G_{11}^\sim G_{11}]G_{21}^\perp \\ &= [(B_1D_{21}^\perp)^T(-sI - A_b^T)^{-1} \quad I] \begin{bmatrix} Q_b & S_b \\ S_b^T & I \end{bmatrix} \\ &\quad \bullet \begin{bmatrix} (sI - A_b)^{-1}B_1D_{21}^\perp \\ I \end{bmatrix} > 0 \quad \forall s = j\omega\end{aligned}$$

we have $\bar{\sigma}(D_{11c1}^T) < 1$ and that the following Riccati equation has a symmetric positive semidefinite solution

$$\begin{aligned}&(A_b - B_1D_{21}^\perp S_b^T)Y + Y(A_b - B_1D_{21}^\perp S_b^T)^T \\ &-Y(Q_b - S_bS_b^T)Y + B_1D_{21}^\perp(B_1D_{21}^\perp)^T = 0 \quad (Ric2)\end{aligned}$$

such that

$$\lambda[\hat{A}_b] \subset \bar{\mathbf{C}}^-, \; \hat{A}_b = A_b - B_1D_{21}^\perp S_b^T - Y(Q_b - S_bS_b^T),$$

$\lambda[\hat{A}_b] \in \mathbf{C^0}$ is uncontrollable from Y and $m.uc.(\hat{A}_b, Y) = m.uc.(A_b, B_1D_{21}^\perp)$.

Remark 1 *We remark that when we solve (Ric 1) and (Ric 2) using the method of Lemma 1, the dimensions of the deflated Riccati equations are equivalent to (n − no. of $\bar{\mathbf{C}}^-$ invariant zeros of G_{12}) and (n − no. of $\bar{\mathbf{C}}^-$ invariant zeros of G_{21}) respectively.*

Further, as [8] we can prove based on the stability of $\Phi(s)$ that

$$\begin{bmatrix} X & XY \\ YX & Y \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} P' \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \qquad (12)$$

in which $P' > 0$. So $P \geq 0$ and $Ker\ P = Ker \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$. Therefore, $\rho(XY) < 1$ follows from these two conditions [8].

So far we have obtained the following necessary condition.

Proposition 1 *The generalized $\mathcal{H}^\infty$ control problem has a solution only if*

1. $max\{\bar{\sigma}(D_{11r1}), \bar{\sigma}(D_{11c1}^T)\} < 1$,
2. *(Ric 1) and (Ric 2) have positive semidefinite quasi-stabilizing solutions X, Y,*
3. $\rho(XY) < 1$.

Owing to (10) all $\bar{\mathbf{C}}^-$ invariant zeros of G_{12} and the mirror image of its $\mathbf{C}^+$ invariant zeros belong to $\lambda[\hat{A}_a]$ [7]. Further, the two conditions that $\lambda[\hat{A}_a] \in \mathbf{C}^0$ is unobservable from X and $m.uo.(X, \hat{A}_a) = m.uo.(D_{12}^\perp C_1, A_a)$ imply that all $\mathbf{C}^0$ eigenvalues of $\hat{A}_a$ are exactly the $\mathbf{C}^0$ invariant zeros of G_{12}, i.e. the $\mathbf{C}^0$ uncontrollable modes of (A, B_2) due to A3. For the same reason, $\lambda[\hat{A}_b]$ contains all $\bar{\mathbf{C}}^-$ invariant zeros of G_{21} and the mirror image of its $\mathbf{C}^+$ invariant zeros and all $\lambda[\hat{A}_b] \in \mathbf{C}^0$ are the $\mathbf{C}^0$ unobservable modes of (C_2, A).

3.2 Sufficient Condition

we give a brief proof that the condition of Proposition 1 is also sufficient (see [7] for details).

1) Under the condition of Proposition 1, both $\Gamma_a(j\omega)$ and $\Gamma_b(j\omega)$ are positive definite so that they can be spectrally factorized. We find the normalized spectral factors satisfying

$$I = G_{12}^\perp(s)[I - G_{11}(s)G_{11}^\sim(s)](G_{12}^\perp(s))^\sim \quad (13)$$

$$I = (G_{21}^\perp(s))^\sim[I - G_{11}(s)^\sim G_{11}(s)]G_{21}^\perp(s) \quad (14)$$

by dividing both sides of Γ_a, Γ_b with obtained spectral factors.

2) Rewrite the closed loop transfer function matrix $\Phi(s)$ as its homographic transformation form[5]

$$\Phi(s) = HM([G_a \quad V], K) = DHM(\begin{bmatrix} G_b \\ W \end{bmatrix}, K) \quad (15)$$

in which

$$G_a(s) = \begin{bmatrix} G_{12} - G_{11}G_{21}^R G_{22} & G_{11}G_{21}^R \\ -G_{21}^R G_{22} & G_{21}^R \end{bmatrix}$$
$$G_b(s) = \begin{bmatrix} G_{12}^L & G_{12}^L G_{11} \\ -G_{22}G_{12}^L & G_{21} - G_{22}G_{12}^L G_{11} \end{bmatrix} \quad (16)$$
$$V(s) = \begin{bmatrix} G_{11}G_{21}^\perp \\ G_{21}^\perp \end{bmatrix} \quad W(s) = [G_{12}^\perp \quad G_{12}^\perp G_{11}].$$

3) The fact that $[W^\sim \quad V]$ is $(J_{m,r}, J_{m-p,r-q})$-unitary is easily verified using the definition of W and V, (13) and (14). Furthermore, we prove based on the characterization of J-lossless matrix[2] and Riccati equations that $[W^\sim \quad V]$ is $(J_{m,r}, J_{m-p,r-q})$-lossless.

4) Construct based on the characterization of J-lossless matrix a matrix $\Theta(s)$ such that $[\Theta\ W^\sim\ V]$ is $(J_{m,r}, J_{p,q} \oplus J_{m-p,r-q})$-lossless.

5) According to [5], the feedback system of Fig. 1 is equivalent to that of Fig. 2 in which $G_a = \Theta\Pi^{-1} + VR$ and

$$\Pi = J_{p,q}G_b J_{m,r}\Theta, \quad \Pi^{-1} = J_{p,q}\Theta^\sim J_{m,r}G_a \quad (17)$$

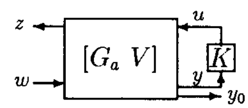

Fig. 2 Chain scattering system

Define $\bar{y}_0 = y_0 + R[u^T \ y^T]^T$, then the system of Fig. 3 is equivalent to that of Fig. 2 in the sense that the input/output relation of (z, y, w, u) is identical [5].

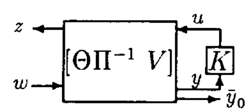

Fig. 3 Equivalent chain scattering system

Straightforward calculation yields

$$\Pi(s) = [\hat{A}_a, B_\pi, C_\pi, D_\pi] \quad (18)$$

$$\Pi^{-1}(s) = [\hat{A}_b, B_{\pi-}, C_{\pi-}, D_{\pi-}] \quad (19)$$

6) Choose a controller as

$$K = HM(\Pi, S) \quad , \quad S \in \mathcal{BH}_{p \times q}^\infty, \quad (20)$$

then according to the multiplicative property of chain scattering system [5] and owing to the $(J_{m,r}, J_{p,q} \oplus -I)$- losslessness of $[\Theta \quad V]$, the closed loop transfer function becomes [8]

$$\begin{aligned} \Phi &= HM([\Theta\Pi^{-1} \quad V], HM(\Pi, S)) \\ &= HM([\Theta \quad V], S) \in \mathcal{BH}_{m \times r}^\infty . \end{aligned}$$

The cancellation of $\Pi(s)$ in the feedback loop implies all uncontrollable modes of (A, B_2) and unobservable modes of (C_2, A) on $\mathbf{C}^0$ are cancelled in the system of Fig. 1 as they are exactly the $\mathbf{C}^0$ eigenvalues of $\hat{A}_a$ and $\hat{A}_b$ respectively. There is no more unstable cancellation since the spectra of $\hat{A}_a$ and $\hat{A}_b$ are both in $\bar{\mathbf{C}}^-$. Further, there is no unstable cancellation in $HM([\Theta\ V], S)$ owing to the property of J-lossless chain scattering system[8]. Therefore we have proved that the system of Fig. 1 do not have $\mathbf{C}^+$ poles and the uncontrollable modes of (A, B_2) and unobservable modes of (C_2, A) on $\mathbf{C}^0$ are the only unstable poles of the closed loop system that are hidden modes as well. So C1, C3, C4 are satisfied. Then C2 is also satisfied. Therefore, (20) is an $\mathcal{H}^\infty$ controller.

Summarizing these results, We obtain the necessary and sufficient condition for the generalized $\mathcal{H}^\infty$ control.

Theorem 1 *The generalized $\mathcal{H}^\infty$ control problem has a solution iff*

1. $max\{\bar{\sigma}[D_{11r1}], \bar{\sigma}[D_{11c1}^T]\} < 1$,
2. *(Ric 1) and (Ric 2) have positive semidefinite quasi-stabilizing solutions X, Y,*
3. $\rho(XY) < 1$.

Compared with the solvability condition of the standard $\mathcal{H}^\infty$ problem, the difference lies in the requirement for the solution of Riccati equations.

3.3 Characterization of $\mathcal{H}^\infty$ controllers

In the previous subsection, we have shown that each $K(s)$ given by (20) is an $\mathcal{H}^\infty$ controller for the generalized $\mathcal{H}^\infty$ problem. Here we prove the converse that every $\mathcal{H}^\infty$ controller is expressed by (20).

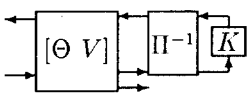

Fig. 4

When the generalized $\mathcal{H}^\infty$ problem has a solution $K(s)$, the whole system can be described as Fig. 4. Absorbing $\Pi^{-1}(s)$ into $K(s)$ and defining

$$S(s) = HM(\Pi^{-1}, K),$$

then the whole system turns into Fig. 5. All uncontrollable modes of $(A,\ B_2)$ and unobservable modes of $(C_2,\ A)$ on $\mathbf{C}^0$ which are contained in the poles and zeros of $\Pi^{-1}(s)$ must have been cancelled in this process. So there should be no further unstable cancellation in the system of Fig. 5. Therefore $S(s) \in \mathcal{BH}^\infty_{p\times q}$ due to $\Phi(s) \in \mathcal{BH}^\infty_{m\times r}$ and the $(J_{m,r}\ ,\ J_{p,q} \oplus -I_{r-q})$-losslessness of $[\Theta\quad V]$[8]. Hence the $\mathcal{H}^\infty$ controller $K(s)$ is of the form of

$$K(s) = HM(\Pi\,,\,S),\quad S(s) \in \mathcal{BH}^\infty_{p\times q}$$

according to the property of HM transformation[5].

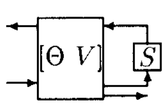

Fig. 5

As Glover-Doyle[3] we define the following notations.

$$D_{1\cdot} = [D_{11}\quad D_{12}]\quad D_{\cdot 1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \tag{21}$$

$$R = D_{1\cdot}^T D_{1\cdot} - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\quad \tilde{R} = D_{\cdot 1} D_{\cdot 1}^T - \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \tag{22}$$

$$F = \begin{bmatrix} F_{11} \\ F_{12} \\ F_2 \end{bmatrix} \begin{matrix} r-q \\ q \\ p \end{matrix} = -R^{-1}(D_{1\cdot}^T C_1 + B^T X) \tag{23}$$

$$H = \begin{matrix} m-p & p & q \\ \end{matrix} \atop \begin{bmatrix} H_{11} & H_{12} & H_2 \end{bmatrix} = -(B_1 D_{\cdot 1}^T + Y C^T)\tilde{R}^{-1} \tag{24}$$

Theorem 2 *All $\mathcal{H}^\infty$ controllers are given by*

$$K(s) = HM(\Pi\,,\,S)\quad ,\quad S(s) \in \mathcal{BH}^\infty_{p\times q} \tag{25}$$

$$: \Pi(s) = \left[\begin{array}{c|cc} A+BF & B_{\pi 1} & B_{\pi 2} \\ \hline C_{\pi 1} & D_{\pi 11} & D_{\pi 12} \\ C_{\pi 2} & D_{\pi 21} & D_{\pi 22} \end{array}\right]$$

where

$$\begin{aligned} B_{\pi 1} &= Z(B_2 + H_{12})\Delta_1 \\ B_{\pi 2} &= -Z(B_2 U + H_{12} U + H_2)\Delta_2^{-1} \\ C_{\pi 1} &= F_2 \\ C_{\pi 2} &= F_{12} + D_{22} F_2 + C_2 \\ Z &= (I - YX)^{-1}. \end{aligned} \tag{26}$$

When $|I - D_{22}U| \neq 0$, the $\mathcal{H}^\infty$ controllers are equivalently expressed in the linear fractional transformation form below.

$$K(s) = \bar{K}_{11} + \bar{K}_{12} S(I - \bar{K}_{22} S)^{-1} \bar{K}_{21}\,,\quad S \in \mathcal{BH}^\infty \tag{27}$$

$$\bar{K}(s) = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A_k & B_{k1} & B_{k2} \\ \hline C_{k1} & D_{k11} & D_{k12} \\ C_{k2} & D_{k21} & D_{k22} \end{array}\right] \tag{28}$$

in which

$$\begin{aligned} A_k &= A + BF - B_{k1}(F_{12} + D_{22}F_2 + C_2) \\ B_{k1} &= -Z(B_2 U + H_{12} U + H_2)(I - D_{22}U)^{-1} \\ B_{k2} &= Z(B_2 + H_{12})\Delta_1 - B_{k1} D_{22} \Delta_1 \\ C_{k1} &= F_2 - U\Delta_2^{-1} C_{k2} \\ C_{k2} &= -\Delta_2 (I - D_{22}U)^{-1}(F_{12} + D_{22}F_2 + C_2) \\ D_{k11} &= -U(I - D_{22}U)^{-1} \\ D_{k12} &= (I + U(I - D_{22}U)^{-1} D_{22})\Delta_1 \\ D_{k21} &= \Delta_2 (I - D_{22}U)^{-1} \\ D_{k22} &= -\Delta_2 (I - D_{22}U)^{-1} D_{22} \Delta_1\,. \end{aligned} \tag{29}$$

For the well-posedness of the $\mathcal{H}^\infty$ controller, $S(\infty)$ must satisfy

$$|I - D_{22}(U - \Delta_1 S(\infty)\Delta_2)| \neq 0\,. \tag{30}$$

Under the same condition, the $\mathcal{H}^\infty$ feedback system is well-posed as well.

(Proof) (26) follows from direct calculation. (29) is derived from [2]

$$\bar{K}(s) = \begin{bmatrix} \Pi_{12}\Pi_{22}^{-1} & \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} \\ \Pi_{22}^{-1} & -\Pi_{22}^{-1}\Pi_{21} \end{bmatrix}.$$

The well-posedness condition of the $\mathcal{H}^\infty$ controller is $|(\Pi_{21}S + \Pi_{22})(\infty)| \neq 0$ which is equivalent to (30). Further since

$$[I - D_{22}K(\infty)][I - D_{22}(U - \Delta_1 S(\infty)\Delta_2)] = I\,,$$

the generalized $\mathcal{H}^\infty$ system is well-posed as well. ■

When $D_{22} \neq 0$, there is the possibility of $|I - D_{22}U| = 0$. In this case the central controller (S(s)=0) does not exist and the $\mathcal{H}^\infty$ controllers can not be expressed in the linear fractional transformation form. However, even in such case it is possible that the $\mathcal{H}^\infty$ controller does exist for some non-zero $S(s)$. This is a merit of the homographical transformation representation of $K(s)$.

The following theorem shows a pole-zero cancellation phenomenon similar to the standard $\mathcal{H}^\infty$ systems [10, 8].

Theorem 3 *When there does not exist cancellation inside the $\mathcal{H}^\infty$ controller given in (20), all invariant zeros of G_{12} and G_{21} in $\bar{\mathbf{C}}^-$ are hidden modes of the generalized $\mathcal{H}^\infty$ control system.*

(Proof) From the characterization of the $\mathcal{H}^\infty$ controllers, we see that in the generalized $\mathcal{H}^\infty$ control system there occurs inevitablly the cancellation of $\Pi(s)$ and $\Pi^{-1}(s)$ if there is not cancellation of $\Pi(s)$ and $S(s)$ inside the controller. Since the invariant zeros of G_{12} and G_{21} in $\bar{\mathbf{C}}^-$ are the eigenvalues of $\hat{A}_a$ and $\hat{A}_b$, they must be cancelled in the chain scattering system. As a hidden mode of a chain scattering matrix is the same as that of the associated scattering matrix[2], the cancellation of $\Pi(s)$ in the chain scattering system then implies that all $\bar{\mathbf{C}}^-$ invariant zeros of G_{12} and G_{21} are hidden modes of the generalized $\mathcal{H}^\infty$ control system. ■

It is obvious that to make the $\mathcal{L}^\infty$ norm of a transfer function smaller, its poles have to be placed farther away from the imaginary axis. $\mathcal{H}^\infty$ control attenuates the $\mathcal{H}^\infty$ norm of the closed loop transfer function, therefore ideally $\mathcal{H}^\infty$ control would offer fast transient response if there were no hidden modes in the closed loop system. Nevertheless, it is often complained that $\mathcal{H}^\infty$ control usually yields bad time response. This is due to the pole-zero cancellation in the $\mathcal{H}^\infty$ control system. Theorem 3 tells us what the hidden modes are, it thus provides a guideline for the selection of weighting functions and control criteria, i.e. to prevent the appearance of slow modes in the invariant zeros of G_{12} and G_{21}.

4 Conclusion

In this paper, we have formulated and solved a generalized $\mathcal{H}^\infty$ control problem. In the generalized $\mathcal{H}^\infty$ problem, the internal stability of the whole system of Fig. 1 is alleviated to allow pole-zero cancellation on the imaginary axis outside the feedback loop, thus admit unstable poles of the whole system. This generalization of $\mathcal{H}^\infty$ control problem deletes the superfluous internal stability condition of the standard $\mathcal{H}^\infty$ control, which enables the incorporation of servosystem synthesis into $\mathcal{H}^\infty$ theory, therefore enlarges the application field for $\mathcal{H}^\infty$ theory and enhances the usability of $\mathcal{H}^\infty$ theory.

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