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Robust Output Feedback Compensator Design for Multivariable Systems

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Abstract

In this note, we consider a robust linear shift-invariant feedback compensator design for discrete-time multivariable systems which have both matched and mismatched uncertainties. In order to attack the problem of designing robust compensators guaranteeing uniform ultimate boundedness of every closed-loop system response within a neighborhood of the zero state based solely on the knowledge of the upper norm-bounds of uncertainties, we use an approach which is effective on studying augmented feedback control systems with both mismatched and matched uncertainties. We draw some robust stability conditions using the approach and give an example.

I. Introduction

Over the years, the problem of designing a robust controller which guarantees the desired performance and stability of continuous-time multivariable systems whose mathematical models are subject to uncertainties has been occupied the attention of system theorists. Many researchers have attacked the problem from the deterministic point of view by using a Lyapunov approach. The salient feature of their approaches is the fact that it is a deterministic treatment of uncertainties in that a certain deterministic performance is required in the presence of uncertain information.[1-4]

Recently, Corless and Manela [5], and Magana and Zak [6] have extended the results for continuous-time systems to apply to discrete-time uncertain dynamical systems described by difference equations.

But both results are based on the assumption that uncertainties satisfy the so-called "matching conditions" and the assumption that the actual system state should be available directly. In addition, in their results if the nominal systems of dynamic systems are not stable then a preliminary stabilization of nominal systems should be performed. In most practical situations, dynamic systems may have uncertainties which do not satisfy matching conditions and the actual system state is not available directly.

In this note, taking account of these problems, we propose a robust linear shift-invariant feedback compensator design methodology for discrete-time multivariable systems which have both mismatched and matched uncertainties. We use an approach, which is more effective than the Lyapunov approach in studying augmented feedback control systems with both mismatched and matched uncertainties, to draw some conditions for uniform boundedness and uniform ultimate boundedness of the closed-loop system and our analysis is restricted to uncertain multivariable systems where the nominal systems are linear. According to the proposed methodology one does not have to stabilize the nominal system preliminarily. To guarantee uniform ultimate boundedness of all possible system responses within a neighborhood of the zero state one has only to locate the nominal closed-loop poles inside the disc in the z-plane the radius of which is determined by norm-bounds on the uncertainties and/or norms involving the parameter of both compensator and system model. Therefore, the control system design can be well performed through eigenstructure assignment.

Notations : If x is a real vector, then $\|x\|_p$ is the norm defined by $\|x\|_p = \{\sum |x_i|^p\}^{1/p}$ where x_i denotes the element of the vector x and $p = 1, 2, \infty$. If A is a matrix, then $\|A\|_{ip}$ is the induced matrix norm corresponding to the vector norm. Details on the norms may be found in [7].

II. Problem Statement and Background Results

Let the actual plant to be controlled be represented by the following difference equations with $x(0) = x_0$

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \eta_1(k, x(k), u(k)) \\ y(k) &= Cx(k) + \eta_2(k, x(k)) \end{aligned} \tag{1}$$

where $k \in Z$, Z is the set of nonnegative integers, and $x(k) \in R^n$, $u(k) \in R^m$, $y(k) \in R^r$ are the state, input, and output respectively. A, B, and C are constant real matrices with appropriate dimensions and $\eta_1(k, x(k), u(k))$ and $\eta_2(k, x(k))$ are uncertainties with the following known upper norm-bounds:

$$\begin{aligned} \|\eta_1(k, x(k), u(k))\|_p &\le \beta_1 + \beta_2 \|x(k)\|_p + \beta_5 \|u(k)\|_p \\ \|\eta_2(k, x(k))\|_p &\le \beta_3 + \beta_4 \|x(k)\|_p \end{aligned} \tag{2}$$

where β_i, $i = 1, \ldots, 5$ are nonnegative constants.

Without loss of generality, we assume that the triple (A, B, C) is controllable and observable. Suppose that only the output vector y is directly available. Let the control system be the following output feedback compensator:

$$v(k+1) = K_{21} y(k) + K_{22} v(k) \qquad v(0) = v_0 \tag{3}$$

$$u(k) = K_{11} y(k) + K_{12} v(k) \tag{4}$$

where K_{11}, K_{12}, K_{21}, and K_{22} have appropriate dimensions, and (3) is a dynamic compensator of order s; $0 \le s \le n$. The extreme case $s=0$ represents static gain output feedback.

Thus, our design problem is formulated as choosing the parameters K_{11}, K_{12}, K_{21}, and K_{22} of (3) and (4) such that all the closed-loop system responses of (1), (3) and (4) satisfy uniform ultimate boundedness within a neighborhood of the zero state.

Now, we will state two technical lemmas which will be used in the next section and the proofs of the lemmas are omitted because of the space limitations.

Lemma 1: For the dynamical system

$$x(k+1) = f(k,x(k)) \qquad x(0) = x_0$$

Suppose that for some finite constants $a>0$, $1>r\ge 0$, $\delta>0$ the following inequality holds:

$$\|x(k)\|_p \le ar^k\|x_0\|_p + \delta(1-r^k), \ \forall k\ge 0 \tag{5}$$

Then the following properties hold:

1) Uniform Boundedness: Given any $S\in[0,\infty)$, there exists a $d(S)<\infty$ such that $\|x_0\|_p\le S$ implies $\|x(k)\|_p \le d(S)$, $\forall k\ge 0$

2) Uniform Ultimate Boundedness: Given any $\bar\delta>\delta$ and any $S\in[0,\infty)$, there is a $T(\bar\delta,S)\in[0,\infty)$ such that $\|x_0\|_p\le S$ implies $\|x(k)\|_p\le\bar\delta$, $\forall k\ge T(\bar\delta,S)$

3) Uniform Stability: Given any $\bar\delta>\delta$, there is a $D(\bar\delta)>0$ such that $\|x_0\|_p\le D(\bar\delta)$ implies $\|x(k)\|_p\le\bar\delta$, $\forall k\ge 0$

Lemma 2: Suppose a function $F(k,\tau,x): Z\times Z\times R\to R$ satisfies

$$x_1\le x_2 \ \rightarrow\ F(k,\tau,x_1)\le F(k,\tau,x_2) \tag{6}$$

Let $x(k)$ be the solution to the inequality

$$x(k)\le x_0(k)+\sum_{\tau=0}^{k-1}F(k,\tau,x(\tau)) \tag{7}$$

Then the solution $w(k)$ of

$$w(k) = x_0(k)+\sum_{\tau=0}^{k-1}F(k,\tau,w(\tau)) \tag{8}$$

satisfies

$$x(k)\le w(k), \ \forall k\ge 0$$

III. Robust Output Compensator Control

Let $\bar x^T(k) = [x^T(k)\ v^T(k)]$, then the closed-loop system is given by

$$\begin{aligned}\bar x(k+1) &= \bar A\bar x(k) + \bar\eta_1(k,\bar x(k),u(k))\\ y(k) &= \bar C\bar x(k) + \bar\eta_2(k,\bar x(k))\end{aligned} \qquad \bar x(0) = \begin{bmatrix}x_0\\ v_0\end{bmatrix} \tag{9}$$

where

$$\bar A = \begin{bmatrix}A & 0\\ 0 & 0_s\end{bmatrix} + \begin{bmatrix}B & 0\\ 0 & I_s\end{bmatrix}\begin{bmatrix}K_{11} & K_{12}\\ K_{12} & K_{22}\end{bmatrix}\begin{bmatrix}C & 0\\ 0 & I_s\end{bmatrix}$$

$$= \begin{bmatrix}A+BK_{11}C & BK_{12}\\ K_{21}C & K_{22}\end{bmatrix} \qquad \bar C = [C\ \ 0] \tag{10}$$

$$\bar\eta_1(k,\bar x(k),u(k)) = \begin{bmatrix}\eta_1(k,x(k),u(k))+BK_{11}\eta_2(k,x(k))\\ K_{21}\eta_2(k,x(k))\end{bmatrix}$$

$$\bar\eta_2(k,\bar x(k)) = \eta_2(k,x(k)). \tag{11}$$

Associated with (9) we get an approximate closed-loop feedback system as follows:

$$\begin{aligned}\bar x(k+1) &= \bar A\bar x(k)\\ y(k) &= \bar C\bar x(k)\end{aligned} \qquad \bar x(0) = \begin{bmatrix}x_0\\ v_0\end{bmatrix}. \tag{12}$$

Let us define the transition matrix $\Phi(k)$ of (12) then it is clear that

$$\|\Phi(k)\|_{ip} = \|\bar A^k\|_{ip} \le \|\bar A\|_{ip}^k \quad k=0,1,\cdots \tag{13}$$

Let

$$\rho_1 = \beta_1 + \beta_3(\|BK_{11}\|_{ip}+\|K_{21}\|_{ip})$$

$$\rho_2 = \beta_2 + \beta_4(\|BK_{11}\|_{ip}+\|K_{21}\|_{ip}) + \beta_5\|[K_{11}C\ \ K_{12}]\|_{ip}$$

Now, we are ready to present a theorem which can be used for establishing a robust output feedback compensator design methodology.

Theorem 1: Consider (9). If we choose the control parameters of (3) and (4) such that the following inequality is satisfied:

$$\|\bar A\|_{ip} + \rho_2 < 1, \tag{14}$$

then the following properties hold:

1) Uniform Boundedness: Given any $S\in[0,\infty)$, there exists a $d(S)<\infty$ such that $\|\bar x_0\|_p\le S$ implies $\|\bar x(k)\|_p\le d(S)$, $\forall k\ge 0$

2) Uniform Ultimate Boundedness: Given any $\bar\delta > \delta_1 = \rho_1/(1-\|\bar A\|_{ip}-\rho_2)$ and any $S\in[0,\infty)$, there is a $T(\bar\delta,S)\in Z$ such that $\|\bar x_0\|_p\le S \rightarrow \|\bar x(k)\|_p\le\bar\delta$, $\forall k\ge T(\bar\delta,S)$

3) Uniform Stability: Given any $\bar\delta>\delta_1$, there is a $D(\bar\delta)>0$ such that $\|\bar x_0\|_p\le D(\bar\delta)\rightarrow\|\bar x(k)\|_p\le\bar\delta$, $\forall k\ge 0$

Proof : By *Lemma 1*, it suffices to show that there exist finite finite constants $a>0$ and $1>r\ge 0$ such that under (14)

$$\|\bar x(k)\|_p \le ar^k\|\bar x_0\|_p + \delta_1(1-r^k), \ \forall k\ge 0 \tag{15}$$

From (9), we get the following inequality :

$$\|\bar x\|_p \le \|\Phi(k)\|_{ip}\|\bar x_0\|_p + \sum_{j=0}^{k-1}\|\Phi(k-j-1)\|_{ip}\|\bar\eta_1(j,\bar x(j),u(j))\|_p$$

Through some algebraic manipulation and using *Lemma* 2 we obtain

$$\|\bar{x}(k)\|_p \le \|\bar{x}_0\|_p (\|\bar{A}\|_{ip} + \rho_2)^k + \delta_1(1-(\|\bar{A}\|_{ip}+\rho_2)^k) \quad (16)$$

Thus, if we choose $a = 1$ and $r = \|\bar{A}\|_{ip} + \rho_2 < 1$, then (15) is satisfied. □

Since the transition matrix $\Phi(k)$ of (12) also satisfies for some finite constants $\alpha \ge 0$, $m > 0$

$$\|\Phi(k)\|_{ip} \le m\,\alpha^k \qquad k = 0, 1, \cdots \quad (17)$$

using the similar manipulation in the proof of Theorem 1 we can get another expression about the upper norm-bound of $\|\bar{x}(k)\|_p$ as follows:

$$\|\bar{x}(k)\|_p \le m\,\|\bar{x}_0\|_p (\alpha + m\,\rho_2)^k + \frac{m\,\rho_1}{1-\alpha-m\,\rho_2}\left[1-(\alpha+m\,\rho_2)^k\right] \quad (18)$$

Thus, we can establish the following theorem.

Theorem 2: Consider (9). If we choose the control parameters of (3) and (4) such that the following inequality is satisfied:

$$\alpha + m\,\rho_2 < 1, \quad (19)$$

then the following properties hold:

1) Uniform Boundedness: Given any $S \in [0,\infty)$, there exists a $d(S) < \infty$ such that $\|\bar{x}_0\|_p \le S$ implies $\|\bar{x}(k)\|_p \le d(S)$, $\forall k \ge 0$

2) Uniform Ultimate Boundedness: Given any $\bar{\delta} > \delta_2 = m\rho_1/(1-\alpha-m\rho_2)$ and any $S \in [0,\infty)$, there is a $T(\bar{\delta},S) \in Z$ such that $\|\bar{x}_0\|_p \le S \rightarrow \|\bar{x}(k)\|_p \le \bar{\delta}$, $\forall k \ge T(\bar{\delta},S)$

3) Uniform Stability: Given any $\bar{\delta} > \delta_2$, there is a $D(\bar{\delta}) > 0$ such that $\|\bar{x}_0\|_p \le D(\bar{\delta}) \rightarrow \|\bar{x}(k)\|_p \le \bar{\delta}$, $\forall k \ge 0$

$\|\bar{A}\|_{ip}$ as well as α is approximately equal to $\max|\lambda_i(\bar{A})|$, i.e. they are upper bounds of $|\lambda_i(\bar{A})|$ where $\lambda_i(\bar{A})$, $i=1,2,\cdots,s+n$ denotes the eigenvalues of $\bar{A}$. So, robustness margins are given by the eigenvalue nearest to the unit disc in the z-plane.

Because ρ_2 of (14) or (19) are dependent on the norms of the parameters K_{11}, K_{12}, K_{21}, and K_{22}, it may not be easy to satisfy one of (14) and (19). But, if not only the upper norm-bound of output uncertainties but β_5 are small, i.e. $\beta_3 \approx 0$, $\beta_4 \approx 0$, and $\beta_5 \approx 0$, then $\rho_1 \approx \beta_1$ as well as $\rho_2 \approx \beta_2$, and it is sure that by choosing the gain matrices of (3) and (4) appropriately, because we can easily satisfy one of the two inequalities (14) and (19) without seriously caring about the norms of the gain matrices, we can guarantee uniform ultimate boundedness of all possible closed-loop system responses within a neighborhood of the zero state.

Consider the special case that $\|\eta_1(k,x(k),u(k))\|_p$ of (2) is bounded by a linear function of $\|x\|_p$ as well as $\|u\|_p$ and $\|\eta_2(k,x(k))\|_p$ of (2) is bounded by a linear function of $\|x\|_p$, i.e. $\beta_1 = \beta_3 = 0$, then (16) and (18) becomes respectively

$$\|\bar{x}(k)\|_p \le \|\bar{x}_0\|_p (\|\bar{A}\|_{ip} + \rho_2)^k \quad (20)$$

and

$$\|\bar{x}(k)\|_p \le m\,\|\bar{x}_0\|_p (\alpha + m\,\rho_2)^k \quad (21)$$

Thus, the following consequent corollary can be established.

Corollary 1: Consider (9) with $\beta_1 = \beta_3 = 0$. If we choose of (3) and (4) such that one of the two following inequalities is satisfied:

$$\|\bar{A}\|_{ip} + \rho_2 < 1, \quad (22)$$

$$\alpha + m\,\rho_2 < 1 \quad (23)$$

where

$$\rho_2 = \beta_2 + \beta_4(\|BK_{11}\|_{ip} + \|K_{21}\|_{ip}) + \beta_5\|[K_{11}C \;\; K_{12}]\|_{ip}$$

Then the closed-loop system (9) with $\beta_1 = \beta_3 = 0$ is asymptotically stable.

Proof : Immediate from (20), (21), (22) and (23). □

The inequality (23) is similar to the results of Sobel *et al.* [8]. Because their results are reduced to the special cases of our results, we can say that our results are general ones.

The above inequalities (14), (19), (22) and (23) are only sufficient conditions. So, we cannot say that robust controllers necessarily satisfy the inequalities. From the above inequality conditions, it is recommended in robust controller design that one should choose the parameters of a controller with minimum induced matrix norms. According to the chosen norm, the above sufficient conditions can be more conservative or less conservative, i.e. the sharpness of the conditions will varies with the chosen norm. So, we may satisfy the above sufficient conditions with some norms while with others we may fail to do. Especially, we may scale the eigenvectors of (12) by using a diagonal matrix V where $V = \mathrm{diag}(V_i)$ and V_i, $i = 1, \cdots, s+n$ is positive entries of V, then $\bar{x}_{scaled}(k) = V^{-1}\bar{x}(k)$ and the closed-loop system of (9) becomes with $\bar{x}_{scaled}(0) = V^{-1}\bar{x}(0)$

$$\bar{x}_{scaled}(k+1) = V^{-1}\bar{A}V\bar{x}_{scaled}(k) + V^{-1}\bar{\eta}_1(k, V\bar{x}_{scaled}(k), u(k))$$
$$y(k) = \bar{C}V\bar{x}_{scaled}(k) + \bar{\eta}_2(k, V\bar{x}_{scaled}(k)) \quad ,$$

and the sufficient conditions (14), (19), (22) and (23) can be replaced by less conservative conditions respectively.

From the preceding analysis, different controller design methods can be established. It can be seen from (10) that the compensator design problem is equivalent to a static output feedback problem, which has been treated by several authors. Especially, Kwon and Youn, in [9], drew the necessary and sufficient conditions for eigenstructure assignment by output feedback and gave a simple procedure for eigenstructure assignment by output feedback. Since to satisfy one of the robust stability conditions we have only to locate the nominal closed-loop poles inside the disc in the z-plane the radius of which is determined by norm-bounds on the uncertainties and/or norms involving the parameters of both compensator and system model, the robust controller design can be well performed through eigenstructure assignment by output feedback.

IV. Example

To illustrate the preceding results, we give an example.

Example : Consider the following dynamic system:

$$x(k+1) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.1 & 0.1 \\ 0 & 2 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(k)$$

$$+ \begin{bmatrix} 0.1x_3 + 0.1cos(u_2) \\ 0.1sin(x_2) - 0.1cos(u_1) \\ 0.1x_1 sin(u_2) \end{bmatrix} \quad x_0 = \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} -0.1x_1 \\ 0.1x_2 sin(u_1) \end{bmatrix}$$

Find a robust static gain output feedback controller which assures uniform ultimate boundedness of every closed-loop system responses $x(k)$ within a neighborhood of $x(k)=0$.

Solution: From the above difference equations, we get $\|\eta_1(k,x(k),u(k))\|_\infty \le 0.1 + 0.1\|x\|_\infty$ and $\|\eta_2(k,x(k))\|_\infty \le 0.1\|x\|_\infty$, i.e. $\beta_1 = 0.1$, $\beta_2 = 0.1$, $\beta_3 = 0$, $\beta_4 = 0.1$, and $\beta_5 = 0$. Through eigenstructure assignment by output feedback, we get a controller as

$$u = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} y$$

then $\|\bar{A}\|_{i\infty} = 0.2$, $\rho_1 = 0.1$, $\rho_2 = 0.1 + 0.1 \cdot 2$, $\|\bar{A}\|_{i\infty} + \rho_2 = 0.2 + 0.3 < 1$, and the following inequality holds:

$$\|x(k)\|_\infty \le 4.8 \cdot 0.5^k + 0.2, \quad \forall k \ge 0$$

V. Conclusion

In this note, we propose a robust linear shift-invariant compensator design methodology for discrete-time multivariable systems which have both matched and mismatched nonlinear time-varying model uncertainties with known upper norm-bounds. In order to design a robust output feedback compensator guaranteeing uniform ultimate boundedness of every system response within a neighborhood of the zero state, we use an approach which is effective on studying augmented feedback control systems with both mismatched and matched uncertainties. Through the approach we draw the sufficient conditions for robust stability, and to satisfy one of the robust stability conditions we have only to locate the nominal closed-loop poles inside the disc in the z-plane the radius of which is determined by norm-bounds on the uncertainties and/or norms involving the parameters of both compansator and system model. Thus, a simple design procedure can be established based on eigenstructure assignment by output feedback.

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