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A study on the construction of balanced realization

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Abstract The paper considers the algorithms of balanced realization from SISO transfer functions. Some methods which have been proposed to find a balanced realization from the companion form realization, are investigated. Then a new method is proposed which finds a balanced realization from the discrete Schwarz form realization. The process of computing the elements of Schwarz matrix from the transfer function is equivalent to the Schur-Cohn stability test procedure. Comparison of the proposed method with the previous works is also discussed.

1 Introduction

Balanced realizations are characterized by having equal and diagonal controllability and observability Gramians with the relative size of each diagonal entry being an indication of the importance of the corresponding state to the input-output behavior. Balanced realizations have been under investigation in connection with model reduction problems.

Recently, some algorithms of balanced realization from an SISO transfer function have been proposed, which require Cholesky decomposition of the solution of Lyapunov equation(Young 1985). All the algorithms proposed in the literature so far can be considered as the methods which find the transformation matrix of balanced realization from the companion form realization such as the observability canonical form. The main numerical steps are the evaluation of a polynomial in a companion matrix, a Cholesky decomposition and a singular-value decomposition.

In this paper, a new algorithm is proposed which finds transformation matrix of the balanced realization from the discrete Schwarz form realization. The process of computing the elements of Schwarz matrix from the transfer function is equivalent to the Schur-Cohn stability test procedure. It is found that the method proposed by Therapos(1985) also involves the variables which appear in the computing process. The comparison of our proposed method with that of Therapos is also carried out, focusing on the meanings of the various variables which appear in the both algorithms.

Section 2 gives a briefview on the balanced realization. In Section 3, the method of Young(1985) which require Cholesky decomposition of the solution of Lyapunov equation is described. The discrete Schwarz matrix form is explained in Section 4. Section 5 describes the algorithms of Therapos(1985) based on the Bezout matrix. Then a new algorithm based on the Schwarz form realization is proposed in Section 6. This section is the major point in the paper. Finally in section 7, the relation between these algorithms is discussed.

2 A briefview on the balanced realization

Consider a discrete single-input single-output (SISO) system, described by its transfer function

$$G(z) = \frac{b(z)}{a(z)} = \frac{\tilde{a}(z)}{\tilde{b}(z)} \tag{1}$$

where the polynomials $a(z), b(z)$ are

$$a(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \tag{2}$$

$$b(z) = b_1 z^{n-1} + \cdots + b_{n-1}z + b_n \tag{3}$$

and $\tilde{a}(z) = z^n a(z^{-1}), \tilde{b}(z) = z^n b(z^{-1})$.

Assume that transfer function G(z) is stable and has no pole-zero cancellations. If a realization of the transfer function G(z) is described by $(A, \mathbf{b}, \mathbf{c}^T)$, then the solutions X, Y of the Lyapunov equations

$$X - AXA^T = \mathbf{b}\mathbf{b}^T \tag{4}$$

$$Y - A^T Y A = \mathbf{c}\mathbf{c}^T \tag{5}$$

are called the controllability gramian and the observability gramian of $(A, \mathbf{b}, \mathbf{c}^T)$, respectively.

Based on the assumptions stated above,X, Y are both symmetric positive-definite matrices. Let the other realization of $G(z)$ is $(\bar{A}, \bar{\mathbf{b}}, \bar{\mathbf{c}}^T)$, then $\bar{A} = TAT^{-1}, \bar{\mathbf{b}} = T\mathbf{b}, \bar{\mathbf{c}}^T = \mathbf{c}^T T^{-1}$. The relations between $\bar{X}, \bar{Y}$ and X, Y are

$$\bar{X} = TXT^T, \bar{Y} = T^{-T} Y T^{-1} \tag{6}$$

Choose a transformation matrix T making $\bar{X} = \bar{Y} = \Sigma$ be diagonal matrices, then the corresponding realization $(\bar{A}, \bar{\mathbf{b}}, \bar{\mathbf{c}})$ of G(z) is called the balanced realization.

3 Young's method

For a given rational function $G(z)$, analytic outside the closed unit disc and satisfying $G(\infty) = 0$, this method is proposed for the realization of $G(z)$ by a discrete linear system which is balanced(i.e. its controllability and observability gramians are equal and diagonal). The main numerical steps are the evaluation of a polynomial in a companion matrix, a Cholesky decomposition and a singular-value decomposition.

Let a companion matrix of polynomial $a(z)$ is C, then

$$C = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & \ddots & O & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \tag{7}$$

Let the i rows of identity matrix be $\mathbf{e}_i$ and let the vector $\mathbf{h} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]^T$. Let the matrix of H and $\tilde{I}$ be

$$H = \begin{bmatrix} 0 & & & O \\ 1 & 0 & & \\ & \ddots & \ddots & \\ O & & 1 & 0 \end{bmatrix}, \tilde{I} = \begin{bmatrix} O & & 0 & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & O \end{bmatrix} \tag{8}$$

respectively.Then a realization of transfer function $G(z)$ is $(C^T, \tilde{I}\mathbf{h}, \mathbf{e}_n^T)$, the transformation matrix $\mathcal{R}$ can be defined by the next algorithm.

i) Form

$$D = \tilde{a}(H)^{-1}a(H)$$

ii) Use the Cholesky decomposition:

$$I - DD^T = \Gamma^{-1} = L_1 L_1^T$$

iii) Perform the singular-value decomposition:

$$L_1^T f(C^T) L_1^{-T} = \xi \Sigma \eta^T$$

iv) Set the transformation matrix

$$\mathcal{R} = \tilde{a}(H) L_1 \xi \Sigma^{\frac{1}{2}}$$

v) Obtain the balanced realization:

$$\begin{cases} \bar{A} = \mathcal{R}^{-1} C^T \mathcal{R} \\ \bar{\mathbf{b}} = \Sigma^{\frac{1}{2}} \eta^T L_1^T \mathbf{e}_1 = \mathcal{R}^{-1} \tilde{I}\mathbf{h} \\ \bar{\mathbf{c}}^T = \mathbf{e}_n^T \mathcal{R} \end{cases}$$

where $f(C)$ is defined by

$$f(C) = \tilde{a}(C)^{-1} b(C) = b(C)\tilde{a}(C)^{-1}. \tag{9}$$

4 Discrete Schwarz matrix

Consider a certain matrix associated with a prescribed polynomial (real or complex) which has come to be known as the Schwarz matrix(1984). This matrix contains information about the root distribution of the polynomial, relative to the unit circle, i.e. on occasions, the matrix indicates the number of zeros of the polynomial in the region $|s| < 1$ for example.

From the coefficient a_i of the polynomials $a(z)$, the ϕ_i can be calculated by the next recursive relations

$$a_{i,n} = a_i, i = 1, 2, \ldots, n, \quad \lambda_n = 1$$

$$\begin{cases} \phi_m = a_{m,m} \\ (1 - \phi_m^2) a_{i,m-1} = a_{i,m} - \phi_m a_{m-i,m} \\ (1 - \phi_m^2)\lambda_{m-1} = \lambda_m \end{cases} \tag{10}$$

$$\begin{aligned} m &= n, n-1, \ldots, 1 \\ i &= 1, 2, \ldots, m-1 . \end{aligned}$$

By use of ϕ_i, Schwarz matrix Φ of $a(z)$ is defined such that

$$\Phi = \begin{bmatrix} -\phi_{n-1}\phi_n & 1-\phi_{n-1}^2 & 0 & \cdots & 0 \\ -\phi_{n-2}\phi_n & -\phi_{n-2}\phi_{n-1} & 1-\phi_{n-2}^2 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\phi_1\phi_n & -\phi_1\phi_{n-1} & -\phi_1\phi_{n-2} & & 1-\phi_1^2 \\ -\phi_n & -\phi_{n-1} & -\phi_{n-2} & \cdots & -\phi_1 \end{bmatrix} \tag{11}$$

Let

$$\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_n), \tag{12}$$

$$W = \begin{bmatrix} 1 & & & & \\ a_{1,1} & 1 & & O & \\ a_{2,1} & a_{1,2} & \ddots & & \\ \vdots & \vdots & \ddots & 1 & \\ a_{n-1,n-1} & a_{n-2,n-1} & \cdots & a_{1,n-1} & 1 \end{bmatrix} \tag{13}$$

then the matrix Φ can be expressed as

$$\Phi = \widetilde{W} C \widetilde{W}^{-1}, \tag{14}$$

where $\widetilde{W} = \tilde{I} W \tilde{I}$. This shows that the companion matrix C and the discrete Schwarz matrix are similar.

5 Therapos' method

The Bezout matrix $B(a, b)$ associated with the polynomials $a(z)$ and $b(z)$ is defined as

$$B(a,b) = \tilde{I}\tilde{a}(H)b(C) = \tilde{I}[\tilde{a}(H)b(H^T) - \tilde{b}(H)a(H^T)]. \tag{15}$$

The polynomials $a(z)$ and $b(z)$ have no zeros in common if and only the associated Bezout matrix $B(a, b)$ is nonsingular. By making use of the Bezout matrix $B(a, b)$ and matrices Λ, W, Therapos proposed a method which constructs the transformation matrix T of balanced realization from the realization $(C, \mathbf{e}_n, \mathbf{h}^T\tilde{I})$:

i) Form

$$\Pi = N^{-1}W\tilde{I} \quad (N = \Lambda^{\frac{1}{2}}\tilde{I})$$

ii) Compute Bezout matrix $B(a,b)$.

iii) Use matrix $B(a,b)$ to form

$$\Pi^{-T}B(a,b)\Pi^{-1} = \tilde{I}V\Theta\Sigma V^T\tilde{I}$$

iv) Set the transformation matrix:

$$T^T = RN\tilde{I}W^{-T}\tilde{I} \quad (T = \tilde{I}W^{-1}\tilde{I}N^TR^T)$$

v) Obtain the balanced realization:

$$\begin{cases} \bar{A} = T^{-1}CT \\ \bar{\mathbf{b}} = T^{-1}\mathbf{e}_n \\ \bar{\mathbf{c}}^T = \mathbf{h}^T\tilde{I}T \end{cases}$$

where matrix Θ is diagonal matrix made of $+1$ or-1, called sign matrix.

6 The realization algorithms based on Φ

Define a matrix M as

$$M = \tilde{a}(C)W^{-1}\Lambda^{\frac{1}{2}}. \quad (16)$$

Let $\mathbf{g} = \tilde{a}(H)^{-1}\mathbf{h}$, then $(C, \mathbf{g}, \mathbf{e}_1^T)$ is a realization of $G(z)$. Let the controllability matrix and the observability matrix of this realization be Γ and$P$, respectively, which are the solutions of the Lyapunov equations:

$$\Gamma - C^T\Gamma C = \mathbf{e}_1\mathbf{e}_1^T \quad (17)$$

$$P - CPC^T = \mathbf{g}\mathbf{g}^T \quad (18)$$

Then Γ and P have the closed-form expression as follows:

$$\Gamma = (MM^T)^{-1} \quad (19)$$

$$P = f(C)MM^Tf(C^T) \quad (20)$$

Since

$$f(C) = \tilde{a}(C)^{-1}b(C) = b(C)\tilde{a}(C)^{-1} \quad (21)$$

and $M^T\Gamma M = I$, we have from equation (17)

$$I - M^TC^TM^{-T}M^{-1}CM = M^T\mathbf{e}_1\mathbf{e}_1^TM \quad (22)$$

and by using equation (14), we obtain

$$\begin{aligned} & M^{-1}PM^{-T} - M^{-1}CMM^{-1}PM^{-T}M^TC^TM^{-T} \\ = \; & M^{-1}\mathbf{g}\mathbf{g}^{-T}M^T \end{aligned} \quad (23)$$

Then the observability matrix of a realization $(M^{-1}CM, M\mathbf{g}, \mathbf{e}_1^TM^T)$ is identity matrix and the controllability matrix is

$$M^{-1}f(C)MM^Tf(C^T)M^{-T}. \quad (24)$$

Suppose further that $M^{-1}f(C)M$ has the singular value decomposition

$$M^{-1}f(C)M = V\Sigma U^T \quad (25)$$

where U and V are orthonormal matrices. Let

$$R = \Sigma^{-\frac{1}{2}}V^T. \quad (26)$$

and

$$\hat{A} = RM^{-1}CMR^{-1}, \hat{\mathbf{b}} = RM^{-1}\mathbf{g}, \hat{\mathbf{c}}^T = \mathbf{e}_1^TMR^{-1} \quad (27)$$

Then $(\hat{A}, \hat{\mathbf{b}}, \hat{\mathbf{c}})$ is a balanced realization of $G(z)$, with controllability and observability matrices both equal to Σ, that is,

$$\Sigma - \hat{A}^T\Sigma\hat{A} = \hat{\mathbf{c}}\hat{\mathbf{c}}^T \quad (28)$$

$$\Sigma - \hat{A}\Sigma\hat{A}^T = \hat{\mathbf{b}}\hat{\mathbf{b}}^T \quad (29)$$

It can be seen from equation(27) that the matrix RM^{-1} is transformation matrix from the realization $(C, \mathbf{g}, \mathbf{e}_1^T)$ to the realization $(\hat{A}, \hat{\mathbf{b}}, \hat{\mathbf{c}}^T)$.

From the above discussions, the algorithms which finds the transformation matrix S of balanced realization from the realization $(C, \mathbf{g}, \mathbf{e}_1^T)$ can be described by

i) Form the matrix Λ, W

ii) Compute Bezout matrix $B(a,b)$

iii) Perform the singular-value decomposition:

$$M^{-1}f(C)M = \Lambda^{\frac{1}{2}}W^{-T}\tilde{I}B(a,b)W^{-1}\Lambda^{\frac{1}{2}} = V\Sigma U^T$$

iv) Set the transformation matrix:

$$S = RM^{-1} = R\Lambda^{\frac{1}{2}}W^{-T}\tilde{a}(H)$$

v) Obtain the balanced realization:

$$\begin{cases} \hat{A} = SCS^{-1} \\ \hat{\mathbf{b}} = S\mathbf{g} \\ \hat{\mathbf{c}}^T = \mathbf{e}_1^TS^{-1} \end{cases}$$

By using $\widetilde{W} = \tilde{I}W\tilde{I}$, the matrix M can be written as

$$\begin{aligned} M &= \tilde{a}(H)^{-1}\tilde{a}(H)\tilde{a}(C)W^{-1}\Lambda^{\frac{1}{2}} \\ &= \tilde{a}(H)^{-1}W^T\Lambda^{-\frac{1}{2}}. \end{aligned} \quad (30)$$

Since

$$\tilde{I}\tilde{a}(H)C = C^T\tilde{I}\tilde{a}(H) \quad (31)$$

it follows that

$$\begin{aligned} M^{-1}CM &= \Lambda^{\frac{1}{2}}W^{-T}\tilde{a}(H)C\tilde{a}(H)^{-1}W^T\Lambda^{-\frac{1}{2}} \\ &= \Lambda^{\frac{1}{2}}\tilde{I}\Phi^T\tilde{I}\Lambda^{-\frac{1}{2}} \\ &= N\Phi^TN^{-1} \end{aligned} \quad (32)$$

where

$$N = \Lambda^{\frac{1}{2}}\tilde{I} \quad (33)$$

Since $\tilde{a}(H)\mathbf{g} = \mathbf{h}$, it follows that

$$\begin{aligned} M^{-1}\mathbf{g} &= \Lambda^{\frac{1}{2}}W^{-T}\tilde{a}(H)\mathbf{g} \\ &= \Lambda^{\frac{1}{2}}W^{-T}\mathbf{h} \qquad (34) \\ &= N\widetilde{W}^{-T}\tilde{I}\mathbf{h} \end{aligned}$$

Let $\mathbf{e}_1 W^T = [1, \phi_1, \phi_2, \cdots, \phi_{n-1}]$, then

$$\begin{aligned} \mathbf{e}_1^T M &= \mathbf{e}_1^T W^T \Lambda^{-\frac{1}{2}} \\ &= [\phi_{n-1}, \cdots, \phi_1, 1]N^{-1} \end{aligned} \qquad (35)$$

and from (27), the realization$(\widehat{A}, \widehat{\mathbf{b}}, \widehat{\mathbf{c}})$ can be written as

$$\begin{cases} \widehat{A} = RN\Phi^T N^{-1}R^{-1} \\ \widehat{\mathbf{b}} = RN\widetilde{W}^{-T}\tilde{I}\mathbf{h} \\ \widehat{\mathbf{c}}^T = [\phi_{n-1}, \phi_{n-2}, \cdots, \phi_1, 1]N^{-1}R^{-1} \end{cases} \qquad (36)$$

Finally, we will describe the calculation of the matrix $M^{-1}f(C)M$. According to the equations (16),(21), we have

$$f(C)M = b(C)W^{-1}\Lambda^{\frac{1}{2}} \qquad (37)$$

And we can obtain from (30)

$$\begin{aligned} M^{-1}f(C)M &= \Lambda^{\frac{1}{2}}W^{-T}\tilde{a}(H)b(C)W^{-1}\Lambda^{\frac{1}{2}} \\ &= N\widetilde{W}^{-T}B(a,b)\tilde{I}\widetilde{W}^{-1}N^T \end{aligned} \qquad (38)$$

It is easily seen that the (i, j)-elements of$B(a, b)$ are

$$\begin{aligned} &[a_{n-i}, b_{n-j+1}] + [a_{n-i-1}, b_{n-j+2}] \\ &+ \cdots + [a_{n-i-k}, b_{n-j+k+1}] \end{aligned} \qquad (39)$$

where $k = \max(i, j) - 1$ and

$$[a_i, b_j] = a_i b_j - a_j b_i, \quad a_0 = 1, b_0 = 0 \qquad (40)$$

Hence, using $S = RN(\tilde{I}W\tilde{I})^{-T}\tilde{I}\tilde{a}(H)$, $\quad \Phi = \widetilde{W}C\widetilde{W}^{-1}$, the method which find balanced realization from the polynomials $a(z), b(z)$ can be described by following algorithm:

i) Form Λ, W and $\Phi^* = \tilde{I}\Phi^T\tilde{I}$

ii) Compute Bezout matrix $B(a, b)$

iii) Perform the singular-value decomposition:

$$M^{-1}f(C)M = \Lambda^{\frac{1}{2}}W^{-T}\tilde{I}B(a,b)W^{-1}\Lambda^{\frac{1}{2}} = V\Sigma U^T$$

iv) Set the transformation matrixT:

$$\mathcal{T} = RN\tilde{I} = \Sigma^{-\frac{1}{2}}V^T\Lambda^{\frac{1}{2}}$$

v) Obtain the balanced realization:

$$\begin{cases} \widehat{A} = \mathcal{T}\Phi^* \mathcal{T}^{-1} \\ \widehat{\mathbf{b}} = \mathcal{T}W^{-1}\mathbf{h} \\ \widehat{\mathbf{c}}^T = [1, \phi_1, \cdots, \phi_{n-1}]\mathcal{T}^{-1} \end{cases}$$

7 Discussion

We have the following relations between $\mathcal{R}, T$ and S:

$$\mathcal{R} = B(a,b)T \qquad (41)$$

$$S = T^T\tilde{I}\tilde{a}(H) \qquad (42)$$

By using $\tilde{I}\tilde{a}(H)C = C^T\tilde{I}\tilde{a}(H)$ and $B(a,b) = \tilde{I}\tilde{a}(H)b(C)$ $= b(C^T)\tilde{I}\tilde{a}(H)$, we can get $\mathcal{R} = b(C^T)S$. Let $\mathcal{H}$ is Hankel matrix of $G(z)$, then $\mathcal{H}$ and $B(a, b)$ have the relations

$$\mathcal{H} = \tilde{I}\tilde{a}(H)^{-1}B(a,b)\tilde{a}(H^T)^{-1}\tilde{I} \qquad (43)$$

and

$$T^T B(a,b)T = S\mathcal{H}S^T = \Theta. \qquad (44)$$

This relation implies the important properties of balanced realizations

$$\Theta\widehat{A} = \widehat{A}^T\Theta, \quad \Theta\widehat{\mathbf{b}} = \widehat{\mathbf{c}} \qquad (45)$$

Then it follows that

$$\bar{A} = \Theta\widehat{A}\Theta, \ \bar{\mathbf{b}} = \widehat{\mathbf{c}} = \Theta\widehat{\mathbf{b}}, \ \bar{\mathbf{c}} = \widehat{\mathbf{b}} = \Theta\widehat{\mathbf{c}}$$

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